Journal of Probability and Statistical Science
24(1), 1-15 Feb. 2026

On the Asymptotic Distribution of the Wilcoxon Signed Rank Test Statistic

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ABSTRACT

The Wilcoxon signed rank test statistic (T^+) is one of the most widely used nonparametric test statistic for one-sample and paired-sample data. As the sample size increases, it is well known that this test statistic can be approximated by a normal distribution [1]. The Gibbons and Chakraborti assumes the independence between the sign indicator Z_i and the rank $r(D_i)$ and $T^+ = \sum\sum_{1\le i\le j\le N} T_{ij}$ in their proof of asymptotic normality of Wilcoxon signed rank test statistic T^+ [1], however these assumptions are not rigorously proved. In this work, we demonstrate the rigorous proof of both of these assumptions as two theorems, thereby completing Gibbons and Chakraborti's derivation of asymptotic distribution of the Wilcoxon signed rank test statistics T^+ and provide applications for these two theorems.

1. Introduction

The Wilcoxon signed rank test statistic (T^+) is widely used in one-sample nonparametric tests [2] for inference about the sample median, assuming only that the underlying distribution is symmetric. The test uses both the sign of the differences D_i between observations an the hypothesized median, under the null hypothesis, as well as the magnitude of those differences.

For large sample sizes, the distribution of this statistic can be approximated by a normal distribution. Claypool, P. L. and Holbert, D. evaluated accuracy of normal approximations to the distribution of Wilcoxon signed rank statistics through computational studies [3].

To date authors such as Gibbons and Chakraborti [1] have outlined the calculation of the mean and variance of the Wilcoxon signed rank test statistic. However, a detailed justification of the independence between the sign indicator Z_i and the rank of D_i,$r(D_i)$ has typically not been provided. Authors have also assumed that $T^+ = \sum\sum_{1\le i\le j\le N} T_{ij}$ in the computation of the variance of T^+ without rigorous mathematical proof, where T_{ij} is an indicator of the sign of $D_i + D_j$and N is the sample size.

In addition to Gibbons and Chakraborti's work [1], Brian Albright recently provided the calculation of the distribution of the sum of signed ranks and developed an exact recursive algorithm for the distribution as well as an approximation of the distribution using the normal [4]. His proof of approximating Wilcoxon signed rank statistics by normal distribution is similar to the proof by Gibbons and Chakraborti [1] in that independence between the sign indicator Z_i and the rank of D_i was implicitly assumed however was not rigorously proved.

In this work, we fill in the details for the justification of the asymptotic distribution of Wilcoxon signed rank test statistic, presenting two proofs:

(1) The proof of independence between Z_i and $r(D_i)$ with the assumption that the distribution is symmetric; and (2) the proof that $T^+ = \sum\sum_{1\le i\le j\le N} T_{ij}$ with probability 1.

Wilcoxon signed rank test:

Let $X_1, \ldots, X_N$ be a random sample from a continuous cdf F_X with median M.

Assume that the distribution is symmetric about M, that is

$$f_X(M-x) = f_X(M+x)\ \forall\ x$$

- Received January 23, 2026, in final form February 2026.
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Where f_X is pdf of X.
We would like to test the hypothesis: $H_0: M = M_0$
Denote: $D_i = X_i - M$ and $Z_i = \begin{cases} 1 \ if \ \ D_i > 0 \\ 0 \ if \ \ D_i \leq 0 \end{cases}$
Wilcoxon signed rank test
$T^+ = \sum_{i=1}^{N} Z_i r(|D_i|)$ and $T^- = \sum_{i=1}^{N}(1 - Z_i) r(|D_i|)$
Approximating the asymptotic distribution of the Wilcoxon signed rank test statistic:

$$E(T^+|H_0) = \sum_{i=1}^{N} E\big(Z_i r(|D_i|)\big) = \sum_{i=1}^{N} E(Z_i) E\big(r(|D_i|)\big) = \sum_{i=1}^{N} \frac{1}{2}\left(\frac{N+1}{2}\right) = \frac{N(N+1)}{4}$$

In here the second equality is true if and only if Z_i and $r|D_i|$ are independent under null hypothesis
In order to derive the variance of T^+ under null hypothesis,

$$Var(T^+|H_0) = \sum_{i=1}^{N} \frac{[r(|D_i|)]^2}{4} = \frac{N(N+1)(2N+1)}{24}$$

we need to prove that $T^+ = \sum\sum_{1 \leq i \leq j \leq N} T_{ij}$ with probability 1.

2. The Independence Between $\mathbf{Z_i}$ and $\mathbf{r(D_i)}$

Theorem 1. *Let* $Z_i = \begin{cases} 1 \ if \ D_i > 0 \\ 0 \ if \ D_i \leq 0 \end{cases}$, $D_i = X_i - \theta$ *and* $r(|D_i|) = \sum_{j=1}^{N} I(|D_i| > |D_j|)$ *for any continuous random variable* X_i *that is symmetrically distributed about* $\theta, Z_1, \dots, Z_N$ *are independent of* $r(|D_i|)$

Proof.

Based on the assumption that the distribution of X is symmetrical, we can prove the independence of Z_i *and* $r(|D_i|)$ as follows.
Assume that X is continuous random variable and symmetrically distributed about θ, let us show that Z_i and $r(|D_i|)$ are independent, where $D_i = X_i - \theta, r(X_i) = \sum_{j=1}^{N} I(X_i > X_j)$ $Z_i = \begin{cases} 1 \ if \ \ D_i > 0 \\ 0 \ if \ \ D_i \leq 0 \end{cases}$ and
$I(X_i > X_j) = \begin{cases} 1 \ \ if \ \ X_i > X_j \\ 0 \ \ otherwise \end{cases}$
First let us show that Z_i is independent of $|D_i|$ for all possible i

Since X_i's are identically and independently distributed, we only need to show that Z is independent of $D = X - \theta$ under null hypothesis.

$CDF \ of |D| \ is \ P(|D| \geq x) \ where \ x \geq 0, and \ Z \sim Ber\left(\frac{1}{2}\right) \ under \ H_0$

$P(Z = 1) = E(Z) = \frac{1}{2}$

$P(Z = 1)P(|D| \geq x) = \frac{1}{2}P(|X - \theta| \geq x) = \frac{1}{2}P[\{X - \theta \geq x\} \cup \{X - \theta \leq -x\}]$

$= \frac{1}{2}\big[\big(P(X - \theta \geq x) + P(X - \theta \leq -x)\big)\big] = P(X - \theta \geq x) = P(X - \theta \geq x, X - \theta \geq 0)$

$= P(X - \theta \geq x, Z = 1) = P(|X - \theta| \geq x, Z = 1)$

The third equality is because $x \geq 0$ so that $\{X - \theta \geq x\} \cap \{X - \theta \leq -x\} = \emptyset$

The fourth equality is because we assume that X is and symmetrically distributed about θ. The fifth and seventh equalities are because $x \geq 0$.

$P(Z=0)P(|D| \geq x) = \frac{1}{2}P(|X-\theta| \geq x)\frac{1}{2}P[\{X-\theta \geq x\} \cup \{X-\theta \leq -x\}]$

$= \frac{1}{2}\left[\left(P(X-\theta \geq x) + P(X-\theta \leq -x)\right)\right] = P(X-\theta \leq -x) = P(X-\theta \leq -x, X-\theta \leq 0)$

$= P(X-\theta \leq -x, Z=0) = P(|X-\theta| \geq x, Z=0)$

In summary:

$P(|X-\theta| \geq x, Z=z) = P(|X-\theta| \geq x)P(Z=z)$

Thus, we have proved the independence between $|D_i|$ and Z_i under H_0

$Z_i = \begin{cases} 1 & if \quad D_i > 0 \\ 0 & if \quad D_i \leq 0 \end{cases} = \begin{cases} 1 & if \quad X_i > \theta \\ 0 & if \quad X_i \leq \theta \end{cases}$, thus Z_i is function of only X_i.

Since $X_1, \dots, X_N$ are mutually independent, $X_1, \dots, X_N$ are independent of $|D_1|, \dots, |D_N|$

$r(|D_i|) = \sum_{j=1}^{N} I(|D_i| > |D_j|) = g(|D_1|, \dots, |D_N|)$ is function of only $|D_1|, \dots, |D_N|$

Therefore $Z_1, \dots, Z_N$ are independent of $r(|D_i|)$.

3. The Signed Rank Test Statistic $T^+ = \sum\sum_{1 \leq i \leq j \leq N} T_{ij}$.

Theorem 2. $Let\ T^+ = \sum_{i=1}^{N} Z_i r(|D_i|),\ and\ T_{ij} = \begin{cases} 1 & if\ D_i + D_j > 0 \\ 0 & otherwise \end{cases}\ then\ T^+ = \sum\sum_{1 \leq i \leq j \leq N} T_{ij}$.

Proof

$Define\ I[A] = \begin{cases} 1 & if\ A\ is\ true \\ 0 & if\ A\ is\ false \end{cases}$

$$T^+ = \sum_{i=1}^{N} Z_i r(|D_i|) = \sum_{i=1}^{N} I(D_i > 0)\left(\sum_{j=1}^{N} I(|D_i| - |D_j| \geq 0)\right)$$

$$= \sum_{i=1}^{N} I(D_i > 0)\left[\sum_{j=1}^{N} I\left((|D_i| - |D_j|)(|D_i| + |D_j|) \geq 0\right)\right]$$

$$= \sum_{i=1}^{N} I(D_i > 0)\left[\sum_{j=1}^{N} I\left((D_i^2 - D_j^2) \geq 0\right)\right]$$

$$= \sum_{i=1}^{N} I(D_i > 0)\left[\sum_{j=1}^{N} I\left((D_i + D_j)(D_i - D_j) \geq 0\right)\right]$$

$$= \sum_{i=1}^{N} I(D_i > 0)\left[\sum_{j=1}^{N} I\left((D_i + D_j \geq 0, D_i - D_j \geq 0)\ or\ (D_i + D_j \leq 0, D_i - D_j \leq 0)\right)\right]$$

$$= \sum_{i=1}^{N} I(D_i > 0)\left[\sum_{j=1}^{N} \left(I(D_i + D_j \geq 0, D_i - D_j \geq 0) + I(D_i + D_j \leq 0, D_i - D_j \leq 0)\right)\right]$$

We used the fact that $I(A \cup B) = I[A] + I[B]\ if\ A \cap B = \phi$

But,

$If\ D_i > 0\ then\ I\big(D_i + D_j \leq 0, D_i - D_j \leq 0\big) = 0.$
$In\ fact\ D_i + D_j \leq 0, D_i - D_j \leq 0 \rightarrow 2D_i \leq 0 \ \rightarrow D_i \leq 0$
Then, with probability 1,

$$T^+ = \sum_{i=1}^{N}\left[\sum_{j=1}^{N} I(D_i > 0)I\big(D_i + D_j \geq 0, D_i - D_j \geq 0\big)\right]$$

$D_i + D_j \geq 0, D_i - D_j \geq 0\ also\ implies\ that\ D_i \geq 0$
Therefore $I(D_i > 0)I\big(D_i + D_j \geq 0, D_i - D_j \geq 0\big) = I(D_i \neq 0)I\big(D_i + D_j \geq 0, D_i - D_j \geq 0\big)$ and $I(D_i > 0)I(D_i + D_j \leq 0, D_i - D_j \leq 0) = 0$ with probability 1.
And then

$$T^+ = \sum_{i=1}^{N}\left[\sum_{j=1}^{N}(D_i \neq 0)I\big(D_i + D_j \geq 0, D_i - D_j \geq 0\big)\right]$$

$$= \sum_{i=1}^{N}\left[\sum_{j=1}^{N}(D_i \neq 0)I\big(D_i + D_j \geq 0, X_i - X_j \geq 0\big)\right]$$

Because $D_i = X_i - M_0$
But $I(D_i \neq 0) = 1$ occurs with probability 1 since F_X is continuous. Then, with probability 1,

$$T^+ = \sum_{i=1}^{N}\left[\sum_{j=1}^{N} I\big(D_i + D_j \geq 0, X_i - X_j \geq 0\big)\right] = \sum_{i=1}^{N}\left[\sum_{j=1}^{N} I\big(D_{(i)} + D_{(j)} \geq 0, X_{(i)} - X_{(j)} \geq 0\big)\right]$$

$$= \sum_{i=1}^{N}\left[\sum_{j=1}^{N} I\big(D_{(i)} + D_{(j)} \geq 0\big)I\big(X_{(i)} - X_{(j)} \geq 0\big)\right]$$

By Fubini's theorem:

$$T^+ = \sum_{j=1}^{N}\left[\sum_{i=1}^{N} I\big(D_{(i)} + D_{(j)} \geq 0\big)I\big(X_{(i)} - X_{(j)} \geq 0\big)\right]$$

But $X_{(i)} - X_{(j)} \geq 0 \leftrightarrow i \geq j$ then

$$T^+ = \sum_{j=1}^{N}\left[\sum_{i=1}^{N} I\big(D_{(i)} + D_{(j)} \geq 0\big)I(i - j \geq 0)\right]$$

If $i < j$ then $I(i - j \geq 0) = 0$ and If $i \geq j$ then $I(i - j \geq 0) = 1$
Therefore

$$T^+ = \sum_{j=1}^{N}\left[\sum_{i=j}^{N} I\big(D_{(i)} + D_{(j)} \geq 0\big)\right] = \sum_{j=1}^{N}\left[\sum_{i=j}^{N} I\big(D_i + D_j \geq 0\big)\right] = \sum\sum_{1\leq i<j\leq N} T_{ij}$$

with probability 1.

4. Applications

To compute the mean of Wilcoxon signed rank test statistic (T^+), we need to know the relation between T^+ and T_{ij}. By definition of T_{ij} we have,

$$E(T_{ij}) = P(D_i + D_j > 0);\ E(T_{ii}) = P(D_i > 0)$$

By **theorem 2**, we have

$$E(T^+) = E\left(\sum\sum_{1\le i<j\le N} T_{ij}\right)$$

Then, it has been demonstrated by Gibbons et. al [2] that

$$E(T^+) = NE(T_{ij}) + N\frac{(N-1)}{2}E(T_{ij}) = N\cdot P(D_i > 0) + \frac{N(N-1)}{2}\cdot P(D_i + D_j > 0)$$

Under null hypothesis

$$E(T^+|H_0) = N\cdot\frac{1}{2} + \frac{N(N-1)}{2}\cdot\frac{1}{2} = \frac{N(N+1)}{4}$$

To compute the variance of Wilcoxon signed rank test statistic (T^+), we also need to know the relation between T^+ and T_{ij}.

By definition of T_{ij} we have,

$E(T_{ij}) = P(D_i + D_j > 0);\ E(T_{ii}) = P(D_i > 0);$

$Var(T_{ii}) = E(T_{ii}^2) - [E(T_{ii})]^2 = E(T_{ii}) - [E(T_{ii})]^2 = P(D_i > 0) - [P(D_i > 0)]^2$

$Var(T_{ij}) = E(T_{ij}^2) - [E(T_{ij})]^2 = E(T_{ij}) - [E(T_{ij})]^2 = P(D_i + D_j > 0) - [P(D_i + D_j > 0)]^2$

$$\begin{aligned} cov(T_{ii}, T_{ij}) &= E(T_{ii}T_{ij}) - E(T_{ii})E(T_{ij}) \\ &= P(D_i > 0, D_i + D_j > 0) - P(D_i > 0)P(D_i + D_j > 0) \end{aligned}$$

$cov(T_{ij}, T_{ik}) = E(T_{ij}T_{ik}) - E(T_{ij})E(T_{ik})$

$= P(D_i + D_j > 0, D_i + D_k > 0) - P(D_i + D_j > 0)P(, D_i + D_k > 0)$

$cov(T_{ij}, T_{hk}) = 0$

By **theorem 2**, we have

$$Var(T^+) = Var\left(\sum\sum_{1\le i<j\le N} T_{ij}\right)$$

Then, it has been demonstrated by Gibbons et. al [2] that

$$\begin{aligned} Var(T^+) &= Var\left(\sum\sum_{1\le i<j\le N} T_{ij}\right) \\ &= Nvar(T_{ii}) + \binom{N}{2}var(T_{ij}) + 2N(N-1)cov(T_{ii}, T_{ik}) + 2N\binom{N-1}{2}cov(T_{ij}, T_{ik}) \\ &\quad + \binom{N}{4}cov(T_{ij}, T_{hk}) \end{aligned}$$

Under null hypothesis

$$Var(T^+|H_0) = Var\left[\left(\sum\sum_{1\le i<j\le N} T_{ij}\right)\middle| H_0\right] = \frac{N(N+1)(2N+1)}{24}$$

From the generalization of central limit theorem, as $N \to \infty$, the distribution of statistics

$$Z = \frac{T^{+} - \frac{N(N+1)}{4}}{\sqrt{\frac{N(N+1)(2N+1)}{24}}}$$

converges to standard normal distribution under null hypothesis.

References

[1]. Gibbons, J. D. and Chakraborti, S. (2011). *Nonparametric Statistical Inference 5th ed.*, Taylor & Francis Group, New York.

[2]. Wilcoxon, F. (1945). Individual Comparisons by Ranking Methods. *Biometric Bulletin*, 1, 80-3.

[3]. Claypool, P. L. and Holbert, D. (1974). Accuracy of Normal and Edgeworth Approximations to the Distribution of the Wilcoxon Signed Rank Statistic. *Journal of the American Statistical Association,* 69, 255-258.

[4]. Albright, A. (2012). The Distribution of the Sum of Signed Ranks. *The College Mathematics Journal*, 43, 232-236.

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