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A Retrospective Study on Nadarajah–Haghighi Distribution

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ABSTRACT

The Nadarajah–Haghighi (NH) distribution has been applied in diverse sectors in modeling lifespan data. By applying several techniques such as exponentiation, transformation, transmutation, trnasformed-transformer and compounding mechanisms among others, to existing distributions, academicians can construct new models that exhibit greater flexibility than the parent distribution and can better fit broadly real data sets. In this work, a survey of generalizations of NH distributions have been presented. The essential properties of the modified distributions encompassing explicit expressions for probability density, cumulative distribution and survival functions have been presented. A number of conventional estimation techniques have also been reviewed.

Keywords: Nadarajah–Haghighi distribution, zero mode, failure rate, generalization

1. Introduction

In almost every scientific discipline, statistical models are required to describe the trend and to predict the future behavior of their data. The quality of every parametric statistical analysis is influenced hugely by the probability distribution assumed. Pursuant to the mentioned reason, researchers are leaving no stone unturned in constructing new and standard probability distributions with various statistical techniques for many situations including lifespan cases, fatigue life studies, reliabilty problems as well as survival studies. Nonetheless, there is still no convergence among theoretical and applied statisticians, on preference of a particular distribution. To update the characteristics of old models, several statistical approaches have been proposed. A powerful technique for generating new family of distributions, named the exponentiated exponential distribution, as a generalization of exponential distribution, was proposed by [1]. [2] reported and discussed some mathematical and statistical identifying features of the distribution. The authors mentioned that many properties of the new class are akin to those of the Weibull or gamma family. Hence the distribution can be used as an effective replacement to a Weibull or gamma distributions.

Another generalization of the exponential distribution was proposed by [3]. The distribution was named, the Nadarajah–Haghighi (NH) distribution. This distribution can sufficiently play a replacement role for some distributions, namely, gamma, exponentiated exponential, Burr III and the popular Weibull distributions. [3] mentioned some theoretical and practical motivations: the ability to model data with mode fixed at zero and the fact that it can be interpreted as a truncated Weibull distribution; the nexus between the PDF and its hazard rate function (the NH density function can be monotonically decreasing and its hazard rate function can be increasing); the proposed model provided a better fit in terms of the information criteria and goodness of fit test to well-known distributions. Any distribution combined with NH has the potential to accumulates this advantages since it has as special model the NH distribution. The hazard rate function (hrf) of NH distribution can only be constant, monotonically decreasing, or monotonically increasing. Therefore, NH distribution is unsuitable to model lifespan data which display a non-monotonic or other curve of hazard function, such as machine life cycles and human

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mortality. Over a decade, academicians have been introducing various extensions or generalizations of the NH distribution, with a wide range of parameters.
If a random variable X is distributed according to NH distribution $(X \sim NH(\gamma, \theta))$, then its cumulative distribution function (CDF) is presented as

$$G(x) = 1 - \exp\left[1 - \left(1 + \gamma x\right)^{\theta}\right], \; x > 0, \; \gamma > 0, \; \theta > 0 \tag{1}$$

The probability density function (PDF) after differentiation is obtained by

$$g(x) = \gamma\theta\left(1 + \gamma x\right)^{\theta - 1} \exp\left[1 - \left(1 + \gamma x\right)^{\theta}\right] \tag{2}$$

The survival function is given by

$$\overline{G}(x) = \exp\left[1 - \left(1 + \gamma x\right)^{\theta}\right] \tag{3}$$

The hazard function describes the way in which the instantaneous probability of failure for a component changes with time. The hazard rate function is a significant quantity characterizing many random phenomena. The hazard rate function is defined as

$$h(x) = \lim_{\Delta x \to 0} \Pr\left(x \le X < x + \Delta x \mid X \ge x\right) / (\Delta x) = g(x) / \overline{G}(x)$$

where $\overline{G}(x) = 1 - G(x)$ is the survival function. Plausible hazard shapes are, namely, constant, decreasing, increasing, upside-down bathtub (unimodal), bathtub-shaped, decreasing-increasing-decreasing (DID) or other complicated process.
The hazard function of NH becomes

$$h(x) = \gamma\theta\left(1 + \gamma x\right)^{\theta - 1} \tag{4}$$

The quantile function of NH is written as

$$Q(u) = \frac{1}{\gamma}\left\{\left(1 - \log\left\langle 1 - u\right\rangle^{\frac{1}{\theta}}\right) - 1\right\}, \; u \in \left[0, 1\right] \tag{5}$$

where θ is a shape parameter, γ a scale parameter.

The chief aim of this work is to furnish a gentle introduction of the Nadaraja Haghighi distribution and discuss some of its recent developments. The rest of the work is outlined as follows. Section 2 is devoted to some existing generalizations of the Nadaraja-Haghighi distribution. Estimation techniques are reviewed in section 3 and Concluding remarks are presented in section 4.

2. Review of the existing family of Nadaraja-Haghighi distributions

This section is devoted to up-to-date review of the generalizations of families of Nadaraja-Haghighi distributions.

2.1 Unit Nadarajah and Haghighi distribution

By the transformation $Y = \exp(-X)$, [4] proposed and studied unit Nadarajah and Haghighi (UNH) distribution to deal with the inflation of ones. A special significance of UNH distribution is that, it was developed based on the unit interval. This distribution did not consider the positive part of a real line and neither includes extra parameters nor special functions in its construction. Apart from obtaining the mathematical and statistical identifying features, 9 estimation methodologies have been discussed in estimating the parameters. The UNH distribution is applied to real data sets. It is shown that the new distribution outperforms some well-known existing distributions. the authors further used the UNH to construct a control chart. The hazard function of the UNH distribution has decreasing, increasing-decreasing hazard function for different choices of the parameters. The PDF shape of the UNH distribution is decreasing and increasing for different values of the parameters.

The expression of the CDF of UNH is

$$G(y) = \exp\left\{1-\left(1-\delta \ln y\right)^{\gamma}\right\},\ 0<y<1,\ \delta>0,\ \gamma>0 \tag{6}$$

Linking this to equation (6), the PDF represented by $g(y)$ is obtained as

$$g(y) = \frac{\delta\gamma}{y}\left(1-\delta \ln y\right)^{\gamma-1} \exp\left\{1-\left(1-\delta \ln y\right)^{\gamma}\right\} \tag{7}$$

The survival function is written as

$$\overline{G}(y) = 1-\exp\left\{1-\left(1-\delta \ln y\right)^{\gamma}\right\} \tag{8}$$

The failure rate function becomes

$$h(y) = \frac{\delta\gamma\left(1-\delta \ln y\right)^{\gamma-1} \exp\left\{1-\left(1-\delta \ln y\right)^{\gamma}\right\}}{y\left[1-\exp\left\{1-\left(1-\delta \ln y\right)^{\gamma}\right\}\right]} \tag{9}$$

The p^{th} quantile is defined as

$$y_p = \exp\left\langle \frac{1}{\delta}\left\{1-\left[1-\ln(p)\right]^{\frac{1}{\gamma}}\right\}\right\rangle \tag{10}$$

2.2 New Generalized Nadarajah Haghighi Distribution

[5] proposed, studied and implemented three parameter New Generalized Nadarajah-Haghighi (NGNH) distribution. Structural statistical features and estimation technique discussed with impressive simulation outcomes. The NGNH was competitive among other distributions in real applications to data sets. The CDF is presented through the expression

$$G(x;\, B,\, \mathbf{\Omega}) = 1-\frac{B\exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}}{B-\left\langle 1-\exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}\right\rangle} \tag{11}$$

The PDF is given by

$$g(x;\, B,\, \mathbf{\Omega}) = \frac{B(B-1)\alpha\lambda\left(1+\lambda x\right)^{\alpha-1}\exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}}{\left(B-\left\langle 1-\exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}\right\rangle\right)^{2}} \tag{12}$$

The survival function is

$$\overline{G}(x;\, B,\, \mathbf{\Omega}) = \frac{B\exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}}{B-\left\langle 1-\exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}\right\rangle} \tag{13}$$

The hazard rate function is given by

$$h(x;\, B,\, \mathbf{\Omega}) = \frac{(B-1)\alpha\lambda\left(1+\lambda x\right)^{\alpha-1}}{B-1+\exp\left\{1-\left(1+\lambda x\right)^{\alpha}\right\}} \tag{14}$$

The quantile function is obtained as

$$x_p = \frac{\left\{1-\ln\left[\frac{(1-p)(B-1)}{B-1+p}\right]\right\}^{\frac{1}{\alpha}}-1}{\lambda},\ p\in\left[0,\,1\right] \tag{15}$$

2.3 Nadarajah Haghighi Generalised Power Weibull

By using the continuous-continuous approach, a distribution, named Nadarajah Haghighi Generalised Power Weibull, (NHGPW) was developed by [6]. The researchers took advantage of the technique of compounding. The NHGPW distribution was obtained by a hybrid of Nadarajah Haghighi and the generalised power Weibull distribution. The failure rate function of the NHGPW distribution displayed various shapes: increasing, decreasing, constant, bathtub, upside down bathtub (unimodal), modified bathtub or modified unimodal. The probability density function also exhibits different shapes including, decreasing-increasing, decreasing, increasing, increasing-decreasing, decreasing-increasing-decreasing, increasing-decreasing-increasing, positively skewed among others. The mathematical and statistical structure of NHGPW were obtained and estimation of parameters technique explored. The suggested model can be considered an alternative to Kumaraswamy log-logistic Weibull (KLLoGW) distribution, exponentiated generalized exponential Dagum (EGED), exponentiated generalized Fisk (EGFD) distribution, etc. The NHGPW distribution has 9 nested distributions of which two are new distributions (Exponential-Exponential Distribution and The NH-NH distribution). The CDF of NHGPW distribution is expressed as

$$G(x) = 1 - \exp\left\langle -\left\{ (1+\eta x)^{\phi} + \left(1+\upsilon x^{\kappa}\right)^{\theta} - 2 \right\} \right\rangle \tag{16}$$

The PDF of NHGPW after differentiation becomes

$$g(x) = \left[\eta\phi(1+\eta x)^{\phi-1} + \upsilon\kappa\theta x^{\kappa-1}\left(1+\upsilon x^{\kappa}\right)^{\theta-1} \right] \exp\left\langle -\left\{ (1+\eta x)^{\phi} + \left(1+\upsilon x^{\kappa}\right)^{\theta} - 2 \right\} \right\rangle \tag{17}$$

The reliability function is given by

$$\overline{G}(x) = \exp\left\langle -\left\{ (1+\eta x)^{\phi} + \left(1+\upsilon x^{\kappa}\right)^{\theta} - 2 \right\} \right\rangle \tag{18}$$

The failure rate function is obtained as

$$h(x) = \eta\phi(1+\eta x)^{\phi-1} + \upsilon\kappa\theta x^{\kappa-1}\left(1+\upsilon x^{\kappa}\right)^{\theta-1} \tag{19}$$

The quantile function of the NHGPW distribution is obtained by the solving the following equation

$$\left(1+\eta x_u\right)^{\phi} + \left(1+\upsilon x_u^{\kappa}\right)^{\theta} + \log(1-u) - 2 = 0\,,\ u \in [0,\, 1] \tag{20}$$

2.4 Nadarajah-Haghighi Lindley distribution

[7] in a detailed review on compounding method for generating distributions pointed out a different compounding technique by taking the minimum between two continuous distributions. Following this lead, [8] defined a continuous-continuous three-parameter competitive distribution named, Nadarajah-Haghighi Lindley (NHL) distribution. The statistical and mathematical identifying properties have been derived. The estimation of model parameters, simulation analysis as well as applications to real data have been provided. The PDF presents decreasing and reverse J shaped curve, while the NHL distribution can have decreasing, increasing, upside-down bathtub and bathtub-shaped hazard functions.

The CDF of NHL is obtained as

$$G(x) = 1 - \frac{(1+\beta+\beta x)}{1+\beta} \exp\left\{1 - \beta x - (1+\lambda x)^{\alpha}\right\},\ x>0,\ \beta>0\,,\ \alpha>0\,,\ \lambda>0 \tag{21}$$

The PDF of NHL after differentiation is given by

$$g(x) = \frac{(1+\beta+\beta x)\left[\beta + \alpha\lambda(1+\lambda x)^{\alpha-1}\right] - \beta}{1+\beta} \exp\left\{1 - \beta x - (1+\lambda x)^{\alpha}\right\} \tag{22}$$

The survival function becomes

$$\overline{G}(x) = \frac{(1+\beta+\beta x)}{1+\beta} \exp\left\{1 - \beta x - (1+\lambda x)^{\alpha}\right\} \tag{23}$$

The hazard function is obtained as

$$h(x)=\frac{(1+\beta+\beta x)\left[\beta+\alpha\lambda\left(1+\lambda x\right)^{\alpha-1}\right]-\beta}{1+\beta+\beta x} \tag{24}$$

It is significant to state that the CDF of NHL can only be inverted numerically.

2.5 Topp-Leone Nadarajah-Haghighi distribution

[9] proposed the Topp-Leone generated (TLG) family of distributions. [10] proposed a new three-parameter lifetime model based on the TLG family and NH distribution, which was named the Topp Leone Nadarajah-Haghighi (TLNH) distribution. The PDF of TPNH can be decreasing, positively skewed and increasing, while its hazard rate function can be constant, monotonic decreasing, reversed J-shaped and bathtub-shaped. The TPNH distribution outperform some well-known generalized distributions under the same criterion. It can also be used as an effective model for modeling survival data, reliability problems and fatigue life studies.
The CDF of TPNH is described as

$$G(x)=\left\langle 1-\exp\left\{2\left[1-\left(1+\theta x\right)^{\lambda}\right]\right\}\right\rangle^{\delta},\ \theta>0,\ \lambda,\delta>0,\ x>0, \tag{25}$$

The corresponding PDF is written as

$$g(x)=2\delta\lambda\theta\left(1+\theta x\right)^{\lambda-1}\exp\left\{2\left[1-\left(1+\theta x\right)^{\lambda}\right]\right\}\left\langle 1-\exp\left\{2\left[1-\left(1+\theta x\right)^{\lambda}\right]\right\}\right\rangle^{\delta-1} \tag{26}$$

The survival function is expressed as

$$\overline{G}(x)=1-\left\langle 1-\exp\left\{2\left[1-\left(1+\theta x\right)^{\lambda}\right]\right\}\right\rangle^{\delta} \tag{27}$$

The hazard rate function becomes

$$h(x)=\frac{2\delta\lambda\theta\left(1+\theta x\right)^{\lambda-1}\exp\left\{2\left[1-\left(1+\theta x\right)^{\lambda}\right]\right\}\left\langle 1-\exp\left\{2\left[1-\left(1+\theta x\right)^{\lambda}\right]\right\}\right\rangle^{\delta-1}}{1-\left\langle 1-\exp\left\{2\left[1-\left(1+\theta x\right)^{\lambda}\right]\right\}\right\rangle^{\delta}} \tag{28}$$

The quantile function of TLNH is given by

$$x_u=\frac{1}{\theta}\left\{\left\langle 1-0.5\log\left(1-u^{\frac{1}{\delta}}\right)\right\rangle^{\frac{1}{\lambda}}-1\right\},\ 0<u<1 \tag{29}$$

2.6 Nadarajah–Haghighi Lomax Distribution.

[11] presented a new distribution named, Nadarajah–Haghighi Lomax (NHLx) Distribution. The theoretical and practical facets of the NHLx distribution have been investigated by the authors. Some of its functional features includes: it has four parameters; it is lower-bounded; its failure rate function possesses increasing, decreasing, and upside-down bathtub shapes; and its probability density function (PDF) exhibits non-increasing and inverted J-shaped curves.
The CDF of NHLx distribution is described as

$$G(x)=1-\exp\left\{1-\left[1+b\left(\frac{\theta}{x+\theta}\right)^{-\gamma}\right]^{a}\right\},\ x\ge-\theta,\ a,\ b,\ \gamma,\ \theta>0 \tag{30}$$

Corresponding to equation (30), the PDF is obtained as

$$g(x)=\frac{ab\gamma}{\theta}\left(\frac{\theta}{x+\theta}\right)^{-\gamma+1}\left[1+b\left(\frac{\theta}{x+\theta}\right)^{-\gamma}\right]^{a-1}\exp\left\{1-\left[1+b\left(\frac{\theta}{x+\theta}\right)^{-\gamma}\right]^{a}\right\} \tag{31}$$

where γ and a are the shape parameters, whiles b and θ are the scale parameters. We obtained $g(x) = G(x)$ for $x < -\theta$. The asymptotic properties of the PDF largely depend on the γ.
The survival function is given by

$$\overline{G}(x) = \exp\left\{1 - \left[1 + b\left(\frac{\theta}{x+\theta}\right)^{-\gamma}\right]^{a}\right\} \tag{32}$$

By applying the definition, $h(x) = \dfrac{g(x)}{\overline{G}(x)}$, the hazard rate function becomes

$$h(x) = \frac{ab\gamma}{\theta}\left(\frac{\theta}{x+\theta}\right)^{-\gamma+1}\left[1 + b\left(\frac{\theta}{x+\theta}\right)^{-\gamma}\right]^{a-1}. \tag{33}$$

In contrast to the PDF, the asymptotic properties of the failure rate function depend on both γ and a.
The quantile function, after some mathematical development is established as

$$Q(u) = \theta\left\langle\left[\frac{1}{b}\left\{\left(1 - \log(1-u)\right)^{\frac{1}{a}} - 1\right\}\right]^{\frac{1}{\gamma}} - 1\right\rangle,\ u \in (0,\ 1) \tag{34}$$

2.7 Power Inverted Nadarajah–Haghighi Distribution

A novel three-parameter distribution, named power inverted Nadarajah–Haghighi (PINH) distribution is introduced and studied in [12]. This distribution generalizes the inverted Nadarajah–Haghighi distribution. The researchers obtained several important and mathematical features of PINH distribution. Moreover, characterization based on two truncated moments is also obtained, estimation of model parameters and applicability of PINH are discussed.
The CDF is given by

$$G(x) = \exp\left\{1 - \left(1 + \frac{\theta}{x^{a}}\right)^{b}\right\},\ x > 0,\ a,\ b,\ \theta > 0 \tag{35}$$

The PDF is given by

$$g(x) = ab\theta x^{-a-1}\left(1 + \frac{\theta}{x^{a}}\right)^{b-1}\exp\left\{1 - \left(1 + \frac{\theta}{x^{a}}\right)^{b}\right\} \tag{36}$$

where $(a,\ b)$ represents the shape parameters and θ the scaling factor. We obtain $g(x) = G(x)$ for $x < 0$.
The PDF in linear representation form is expressed as

$$g(x) = eab\theta x^{-a-1}\sum_{k=0}^{\infty}\frac{(-1)^{k}}{k!}\left(1 + \frac{\theta}{x^{a}}\right)^{b(k+1)-1} \tag{37}$$

The survival function is given as

$$\overline{G}(x) = 1 - \exp\left\{1 - \left(1 + \frac{\theta}{x^{a}}\right)^{b}\right\} \tag{38}$$

The hazard rate function is obtained as

$$h(x) = \frac{ab\theta x^{-a-1}\left(1 + \frac{\theta}{x^{a}}\right)^{b-1}\exp\left\{1 - \left(1 + \frac{\theta}{x^{a}}\right)^{b}\right\}}{1 - \exp\left\{1 - \left(1 + \frac{\theta}{x^{a}}\right)^{b}\right\}} \tag{39}$$

The quantile function can be obtained by inverting the CDF and is given by

$$Q(u) = \left\{ \frac{\left(1-\log(u)\right)^{\frac{1}{b}} - 1}{\theta} \right\}^{-\frac{1}{a}}, \quad u \in (0, 1) \tag{40}$$

2.8 The Odd Nadarajah-Haghighi (ONH) Family of Distributions

[13] proposed a new class of distributions, namely the Odd Nadarajah-Haghighi (ONH) Family of Distributions. Some mathematical features of the family are discussed. The researchers investigated three special distributions in this family, namely the ONH gamma, beta and Weibull distributions. The new density function can be expressed as a linear combination of exponentiated densities based on the same parent distribution. The authors provided a Monte Carlo simulation analysis to assess the maximum likelihood estimates, discussed estimation of the model parameters by maximum likelihood and an application to a real data set. The Odd NadarajahHaghighi-G (ONH-G for short) family is defined by the CDF

$$G(x) = 1 - \exp\left\{1 - \left[1 + \delta \frac{F(x)}{\overline{F}(x)}\right]^{\gamma}\right\}, \; x \in R, \; \delta > 0, \; \gamma > 0 \tag{41}$$

Corresponding to $G(x)$, the PDF of ONH-G family is given by

$$g(x) = \frac{\delta\gamma}{\overline{F}(x)^2}\left[1 + \delta \frac{F(x)}{\overline{F}(x)}\right]^{\gamma-1} \exp\left\{1 - \left[1 + \delta \frac{F(x)}{\overline{F}(x)}\right]^{\gamma}\right\} \tag{42}$$

The survival function is obtained as

$$\overline{G}(x) = \exp\left\{1 - \left[1 + \delta \frac{F(x)}{\overline{F}(x)}\right]^{\gamma}\right\} \tag{43}$$

The failure rate function is expressed as

$$h(x) = \frac{\delta\gamma}{\overline{F}(x)^2}\left[1 + \delta \frac{F(x)}{\overline{F}(x)}\right]^{\gamma-1} \tag{44}$$

The quantile function after inverting the CDF is obtained as

$$Q(u) = F^{-1}\left\langle \frac{\left[1-\log(1-u)\right]^{\frac{1}{\gamma}} - 1}{\left[1-\log(1-u)\right]^{\frac{1}{\gamma}} + \delta - 1} \right\rangle \tag{45}$$

where F^{-1} is the quantile function of the parent distribution.

2.9 Exponentiated (Lehmann Type-II) Nadarajah Haghighi Distribution

[14] introduced, studied and implemented a novel distribution based on Lehmann Type-II generator and NH distribution. This distribution is referred to by the authors as Exponentiated (Lehmann Type-II) Nadarajah Haghighi (ENH-II) distribution. The PDF as well as the hazard rate functions are more flexible than the original distribution. Various statistical distributional characteristics of the new model are explored. The work used the maximum likelihood estimation procedure to estimate the model parameters. Application to real data set has been demonstrated to show the usefulness and applicability of the model.

The CDF of ENH-II is given by

$$G(x) = 1 - \exp\left\{\phi\left\langle 1 - \left(1 + \upsilon x\right)^{\kappa}\right\rangle\right\} \tag{46}$$

The corresponding PDF is presented as

$$g(x) = \phi\upsilon\kappa\left(1 + \upsilon x\right)^{\kappa-1} \exp\left\{\phi\left\langle 1 - \left(1 + \upsilon x\right)^{\kappa}\right\rangle\right\} \tag{47}$$

The survival function is given by

$$\overline{G}(x)=\exp\left\{\phi\left\langle 1-\left(1+\upsilon x\right)^{\kappa}\right\rangle\right\} \tag{48}$$

Therefore, the hazard rate function of ENH-II is obtained as

$$h(x)=\phi\upsilon\kappa\left(1+\upsilon x\right)^{\kappa-1} \tag{49}$$

The integrated hazard rate function $H(x)$ is a risk indicator; the greater $H(x)$ value, the greater the risk of failure via time.

$$H(x)=\int_0^x h(t)dt=-\log\overline{G}(x)$$

It is noticed that $\overline{G}(x)=\exp\left(-H(x)\right)$ and $g(x)=h(x)\exp\left(-H(x)\right)$

Therefore, the integrated hazard rate function for ENH-II becomes

$$H(x)=-\log\left(\exp\left\{\phi\left\langle 1-\left(1+\upsilon x\right)^{\kappa}\right\rangle\right\}\right) \tag{50}$$

The quantile function is obtained after inverting the CDF. This is expressed as

$$Q(u)=\upsilon^{-1}\left\langle\left(1-\frac{\log(1-u)}{\phi}\right)^{\frac{1}{\kappa}}-1\right\rangle,\ 0<u<1 \tag{51}$$

2.10 Exponentiated Weibull Nadarajah Haghighi (EWNH) model

[15] proposed and studied exponentiated Weibull-H (EW-H) family. In a study, Kamal et al. (2020) used this generator with Nadarajah Haghighi distribution and obtained a new distribution, named exponentiated Weibull Nadarajah Haghighi (EWNH) model. The authors derived analytical formula for several statistical and mathematical quantities. The failure rate function of EWNH distribution exhibits both monotonic and non-monotonic shapes. The researchers mentioned several nested models of EWNH. The estimation of the distribution's unknown parameters, the flexibility and utility of the distribution with a real data set are presented. The CDF of EWNH distribution is given by

$$G(x)=\left\langle 1-\exp\left\{-\left[\frac{1-\exp\left(1-(1+\theta x)^{\gamma}\right)}{\exp\left(1-(1+\theta x)^{\gamma}\right)}\right]^{\beta}\right\}\right\rangle^{\alpha},\ x>0,\ \theta>0,\ \beta>0,\ \alpha>0 \tag{52}$$

$$g(x)=\theta\gamma\beta\alpha(1+\theta x)^{\gamma-1}\frac{\left(1-\exp\left(1-(1+\theta x)^{\gamma}\right)\right)^{\chi-1}}{\left(\exp\left(1-(1+\theta x)^{\gamma}\right)\right)^{\beta}}\exp\left\{-\left[\frac{1-\exp\left(1-(1+\theta x)^{\gamma}\right)}{\exp\left(1-(1+\theta x)^{\gamma}\right)}\right]^{\beta}\right\}$$
$$\times\left\langle 1-\exp\left\{-\left[\frac{1-\exp\left(1-(1+\theta x)^{\gamma}\right)}{\exp\left(1-(1+\theta x)^{\gamma}\right)}\right]^{\beta}\right\}\right\rangle^{\alpha-1} \tag{53}$$

The survival function is expressed as

$$\vec{G}(x)=1-\left\langle 1-\exp\left\{-\left[\frac{1-\exp\left(1-(1+\theta x)^{\gamma}\right)}{\exp\left(1-(1+\theta x)^{\gamma}\right)}\right]^{\beta}\right\}\right\rangle^{\alpha} \tag{54}$$

The hazard rate function is derived as

$$h(x)=\frac{\theta\gamma\beta\alpha(1+\theta x)^{\gamma-1}\frac{\left(1-\exp\left(1-(1+\theta x)^{\gamma}\right)\right)^{\chi-1}}{\left(\exp\left(1-(1+\theta x)^{\gamma}\right)\right)^{\beta}}\exp\left\{-\left[\frac{1-\exp\left(1-(1+\theta x)^{\gamma}\right)}{\exp\left(1-(1+\theta x)^{\gamma}\right)}\right]^{\beta}\right\}\times\left\langle 1-\exp\left\{-\left[\frac{1-\exp\left(1-(1+\theta x)^{\gamma}\right)}{\exp\left(1-(1+\theta x)^{\gamma}\right)}\right]^{\beta}\right\}\right\rangle^{\alpha-1}}{1-\left\langle 1-\exp\left\{-\left[\frac{1-\exp\left(1-(1+\theta x)^{\gamma}\right)}{\exp\left(1-(1+\theta x)^{\gamma}\right)}\right]^{\beta}\right\}\right\rangle^{\alpha}} \tag{55}$$

The quantile function, which is the inverse of the CDF is derived as

$$Q(u)=\theta^{-1}\left\langle\left[1-\log\left(\left[-\log\left(1-u^{\frac{1}{\alpha}}\right)\right]^{\frac{1}{\beta}}+1\right)^{-1}\right]^{\frac{1}{\gamma}}-1\right\rangle,\quad 0<u<1 \tag{56}$$

2.11 The Inverted Nadarajah-Haghighi distribution

[16] defined a new inverted distribution called The inverted Nadarajah-Haghighi (INH) distribution, corresponding to the distribution of $Y = X^{-1}$. The authors demonstrated its ability to model positive real data sets with monotonic decreasing and non-monotonic hazard rate shapes. The academicians [17] in a comparative analysis of INH distribution with ten inverted distributions, found that INH distribution was superior. [18] studied the reliability analysis of the INH distribution with an adaptive Type-I progressive hybrid censoring scheme. [19] considered the inference of unknown parameters using a generalized Type-II hybrid censoring scheme (GT-II HCS) for the INH distribution in the presence of competing risks. Bayes and maximum likelihood (ML) estimation techniques were discussed. The methods of inference were evaluated via simulation analysis and real data sets.

The associated PDF and CDF are obtained, respectively, as follows

$$g(y)=\frac{\theta\gamma}{y^{2}}\left(1+\frac{\gamma}{y}\right)^{\theta-1}\exp\left[1-\left(1+\frac{\gamma}{y}\right)^{\theta}\right] \tag{57}$$

and

$$G(y)=\exp\left[1-\left(1+\frac{\gamma}{y}\right)^{\theta}\right] \tag{58}$$

The reliability function is written as

$$\overline{G}(y)=1-\exp\left[1-\left(1+\frac{\gamma}{y}\right)^{\theta}\right] \tag{59}$$

The hazard rate function is obtained as

$$h(y)=\frac{\frac{\theta\gamma}{y^2}\left(1+\frac{\gamma}{y}\right)^{\theta-1}\exp\left[1-\left(1+\frac{\gamma}{y}\right)^{\theta}\right]}{1-\exp\left[1-\left(1+\frac{\gamma}{y}\right)^{\theta}\right]} \tag{60}$$

The quantile function of INH is derived as

$$Q(u)=\gamma\left\{(1-\log u)^{\frac{1}{\theta}}-1\right\}^{-1},\ 0<u<1 \tag{61}$$

2.12 Odd Lomax inverted Nadarajah-Haghighi distribution

With generalizations and extensions in mind, researchers, [20] derived and obtained a new extension of the Nadarajah-Haghighi distribution in modeling vaccination rate of COVID-19 in some African countries. The four-parameters distribution was obtained by combining the inverted Nadarajah-Haghighi distribution and the odd Lomax-G family. The introduced distribution is named the odd Lomax inverted Nadarajah-Haghighi (OLxINH) distribution. The authors investigated the virtuous features and elegant statistical characteristics. The problem of parameter estimation from frequentist and Bayesian approaches have been addressed in this work.

The CDF of OLxNH is expressed as

$$G(x)=1-\phi^{a}\left\{\phi+\frac{\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}}{1-\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}}\right\}^{-a} \tag{62}$$

The PDF is given by

$$g(x)=a\lambda\phi^{a}\frac{\frac{b}{x^2}\left(1+\frac{b}{x}\right)^{\lambda-1}\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}}{\left(1-\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}\right)^{2}}\left\{\phi+\frac{\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}}{1-\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}}\right\}^{-a-1} \tag{63}$$

The survival function is given by

$$\overline{G}(x)=\phi^{a}\left\{\phi+\frac{\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}}{1-\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}}\right\}^{-a} \tag{64}$$

The hazard rate function becomes

$$h(x)=a\lambda\phi\frac{\frac{b}{x^2}\left(1+\frac{b}{x}\right)^{\lambda-1}\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}}{\left(1-\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}\right)^{2}}\left\{\phi+\frac{\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}}{1-\exp\left\{1-\left(1+\frac{b}{x}\right)^{\lambda}\right\}}\right\}^{-1} \tag{65}$$

The quantile function of OLxNH is given by

$$x_u = b\left[1+\log\left(1+\frac{1}{\phi\left((1-u)^{-\frac{1}{\phi}-1}\right)}\right)\right]^{1-\lambda^{-1}}, \ 0<u<1 \tag{66}$$

2.13 Transmuted Inverted Nadarajah-Haghighi Distribution

Given $f(x)$ and $F(x)$ to be the PDF and CDF, in that order, of the inverted Nadarajah-Haghighi (INH) Distribution introduced by [21], the CDF of the transmuted inverted Nadarajah-Haghighi (TINH) distribution as follows

$$G(x) = \exp\left\{1-\left(1+\frac{\gamma}{x}\right)^{\delta}\right\}\left\langle 1+\xi-\xi\exp\left\{1-\left(1+\frac{\gamma}{x}\right)^{\delta}\right\}\right\rangle, \ x>0, \ \delta>0, \ \gamma>0, \ |\xi|\le 1 \tag{67}$$

Corresponding to the CDF, the PDF after differentiation is obtained as

$$g(x) = \frac{\delta\gamma}{x^2}\left(1+\frac{\gamma}{x}\right)^{\delta-1}\exp\left\{1-\left(1+\frac{\gamma}{x}\right)^{\delta}\right\}\left\langle 1+\xi-2\xi\exp\left\{1-\left(1+\frac{\gamma}{x}\right)^{\delta}\right\}\right\rangle \tag{68}$$

The survival function is expressed as

$$\overline{G}(x) = 1-\exp\left\{1-\left(1+\frac{\gamma}{x}\right)^{\delta}\right\}\left\langle 1+\xi-\xi\exp\left\{1-\left(1+\frac{\gamma}{x}\right)^{\delta}\right\}\right\rangle \tag{69}$$

The hazard function becomes

$$h(x) = \frac{\frac{\delta\gamma}{x^2}\left(1+\frac{\gamma}{x}\right)^{\delta-1}\exp\left\{1-\left(1+\frac{\gamma}{x}\right)^{\delta}\right\}\left\langle 1+\xi-2\xi\exp\left\{1-\left(1+\frac{\gamma}{x}\right)^{\delta}\right\}\right\rangle}{1-\exp\left\{1-\left(1+\frac{\gamma}{x}\right)^{\delta}\right\}\left\langle 1+\xi-\xi\exp\left\{1-\left(1+\frac{\gamma}{x}\right)^{\delta}\right\}\right\rangle} \tag{70}$$

The quantile function of TINH can be derived via inverting the CDF

$$Q(u) = \begin{cases} \gamma\left[\left(1-\log\left[\frac{(\xi+1)-\sqrt{(\xi+1)^2-4\xi u}}{2\xi}\right]\right)^{\frac{1}{\delta}}-1\right]^{-1}, & \text{if } \xi\ne 0 \\ \gamma\left[\left(1-\log u\right)^{\frac{1}{\delta}}-1\right]^{-1}, & \text{if } \xi=0 \end{cases}, \ 0<u<1. \tag{71}$$

2.14 Beta Nadarajah-Haghighi distribution

[22] introduced the beta family of distributions. Motivated by the aforementioned work[23] defined and studied a lifespan distribution of the beta generated family, named the beta Nadarajah-Haghighi (BNHs) distribution, which can be employed in fitting survival data. This distribution houses five nested models. Besides, the distribution exhibits great flexibility in terms of its PDF and hazard rate function. The hazard rate function shows the classical four forms (increasing, de creasing, unimodal and bathtub-shaped) depending on its shape parameters. Rudimentary mathematical features and maximum likelihood estimation of the model parameters are explored. The CDF of BNH is given by

$$G(x) = I_{1-\exp\left\{1-(1+\lambda x)^{\alpha}\right\}}(a, b), \ x>0, \ \alpha>0, \ \lambda>0, \ a>0, \ b>0 \tag{72}$$

The PDF of BNH is obtained as

$$g(x)=\frac{\alpha\lambda}{B(a,\,b)}\left(1+\lambda x\right)^{\alpha-1}\left\{1-\exp\left(1-\left[1+\lambda x\right]^{\alpha}\right)\right\}^{a-1}\left\{\exp\left(1-\left[1+\lambda x\right]^{\alpha}\right)\right\}^{b} \tag{73}$$

The survival function is read as

$$\overline{G}(x)=1-I_{1-\exp\left\{1-(1+\lambda x)^{\alpha}\right\}}(a,\,b) \tag{74}$$

The hazard function is expressed as

$$h(x)=\frac{\alpha\lambda\left(1+\lambda x\right)^{\alpha-1}\left[1-\exp\left(1-\left[1+\lambda x\right]^{\alpha}\right)\right]^{a-1}\left[\exp\left(1-\left[1+\lambda x\right]^{\alpha}\right)\right]^{b}}{B(a,\,b)-B_{1-\exp\left\{1-(1+\lambda x)^{\alpha}\right\}}(a,\,b)} \tag{75}$$

The quantile is described by

$$Q(u)=\lambda^{-1}\left\langle 1-\log\left\{1-I_{u}^{-1}(a,\,b)\right\}\right\rangle^{\alpha^{-1}}-1,\ 0<u<1 \tag{76}$$

where $I_{u}^{-1}(a,\,b)$ denotes the inverse of the incomplete beta function.

2.15 Unit Nadarajah-Haghighi Generated Family of Distributions

The CDF and PDF of a random variable X, that is distributed according to the unit Nadarajah-Haghighi distribution are respectively given by:

$$G(x)=\frac{1-\exp\left\{1-\left(1+x\right)^{\theta}\right\}}{1-\exp\left(1-2^{\theta}\right)},\ 0<x<1,\ \theta>0 \tag{77}$$

and

$$g(x)=\frac{\theta\left(1+x\right)^{\theta-1}\exp\left\{1-\left(1+x\right)^{\theta}\right\}}{1-\exp\left(1-2^{\theta}\right)} \tag{78}$$

Replacing x with $F(x;\ \boldsymbol{\delta})$ as the CDF of a certain parent model relying on a vector of unknown $\boldsymbol{\delta}$, [24] introduced a relatively new flexible model of distributions, named the unit Nadarajah-Haghighi-G (UNH-G) family of distributions.

The PDF and CDF are defined as:

$$g(x)=\frac{\theta f(x;\ \boldsymbol{\delta})\left(1+F(x;\boldsymbol{\delta})\right)^{\theta-1}\exp\left\{1-\left(1+F(x;\ \boldsymbol{\delta})\right)^{\theta}\right\}}{1-\exp\left(1-2^{\theta}\right)} \tag{79}$$

and

$$G(x)=\frac{1-\exp\left\{1-\left(1+F(x;\ \delta)\right)^{\theta}\right\}}{1-\exp\left(1-2^{\theta}\right)},\ x\in\Re \tag{80}$$

The survival function is obtained as

$$\overline{G}(x)=\frac{\exp\left\{1-\left(1+F(x;\ \delta)\right)^{\theta}\right\}-\exp\left(1-2^{\theta}\right)}{1-\exp\left(1-2^{\theta}\right)} \tag{81}$$

The hazard rate function given by

$$h(x)=\frac{\theta f(x;\ \boldsymbol{\delta})\left(1+F(x;\boldsymbol{\delta})\right)^{\theta-1}\exp\left\{1-\left(1+F(x;\ \boldsymbol{\delta})\right)^{\theta}\right\}}{\exp\left\{1-\left(1+F(x;\ \boldsymbol{\delta})\right)^{\theta}\right\}-\exp\left(1-2^{\theta}\right)} \tag{82}$$

The quantile function becomes

$$x_q = Q_F \left\langle \left\{1-\log\left[1-q\left(1-\exp\left(1-2^{\theta}\right)\right)\right]\right\}^{\frac{1}{\theta}} - 1 \right\rangle, \; q \in (0,\ 1) \tag{83}$$

where $Q_F(.)$ denotes the quantile of the parent distribution.

The authors used the generator to developed two special modes, name UNH Weibull and UNH log-logistics.

Denote the parent distribution as Weibull with CDF and PDF $F(x) = 1-\exp\left(-\beta x^{\lambda}\right)$ and $f(x) = \beta\lambda x^{\lambda-1}\exp\left(-\beta x^{\lambda}\right)$ respectively. The CDF of UNHW is given by

$$G(x) = \frac{1-\exp\left\{1-\left(2-\exp\left(-\beta x^{\lambda}\right)\right)^{\theta}\right\}}{1-\exp\left(1-2^{\theta}\right)}, \; x>0,\ \beta>0,\ \theta>0 \tag{84}$$

Corresponding PDF is given by

$$g(x) = \frac{\theta\beta\lambda x^{\lambda-1}\exp\left(-\beta x^{\lambda}\right)\left(2-\exp\left(-\beta x^{\lambda}\right)\right)^{\theta-1}\exp\left\{1-\left(2-\exp\left(-\beta x^{\lambda}\right)\right)^{\theta}\right\}}{1-\exp\left(1-2^{\theta}\right)} \tag{85}$$

It is observed that the PDF can effectively model data that show the attributes of decreasing, nearly symmetric or left skewed.

The survival function of UNHW is

$$\overline{G}(x) = \frac{\exp\left\{1-\left(2-\exp\left(-\beta x^{\lambda}\right)\right)^{\theta}\right\}-\exp\left(1-2^{\theta}\right)}{1-\exp\left(1-2^{\theta}\right)} \tag{86}$$

The failure rate function is derived as

$$h(x) = \frac{\theta\beta\lambda x^{\lambda-1}\exp\left(-\beta x^{\lambda}\right)\left(2-\exp\left(-\beta x^{\lambda}\right)\right)^{\theta-1}\exp\left\{1-\left(2-\exp\left(-\beta x^{\lambda}\right)\right)^{\theta}\right\}}{\exp\left\{1-\left(2-\exp\left(-\beta x^{\lambda}\right)\right)^{\theta}\right\}-\exp\left(1-2^{\theta}\right)} \tag{87}$$

The integrated hazard rate function becomes

$$\omega(x) = -\log\left\langle \frac{\exp\left\{1-\left(2-\exp\left(-\beta x^{\lambda}\right)\right)^{\theta}\right\}-\exp\left(1-2^{\theta}\right)}{1-\exp\left(1-2^{\theta}\right)} \right\rangle$$

(88)

The quantile function of UNHW is given by

$$Q(u) = \left\langle -\frac{1}{\beta}\log\left\{2-\left[1-\log\left(1-u\left(1-\exp\left(1-2^{\theta}\right)\right)\right)\right]^{\frac{1}{\theta}}\right\}\right\rangle^{\frac{1}{\lambda}}, \; u \in (0,\ 1) \tag{89}$$

Assuming that the parent distribution follows the log-logistic distribution with CDF given by $F(x) = \left\{1+(\beta x)^{-\lambda}\right\}^{-1}$. The PDF given as $f(x) = \beta^{-\lambda}\lambda x^{-\lambda-1}\left\{1+(\beta x)^{-\lambda}\right\}^{-2}$

3. Parameter Estimation

This section is devoted to parameter estimation techniques. In most cases, no closed-form solutions exist for the estimators of the parameters, which imposes the use of numerical interactive methods (such as Newton-Raphson, BFGS and Nelder-Mead, among others). Standard routines implemented in software's such as R, *Ox* and SAS can be used to maximize the likelihoods directly.

3.1 Maximum likelihood estimation

Let x_1, x_2, …, x_n show n observed values from the $g(x)$ distribution with parameters β_1, β_2, …, β_k. Let $\Xi = (\beta_1, \beta_2, ..., \beta_k)^T$ be the parameter vector. The total log-likelihood function for Ξ

$$\ell(\Xi) = \sum_{i=1}^{n} \log g(x) \tag{99}$$

The components for the score functions $U_n(\Xi) = \left(\frac{\partial \ell(\Xi)}{\partial \beta_i}\right)$, $i = 1, 2, 3, ..., k$ are

$$U_{\beta_1}(\Xi) = \frac{\partial \ell(\Xi)}{\partial \beta_1}$$
$$U_{\beta_2}(\Xi) = \frac{\partial \ell(\Xi)}{\partial \beta_2}$$
$$U_{\beta_k}(\Xi) = \frac{\partial \ell(\Xi)}{\partial \beta_k}$$

Setting these equations to zero, $U_{\beta_1}(\Xi) = U_{\beta_2}(\Xi) = ... = U_{\beta_k}(\Xi) = 0$, and solving them simultaneously culminates the maximum likelihood estimates $\hat{\Xi}$ of Ξ.

For interval estimation of the parameters, the observed information matrix, whose elements are $U_{r\,s} = \frac{\partial^2 \ell(\Xi)}{\partial r \partial s}$ (for $r\ s = \beta_1, \beta_2, ..., \beta_k$) are required.

Often with lifespan data and reliability analysis, one encounters censoring. The censored likelihood for the model parameters is

$$L(\Xi) = \prod_{i=1}^{n} \{g(x)\}^{\gamma} \{\overline{G}(x)\}^{1-\gamma} \tag{100}$$

where $\overline{G}(x) = 1 - G(x)$

3.2 Least Squares Estimators

Let X denote a random variable with values in ascending order $x_{(1)} < x_{(2)}, <, ..., < x_{(n)}$. It is possible to minimize the following function in order to obtain the least squares of the unknown parameters $\boldsymbol{\theta} = (\theta_1, \theta_2, ..., \theta_k)$ of $G(x;\ \boldsymbol{\theta})$ distribution

$$\mathrm{LS}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \left[G\left(x_{(i)};\ \boldsymbol{\theta}\right) - \frac{i}{n+1} \right]^2 \tag{101}$$

The nonlinear equations can be solved to find the least squares estimators such as

$$\frac{\partial \mathrm{LS}(\boldsymbol{\theta})}{\partial \theta_i} = 0, \ i = 1, 2, 3, ..., k$$

3.3 Weighted Least Squares Estimators

The weighted least squares estimators is given by

$$\mathrm{W}(\boldsymbol{\theta}) = \sum_{i=1}^{n} \frac{(n+1)^2 (n+2)}{i(n-i+1)} \left[G\left(x_{(i)};\ \boldsymbol{\theta}\right) - \frac{i}{n+1} \right]^2 \tag{102}$$

$$\frac{\partial \mathrm{W}(\boldsymbol{\theta})}{\partial \theta_i} = 0\,,\ i = 1, 2, 3, ..., k$$

3.4 Cramer-von Mises Estimators

The Cramer-von Mises estimators is defined as follows:

$$C(\boldsymbol{\theta}) = \frac{1}{12n} + \sum_{i=1}^{n} \left[G\left(x_{(i)};\ \boldsymbol{\theta}\right) - \frac{2i-1}{2n} \right]^2 \tag{103}$$

$$\frac{\partial \mathrm{C}(\boldsymbol{\theta})}{\partial \theta_i} = 0\,,\ i = 1, 2, 3, ..., k$$

3.5 Anderson and Darling Estimators

Anderson and Darling (1952) proposed the AD test. [25] studied the AD estimators' traits. The Anderson-Darling (AD) estimators of the parameters $\boldsymbol{\theta}$ are determined by minimizing the subsequent function

$$A(\boldsymbol{\theta}) = -n - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \left\{ \log\left(G(x_i)\right) + \log\left(\overline{G}(x_{n+1-i})\right) \right\} \tag{104}$$

where $\overline{G}(.) = 1 - G(.)$. To obtain the estimators, the following must be solved

$$\frac{\partial A(\boldsymbol{\theta})}{\partial \theta_i} = 0\,,\ i = 1, 2, 3, ..., k$$

3.6 The Right-Tail Anderson Darling Estimators

The computational form of right-tail AD can be written in the form of

$$R(\boldsymbol{\theta}) = \frac{n}{2} - 2\sum_{i=1}^{n} \log G(x_{i+1}) - \frac{1}{n} \sum_{i=1}^{n} (2i-1) \log\left[\overline{G}(x_{n+1-i})\right] \tag{105}$$

Solving $\frac{\partial R(\boldsymbol{\theta})}{\partial \theta_i} = 0$, $i = 1, 2, 3, ..., k$ creates the right-tail AD estimates.

3.6 Maximum Product Spacing Approach

The maximum product spacing (MPS) method first appeared in [26]. The MPS is defined as

$$D_j(\boldsymbol{\theta}) = G\left(x_{j:k} \mid \boldsymbol{\theta}\right) - G\left(x_{j-1:k} \mid \boldsymbol{\theta}\right),\ j = 1,\ 2, 3,..., k \tag{106}$$

The geometric mean of the spacing is expressed as

$$T(\boldsymbol{\theta}) = \left\{ \prod_{i=1}^{n} D_j(\boldsymbol{\theta}) \right\}^{\frac{1}{k+1}} \tag{107}$$

Or maximizing the function

$$\Delta(\boldsymbol{\theta}) = \frac{1}{1+k} \sum_{i=1}^{1+k} \log D_j(\boldsymbol{\theta}) \tag{108}$$

Solving $\frac{\partial \Delta(\boldsymbol{\theta})}{\partial \theta_i} = 0$ creates the MPS estimates.

3.7 Minimum Spacing Distance Approach

This estimation methodology relies on the Kullback–Leibler information measure. This technique appeared in [27]. Although varied selection of distances exist, this study reviewed minimum spacing absolute distance (MSAD) and minimum spacing absolute-log distance (MSALD). The MSAD estimates are determined by minimizing the subsequent function

$$\text{MSAD} = \sum_{i=1}^{1+n} \left| D_j(\boldsymbol{\theta}) - \frac{1}{n+1} \right| \tag{109}$$

The MASLD estimates are obtained by minimizing the following function with respect to the parameters.

$$\text{MSALD} = \sum_{i=1}^{1+n} \left| \log D_j(\boldsymbol{\theta}) - \log \frac{1}{n+1} \right| \tag{110}$$

3.8 Percentile Based Estimators

Another methodology of estimating the parameters of a developed model is the percentile based. If the cumulative distribution function has a closed form, then one can estimate the unknown parameter by fitting a straight line to the percentile points. The estimates of the parameters can be obtained by minimizing the following function

$$Z(x_{(i)} \mid \boldsymbol{\theta}) = \sum_{i=1}^{n} \left\{ X_{(i)} - G^{-1}(u_{(i)}) \right\}^2 \tag{111}$$

4. Simulation Study

Subsequent to obtaining the maximum likelihood estimates of the distribution parameters, is to investigate the performance and accuracy of these parameters via simulation. Random samples of different sizes are selected from the density function using the inverse CDF method. It is significant to state that there are no definite rules in the selection of the initial values of the parameters. The most important is that the selected values must stay within the domain of each parameter.

Finally, some accuracy measures such as Bias, Mean squared error (MSE) and absolute biases are selected to see the performances of the estimates. The values of these statistical measures are, respectively, obtained as

$$\text{MSE}(\hat{\boldsymbol{\theta}}) = \frac{1}{N} \sum (\hat{\boldsymbol{\theta}} - \theta)^2 \tag{112}$$

$$\text{Bias}(\hat{\boldsymbol{\theta}}) = \frac{1}{N} \sum (\hat{\boldsymbol{\theta}} - \theta) \tag{113}$$

$$\left| \text{Absolute Bias}(\hat{\boldsymbol{\theta}}) \right| = \left| \frac{1}{N} \sum (\hat{\boldsymbol{\theta}} - \theta) \right| \tag{114}$$

where $\boldsymbol{\theta}$ is a vector of parameters.

5. Conclusions

There is a clear manifestation that classical statistical distributions provide poor fit in modeling fatigue problems, reliability data, ecology and survival studies among others. The aforementioned offers a basis for the extension of these conventional distributions. This work reviews the modifications, extensions or generalizations of Nadarajah Haghighi distribution. These generalizations have been constructed using varied techniques: incorporating additional parameter(s); generating new families of distributions, transformations etc. The statistical and mathematical features, including the probability density function, the distribution function, the survival function, hazard rate function, integrated hazard rate function and quantile functions have been reviewed. The hazard functions of the various generalized distributions exhibit diverse range of shapes, namely, decreasing, increasing, constant, unimodal and upside-down bathtub shapes among others. Parameter estimation techniques are also reviewed.

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