

Two-Parameter Marshall Olkin Extended Exponentiated Gamma Distribution: Different Methods of Estimation

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ABSTRACT

In this paper, we investigate ten different methods for estimating the unknown parameters of the Marshall–Olkin Extended Exponentiated Gamma (MOEEG) distribution. We first present several frequentist approaches, including maximum likelihood, ordinary least squares, weighted least squares, Cramér–von Mises, Kolmogorov, and five variants of the Anderson–Darling statistic (standard, right-tail, left-tail, second-order left-tail). We also develop a Bayesian estimator assuming independent Gamma priors for the parameters. The performance of all estimators is evaluated through extensive Monte Carlo simulations based on bias, root mean square error, and goodness-of-fit metrics. Finally, an application to a real dataset is provided to illustrate the practical relevance of the proposed methods.

Keywords: Marshall Olkin, simulations, estimators.

1. Introduction

Gupta et al. (1998) introduced the one-parameter Exponentiated Gamma (EG) distribution in connection with problems in reliability and lifetime analysis. Since then, several studies have explored this distribution in different scientific fields. Building on the EG model, Aguilar et al. (2022) proposed the Marshall–Olkin Extended Exponentiated Gamma (MOEEG) distribution, obtained by compounding the EG distribution with a geometric distribution based on the competing and complementary risks framework.

The choice of estimation method may vary depending on the user’s objectives and the area of application. The novelty of the present study lies in the fact that, to the best of our knowledge, no previous work has compared all these estimation techniques for the MOEEG distribution. However, similar comparative studies have been conducted for other distributions, such as the Zero-Truncated Poisson Exponentiated Gamma (Aguilar et al., 2019), the Marshall–Olkin Extended Lindley (Espírito Santo and Mazucheli, 2015), the Weighted Exponential (Dey et al., 2015), the two-parameter Rayleigh (Dey et al., 2014), the Weibull (Teimouri et al., 2013), and the generalized Logistic distributions (Alkawasbeh and Raqab, 2009).

The main objective of this paper is to investigate ten different estimation methods for the parameters of the MOEEG distribution. Since the estimators do not have closed-form expressions, they are obtained numerically through one-dimensional optimization procedures. We first consider several frequentist estimators—maximum likelihood, ordinary least squares, weighted least squares, Cramér–von Mises, Anderson–Darling (standard, right-tail, left-tail, and second-order left-tail), and Kolmogorov. We also develop Bayesian estimators for the unknown parameters,

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assuming independent Gamma priors for the scale and shape parameters. The performance of all methods is assessed through Monte Carlo simulations, and a real dataset is analyzed to illustrate the results.

The remainder of this paper is organized as follows: Section 2 describes the estimation procedures; Section 3 presents the simulation design; Section 4 details the criteria used for comparing estimators; Section 5 provides the real-data application; and Section 6 concludes the study with final remarks.

1.1 The Marshall–Olkin Extended Exponentiated Gamma (MOEEG) Distribution

The MOEEG distribution, characterized by one shape parameter (α) and one scale parameter (θ), has the following probability density function (PDF), cumulative distribution function (CDF), and hazard rate function, respectively:

$$f(x|\theta, \alpha) = \alpha\theta x e^{-x} (1 - e^{-x(x+1)})^{\theta-1} \left[1 - \bar{\alpha} \left(1 - (1 - e^{-x(x+1)})^\theta\right)\right]^{-2}, \quad (1)$$

$$F(x|\theta, \alpha) = \frac{(1 - e^{-x(x+1)})^\theta}{1 - \bar{\alpha} [1 - (1 - e^{-x(x+1)})^\theta]}, \quad 0 < x, \quad 0 < \alpha, \quad 0 < \theta, \quad \bar{\alpha} = 1 - \alpha, \quad (2)$$

$$h(x|\theta, \alpha) = \frac{\theta x e^{-x} (1 - e^{-x(x+1)})^{\theta-1}}{[1 - (1 - e^{-x(x+1)})^\theta][1 - \bar{\alpha} (1 - (1 - e^{-x(x+1)})^\theta)]}. \quad (3)$$

The hazard rate function of the MOEEG distribution can exhibit a wide variety of shapes, such as increasing, bathtub, or unimodal, depending on the parameter values. In particular, the ratio $h(x|\theta, \alpha)/h(x|\theta)$ is an increasing function of x when $\alpha \geq 1$ and a decreasing function when $0 < \alpha < 1$.

Figures 1 and 2 illustrate the behavior of the MOEEG hazard rate for different combinations of the parameters α and θ . In Figure 1, α is held fixed while θ varies, showing how the scale parameter affects the shape and spread of the hazard function. Conversely, in Figure 2, θ is fixed and α varies, highlighting the role of the shape parameter in controlling the tail behavior and modality of the hazard rate. The figures demonstrate that the MOEEG distribution is highly flexible and can model a wide range of reliability patterns, including increasing, decreasing, and bathtub-shaped hazard functions.

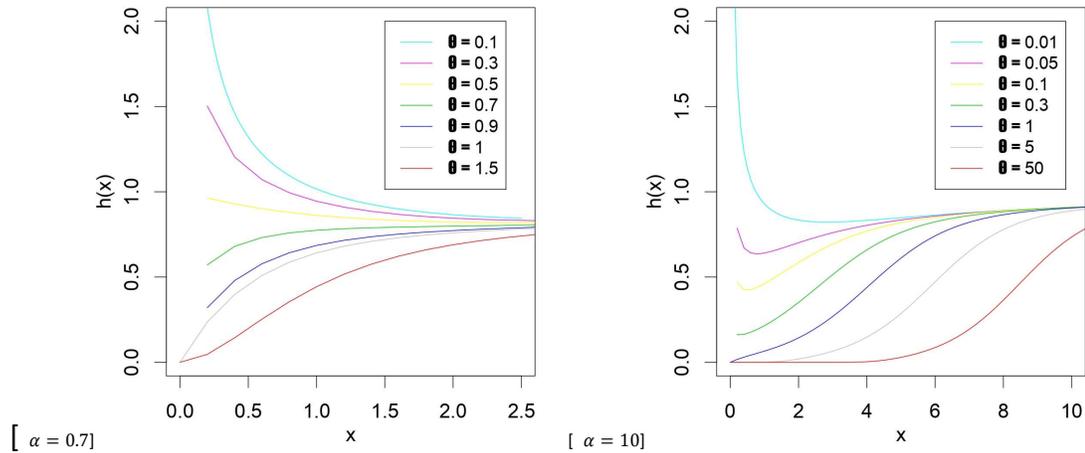


Figure 1: Hazard rate for different values of θ .

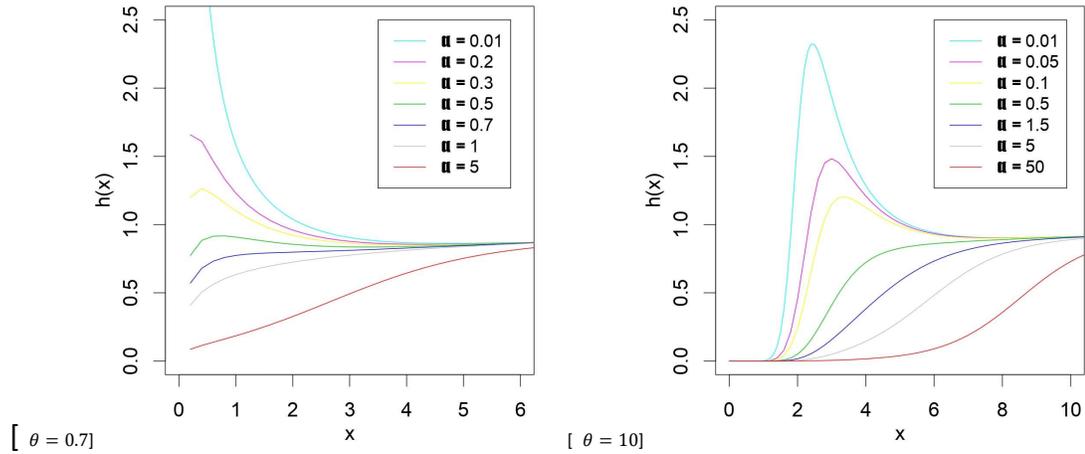


Figure 2: Hazard rate for different values of α .

2. Methods of Estimation

In this section, we present and compare ten different approaches for estimating the parameters of the Marshall–Olkin Extended Exponentiated Gamma (MOEEG) distribution. Because the estimators of the MOEEG model do not have closed-form solutions, all methods require numerical optimization to obtain parameter estimates.

The estimation techniques considered here can be broadly divided into two main categories:

Frequentist methods: These include classical approaches based on minimizing distance measures between empirical and theoretical distributions, as well as likelihood-based procedures. Specifically, we consider the Maximum Likelihood Estimator (MLE), Ordinary Least Squares (LSE), Weighted Least Squares (WLSE), Cramér–von Mises (CME), Kolmogorov (KE), and five variants of the Anderson–Darling (AD) statistic — the standard AD, right-tail (AD–RT), left-tail (AD–LT), and second-order left-tail (AD–2LT) versions.

Bayesian method: This approach incorporates prior information about the parameters through independent Gamma priors for both shape and scale parameters, with estimation carried

out via the Metropolis–Hastings algorithm.

Each method provides a different perspective on parameter estimation, emphasizing distinct aspects such as robustness, tail sensitivity, or goodness-of-fit. To ensure comparability, all procedures are applied to the same simulated datasets, and their performance is later evaluated through numerical criteria such as bias, root mean squared error (RMSE), and goodness-of-fit distances.

The following subsections describe each estimation procedure in detail.

2.1 Maximum Likelihood Method

Let $X_i, i = 1, 2, \dots, n$, be a random sample of size n , such that $X_i \sim \text{MOEEG}(\alpha, \theta)$. The likelihood function is given by

$$L(x|\theta, \alpha) = \prod_{i=1}^n f(x_i|\theta, \alpha).$$

By the properties of the likelihood function, maximizing $L(\theta, \alpha)$ is equivalent to maximizing its natural logarithm. Thus, letting $l(\alpha, \theta|x) = \ln L(\alpha, \theta|x)$, we obtain

$$l(\alpha, \theta|x) = n \ln(\alpha) + n \ln(\theta) + \sum_{i=1}^n \ln(x_i) - \sum_{i=1}^n x_i + (\theta - 1) \sum_{i=1}^n \ln[1 - e^{-x_i}(x_i + 1)] - 2 \sum_{i=1}^n \ln\{(1 - \alpha)[1 - (1 - e^{-x_i}(x_i + 1))^\theta]\}. \tag{4}$$

Differentiating (4) with respect to α and θ yields the components of the score vector:

$$U_\alpha(\alpha, \theta|x) = \frac{\partial l(\alpha, \theta|x)}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^n \frac{1 - [1 - e^{-x_i}(x_i + 1)]^\theta}{1 - (1 - \alpha)[1 - [1 - e^{-x_i}(x_i + 1)]^\theta]}, \tag{5}$$

$$U_\theta(\alpha, \theta|x) = \frac{\partial l(\alpha, \theta|x)}{\partial \theta} = \frac{n}{\theta} + \sum_{i=1}^n \ln[1 - e^{-x_i}(x_i + 1)] - 2 \sum_{i=1}^n \frac{(1 - \alpha)[1 - e^{-x_i}(x_i + 1)]^\theta \ln[1 - e^{-x_i}(x_i + 1)]}{1 - (1 - \alpha)[1 - [1 - e^{-x_i}(x_i + 1)]^\theta]}. \tag{6}$$

The maximum likelihood estimators (MLEs) $\hat{\alpha}_{\text{MLE}}$ and $\hat{\theta}_{\text{MLE}}$ for the parameters α and θ are obtained by solving the nonlinear equations $U_\alpha(\alpha, \theta|x) = 0$ and $U_\theta(\alpha, \theta|x) = 0$.

Given the vector of parameters $\Theta = (\alpha, \theta)$, for large samples, the asymptotic distribution of the MLE is (Migon et al. 2014)

$$(\hat{\Theta} - \Theta) \rightarrow N_2(0, I^{-1}(\Theta)),$$

where $I(\Theta)$ denotes the observed Fisher information matrix, given by

$$I(\Theta) = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}. \tag{7}$$

Its inverse, $I^{-1}(\Theta)$, is expressed as

$$I^{-1}(\Theta) = \begin{pmatrix} \text{var}(\hat{\alpha}_{\text{MLE}}) & \text{cov}(\hat{\alpha}_{\text{MLE}}, \hat{\theta}_{\text{MLE}}) \\ \text{cov}(\hat{\theta}_{\text{MLE}}, \hat{\alpha}_{\text{MLE}}) & \text{var}(\hat{\theta}_{\text{MLE}}) \end{pmatrix}. \tag{8}$$

The second derivatives of the log-likelihood, required for computing the observed Fisher information, are given by:

$$\frac{\partial^2 l(\alpha, \theta|x)}{\partial \alpha^2} = -\frac{n}{\alpha^2} - 2 \sum_{i=1}^n \frac{\{1 - [1 - e^{-x_i}(x_i + 1)]^\theta\}^2}{\{1 - (1 - \alpha)[1 - [1 - e^{-x_i}(x_i + 1)]^\theta\}^2}, \tag{9}$$

$$\frac{\partial^2 l(\alpha, \theta|x)}{\partial \theta^2} = -\frac{n}{\theta^2} - 2 \sum_{i=1}^n \frac{(1 - \alpha)[1 - e^{-x_i}(x_i + 1)]^\theta [\ln(1 - e^{-x_i}(x_i + 1))]^2}{\{1 - (1 - \alpha)[1 - [1 - e^{-x_i}(x_i + 1)]^\theta\}^2}, \tag{10}$$

$$\frac{\partial^2 l(\alpha, \theta|x)}{\partial \theta \partial \alpha} = -2 \sum_{i=1}^n \frac{[1 - e^{-x_i}(x_i + 1)]^\theta \ln[1 - e^{-x_i}(x_i + 1)]}{\{1 - (1 - \alpha)[1 - [1 - e^{-x_i}(x_i + 1)]^\theta\}^2}. \tag{11}$$

Therefore, approximate $(1 - \tau)\%$ confidence intervals for the parameters $\Theta = (\alpha, \theta)$ can be obtained as

$$\hat{\alpha}_{MLE} \pm z_{\frac{\tau}{2}} \sqrt{var(\hat{\alpha}_{MLE})}, \quad \hat{\theta}_{MLE} \pm z_{\frac{\tau}{2}} \sqrt{var(\hat{\theta}_{MLE})},$$

where $z_{\frac{\tau}{2}}$ denotes the $\frac{\tau}{2}$ -th percentile of the standard normal distribution.

2.2 Ordinary Least Squares Method

Let $x_{1:n} < x_{2:n} < \dots < x_{n:n}$ denote the ordered statistics from a random sample of size n drawn from the MOEEG distribution with cumulative distribution function (CDF) $F(x|\alpha, \theta)$. Under the theoretical model, the expected value and variance of the CDF evaluated at the ordered sample are given by

$$E[F(x_{i:n}|\alpha, \theta)] = \frac{i}{n+1}, \quad V[F(x_{i:n}|\alpha, \theta)] = \frac{i(n-i+1)}{(n+1)^2(n+2)}.$$

The ordinary least squares estimators (LSE) of the parameters α and θ , denoted by $\hat{\alpha}_{LSE}$ and $\hat{\theta}_{LSE}$, are obtained by minimizing the objective function

$$M(\alpha, \theta) = \sum_{i=1}^n \left[F(x_{i:n}|\alpha, \theta) - \frac{i}{n+1} \right]^2. \quad (12)$$

The estimators are the solutions to the system of equations $\frac{\partial M}{\partial \alpha} = 0$ and $\frac{\partial M}{\partial \theta} = 0$. Since these equations do not have closed-form expressions, they must be solved numerically. The partial derivatives of $M(\alpha, \theta)$ with respect to α and θ are given by:

$$\begin{aligned} \frac{\partial M}{\partial \alpha} &= 2 \sum_{i=1}^n \frac{[1-e^{-x_i(x_i+1)}]^\theta i \{1-[1-e^{-x_i(x_i+1)}]^\theta\}}{(n+1)\{-\alpha[1-e^{-x_i(x_i+1)}]^\theta + [1-e^{-x_i(x_i+1)}]^\theta + \alpha\}^2} \\ &\quad - 2 \sum_{i=1}^n \frac{[1-e^{-x_i(x_i+1)}]^{2\theta} \{1-[1-e^{-x_i(x_i+1)}]^\theta\}}{\{-\alpha[1-e^{-x_i(x_i+1)}]^\theta + [1-e^{-x_i(x_i+1)}]^\theta + \alpha\}^3}, \end{aligned} \quad (13)$$

$$\begin{aligned} \frac{\partial M}{\partial \theta} &= \sum_{i=1}^n \left\{ \frac{(-2i\alpha^2 + (2i-2n-2)\alpha)\ln(1-e^{-x_i(x_i+1)})[1-e^{-x_i(x_i+1)}]^{2\theta}}{(n+1)[\alpha[1-e^{-x_i(x_i+1)}]^\theta - [1-e^{-x_i(x_i+1)}]^\theta - \alpha]^3} \right. \\ &\quad \left. + \frac{2i\alpha^2[1-e^{-x_i(x_i+1)}]^\theta \ln(1-e^{-x_i(x_i+1)})}{(n+1)[\alpha[1-e^{-x_i(x_i+1)}]^\theta - [1-e^{-x_i(x_i+1)}]^\theta - \alpha]^3} \right\}. \end{aligned} \quad (14)$$

Because both derivatives depend on the parameters in complex nonlinear forms, numerical optimization algorithms such as the BFGS or Nelder–Mead methods are typically employed to obtain the LSEs. These estimators are consistent and asymptotically normal under standard regularity conditions.

2.3 Weighted Least Squares Method

The ordinary least squares estimator relies on several assumptions such as linearity, homoscedasticity, unbiasedness, minimum variance, and consistency. However, in practice, these assumptions—particularly the homoscedasticity condition—are often violated. To mitigate this issue, an alternative approach known as the weighted least squares (WLS) method is employed, in which each observation is weighted according to the inverse of its variance, thereby improving efficiency when variances are unequal.

The weighted least squares estimators (WLSE) of the parameters α and θ , denoted by $\hat{\alpha}_{WLSE}$ and $\hat{\theta}_{WLSE}$, are obtained by minimizing the weighted objective function

$$W(\alpha, \theta) = \sum_{i=1}^n \left[\frac{(n+1)^2(n+2)}{i(n-i+1)} F(x_{i:n}|\alpha, \theta) - \frac{i}{n+1} \right]^2. \quad (15)$$

This expression differs from the ordinary least squares criterion in (12), as the residuals are now multiplied by weights proportional to the precision of the theoretical distribution. Since closed-form solutions for α and θ are not available, the WLSEs must be obtained numerically.

The first-order partial derivatives of $W(\alpha, \theta)$ with respect to α and θ are given by:

$$\frac{\partial W(\alpha, \theta | x)}{\partial \alpha} = \sum_{i=1}^n \left[\frac{(n+1)^2(n+2)}{i(n-i+1)} \frac{F(x_{i:n}|\theta)}{1-\bar{\alpha}S(x_{i:n}|\theta)} - \frac{i}{n+1} \right] \eta_1(\alpha, \theta | x_{i:n}), \tag{16}$$

$$\frac{\partial W(\alpha, \theta | x)}{\partial \theta} = \sum_{i=1}^n \left[\frac{(n+1)^2(n+2)}{i(n-i+1)} \frac{F(x_{i:n}|\theta)}{1-\bar{\alpha}S(x_{i:n}|\theta)} - \frac{i}{n+1} \right] \eta_2(\alpha, \theta | x_{i:n}), \tag{17}$$

where $\bar{\alpha} = 1 - \alpha$ and

$$\eta_1(\alpha, \theta | x_{i:n}) = -\frac{S(x_{i:n}|\theta)F(x_{i:n}|\theta)}{[1-\bar{\alpha}S(x_{i:n}|\theta)]^2},$$

$$\eta_2(\alpha, \theta | x_{i:n}) = \frac{[1-\bar{\alpha}S(x_{i:n}|\theta)]\left[\frac{\partial F(x_{i:n}|\theta)}{\partial \theta}\right] + \bar{\alpha}F(x_{i:n}|\theta)\left[\frac{\partial S(x_{i:n}|\theta)}{\partial \theta}\right]}{[1-\bar{\alpha}S(x_{i:n}|\theta)]^2}.$$

As in the ordinary least squares case, numerical optimization algorithms such as the BFGS or Nelder–Mead methods can be used to obtain the estimates. The WLSEs are asymptotically consistent and efficient when the weighting scheme correctly reflects the heteroscedastic structure of the data.

2.4 Cramér–von Mises Method

The Cramér–von Mises (CME) estimators, denoted by $\hat{\alpha}_{CME}$ and $\hat{\theta}_{CME}$, are obtained by minimizing a minimum-distance goodness-of-fit statistic. This method is based on the squared differences between the theoretical cumulative distribution function (CDF) and the empirical distribution function (EDF) of the sample. The objective function to be minimized is given by

$$C(\alpha, \theta) = \frac{1}{12} + \sum_{i=1}^n \left[F(x_{i:n}|\alpha, \theta) - \frac{2i-1}{2n} \right]^2. \tag{18}$$

The parameters α and θ are estimated by solving the nonlinear system

$$\frac{\partial C}{\partial \alpha} = 0, \quad \frac{\partial C}{\partial \theta} = 0,$$

which must be solved numerically, since analytical solutions are not available.

The partial derivatives of $C(\alpha, \theta)$ with respect to α and θ are expressed as:

$$\frac{\partial C(\alpha, \theta | x)}{\partial \alpha} = \sum_{i=1}^n \left[\frac{F(x_{i:n}|\theta)}{1-\bar{\alpha}S(x_{i:n}|\theta)} - \frac{2i-1}{2n} \right] \eta_1(\alpha, \theta | x_{i:n}), \tag{19}$$

$$\frac{\partial C(\alpha, \theta | x)}{\partial \theta} = \sum_{i=1}^n \left[\frac{F(x_{i:n}|\theta)}{1-\bar{\alpha}S(x_{i:n}|\theta)} - \frac{2i-1}{2n} \right] \eta_2(\alpha, \theta | x_{i:n}), \tag{20}$$

where $\bar{\alpha} = 1 - \alpha$, and the auxiliary functions $\eta_1(\cdot)$ and $\eta_2(\cdot)$ are defined as in the weighted least squares method:

$$\eta_1(\alpha, \theta | x_{i:n}) = -\frac{S(x_{i:n}|\theta)F(x_{i:n}|\theta)}{[1-\bar{\alpha}S(x_{i:n}|\theta)]^2},$$

$$\eta_2(\alpha, \theta | x_{i:n}) = \frac{[1-\bar{\alpha}S(x_{i:n}|\theta)]\left[\frac{\partial F(x_{i:n}|\theta)}{\partial \theta}\right] + \bar{\alpha}F(x_{i:n}|\theta)\left[\frac{\partial S(x_{i:n}|\theta)}{\partial \theta}\right]}{[1-\bar{\alpha}S(x_{i:n}|\theta)]^2}.$$

The CME estimators provide a robust alternative to least squares estimation, emphasizing a global measure of fit between the empirical and theoretical distributions. They are particularly useful when deviations between the model and data occur across the entire support of the distribution.

2.5 Anderson–Darling Methods

The Anderson–Darling (AD) family of estimators provides minimum-distance parameter estimates by comparing the theoretical cumulative distribution function (CDF) with the empirical distribution function (EDF). Different variants of AD emphasize distinct parts of the distribution, such as the central region or the tails.

2.5.1 Standard Anderson–Darling (AD)

The standard Anderson–Darling estimators, denoted by $\hat{\alpha}_{ADE}$ and $\hat{\theta}_{ADE}$, are obtained by minimizing the statistic

$$A(\alpha, \theta) = -n - \frac{1}{n} \sum_{i=1}^n (2i - 1) [\ln F(x_{i:n} | \alpha, \theta) + \ln \bar{F}(x_{n+1-i:n} | \alpha, \theta)], \quad (21)$$

where $\bar{F}(x | \alpha, \theta) = 1 - F(x | \alpha, \theta)$ is the survival function.

The estimators are obtained by solving the system

$$\frac{\partial A}{\partial \alpha} = 0, \quad \frac{\partial A}{\partial \theta} = 0,$$

with derivatives

$$\frac{\partial A(\alpha, \theta | x)}{\partial \alpha} = \sum_{i=1}^n (2i - 1) \left[\frac{\eta_1(\alpha, \theta | x_{i:n})}{F(x_{i:n} | \alpha, \theta)} - \frac{\eta_1(\alpha, \theta | x_{n+1-i:n})}{S(x_{n+1-i:n} | \alpha, \theta)} \right], \quad (22)$$

$$\frac{\partial A(\alpha, \theta | x)}{\partial \theta} = \sum_{i=1}^n (2i - 1) \left[\frac{\eta_2(\alpha, \theta | x_{i:n})}{F(x_{i:n} | \alpha, \theta)} - \frac{\eta_2(\alpha, \theta | x_{n+1-i:n})}{S(x_{n+1-i:n} | \alpha, \theta)} \right], \quad (23)$$

where $\eta_1(\cdot)$ and $\eta_2(\cdot)$ are auxiliary functions defined as in the Weighted Least Squares section.

2.5.2 Right-Tail Anderson–Darling (AD–RT)

The right-tail Anderson–Darling estimators, denoted by $\hat{\alpha}_{ADRE}$ and $\hat{\theta}_{ADRE}$, emphasize discrepancies in the right tail. The statistic is

$$R(\alpha, \theta) = \frac{n}{2} - 2 \sum_{i=1}^n F(x_{i:n} | \alpha, \theta) - \frac{1}{n} \sum_{i=1}^n (2i - 1) \ln \bar{F}(x_{n+1-i:n} | \alpha, \theta). \quad (24)$$

The derivatives are

$$\frac{\partial R(\alpha, \theta | x)}{\partial \alpha} = -2 \sum_{i=1}^n \eta_1(\alpha, \theta | x_{i:n}) + \frac{1}{n} \sum_{i=1}^n (2i - 1) \frac{\eta_1(\alpha, \theta | x_{n+1-i:n})}{S(x_{n+1-i:n} | \alpha, \theta)}, \quad (25)$$

$$\frac{\partial R(\alpha, \theta | x)}{\partial \theta} = -2 \sum_{i=1}^n \eta_2(\alpha, \theta | x_{i:n}) + \frac{1}{n} \sum_{i=1}^n (2i - 1) \frac{\eta_2(\alpha, \theta | x_{n+1-i:n})}{S(x_{n+1-i:n} | \alpha, \theta)}. \quad (26)$$

2.5.3 Left-Tail Anderson–Darling (AD–LT)

The left-tail Anderson–Darling estimators, denoted by $\hat{\alpha}_{ADLE}$ and $\hat{\theta}_{ADLE}$, emphasize discrepancies in the left tail. The statistic is

$$LT = -\frac{3n}{2} + 2 \sum_{i=1}^n F(x_{i:n} | \alpha, \theta) - \frac{1}{n} \sum_{i=1}^n (2i - 1) \ln F(x_{i:n} | \alpha, \theta). \quad (27)$$

The derivatives are

$$\frac{\partial LT(\alpha, \theta | x)}{\partial \alpha} = 2 \sum_{i=1}^n \eta_1(\alpha, \theta | x_{i:n}) - \frac{1}{n} \sum_{i=1}^n (2i - 1) \frac{\eta_1(\alpha, \theta | x_{i:n})}{F(x_{i:n} | \alpha, \theta)}, \quad (28)$$

$$\frac{\partial LT(\alpha, \theta | x)}{\partial \theta} = 2 \sum_{i=1}^n \eta_2(\alpha, \theta | x_{i:n}) - \frac{1}{n} \sum_{i=1}^n (2i - 1) \frac{\eta_2(\alpha, \theta | x_{i:n})}{F(x_{i:n} | \alpha, \theta)}. \quad (29)$$

2.5.4 Second-Order Left-Tail Anderson–Darling (AD–2LT)

The second-order left-tail Anderson–Darling estimators, denoted by $\hat{\alpha}_{AD2LE}$ and $\hat{\theta}_{AD2LE}$, give more weight to extreme deviations in the left tail. The statistic is

$$LT = 2 \sum_{i=1}^n \ln F(x_{i:n} | \alpha, \theta) + \frac{1}{n} \sum_{i=1}^n \frac{2i-1}{F(x_{i:n} | \alpha, \theta)}. \quad (30)$$

The derivatives are

$$\frac{\partial LT(\alpha, \theta | x)}{\partial \alpha} = 2 \sum_{i=1}^n \frac{\eta_1(\alpha, \theta | x_{i:n})}{F(x_{i:n} | \alpha, \theta)} - \frac{1}{n} \sum_{i=1}^n (2i - 1) \frac{\eta_1(\alpha, \theta | x_{i:n})}{(F(x_{i:n} | \alpha, \theta))^2}, \quad (31)$$

$$\frac{\partial LT(\alpha, \theta | x)}{\partial \theta} = 2 \sum_{i=1}^n \frac{\eta_2(\alpha, \theta | x_{i:n})}{F(x_{i:n} | \alpha, \theta)} - \frac{1}{n} \sum_{i=1}^n (2i - 1) \frac{\eta_2(\alpha, \theta | x_{i:n})}{(F(x_{i:n} | \alpha, \theta))^2}. \quad (32)$$

2.6 Kolmogorov (KE)

The Kolmogorov (KE) estimators, denoted by $\hat{\alpha}_{KE}$ and $\hat{\theta}_{KE}$, are obtained by minimizing a minimum-distance goodness-of-fit statistic based on the maximum absolute difference between the theoretical CDF and the empirical distribution function:

$$K(\alpha, \theta) = \max_{1 \leq i \leq n} \left\{ \frac{i}{n} - F(x_{i:n} | \alpha, \theta), F(x_{i:n} | \alpha, \theta) - \frac{i-1}{n} \right\}. \quad (33)$$

The parameter estimates are obtained by solving

$$\frac{\partial K}{\partial \alpha} = 0, \quad \frac{\partial K}{\partial \theta} = 0,$$

which is typically carried out numerically.

2.7 Bayesian Method

Bayesian inference differs from classical (frequentist) methods in that prior information about the parameters is explicitly incorporated into the analysis. In Bayesian inference, the parameters of the distribution are treated as random variables, whereas in classical inference they are considered fixed but unknown. The prior information, expressed through a *prior distribution*, is updated using the observed data via Bayes' theorem to obtain the *posterior distribution*.

For the $MOEEG(\alpha, \theta)$ distribution with unknown parameters, we assume independent Gamma priors for α and θ , with probability density functions

$$g(\alpha) \propto \alpha^{a-1} e^{-\alpha b}, \quad \alpha > 0, \quad g(\theta) \propto \theta^{c-1} e^{-\theta d}, \quad \theta > 0,$$

where the hyperparameters a, b, c , and d are known and non-negative.

Parameter estimation is performed using the Metropolis–Hastings algorithm (Metropolis et al. 1953, Hastings 1970), which allows sampling from posterior distributions that cannot be obtained analytically. Given an auxiliary proposal distribution $q(\theta, \cdot)$, candidate values of the posterior are generated and accepted with probability

$$\alpha(\theta^{(t)}, \theta^{(*)}) = \min \left[1, \frac{p(\theta^{(*)})q(\theta^{(t)}|\theta^{(*)})}{p(\theta^{(t)})q(\theta^{(*)}|\theta^{(t)})} \right] \geq u(0,1),$$

where $u(0,1)$ is a uniform random number between 0 and 1.

The algorithm proceeds as follows:

1. Choose a proposal distribution $q(\theta, \cdot)$ that generates candidate values for θ ;
2. Initialize the iteration counter $t = 1$;
3. Select an initial value $\theta^{(t)}$;
4. Generate a candidate value $\theta^{(*)}$ from $q(\theta, \cdot)$;
5. Compute the acceptance probability $\alpha(\theta^{(t)}, \theta^{(*)})$;
6. If the candidate is accepted: set $\theta^{(t+1)} = \theta^{(*)}$, increment t , and return to step 4;
7. If the candidate is rejected: set $\theta^{(t+1)} = \theta^{(t)}$, increment t , and return to step 4;
8. Repeat steps 4–7 until convergence is reached.

The algorithm was implemented in the *R* programming language. For each dataset of 1,000 samples, 10,000 iterations were performed, with the first 2,500 iterations discarded as burn-in.

3. Simulation

We generated $B = 1,000$ pseudo-random samples from the Marshall–Olkin Extended Exponentiated Gamma (MOEEG) distribution for different parameter values and four distinct sample sizes: $n = 20, 60, 100$, and 200 .

The samples were obtained using the inverse transform method. A vector u of size 100,000 was generated, with $u \sim U(0,1)$. Since the MOEEG distribution does not have a closed-form inverse CDF $x = F^{-1}(u)$, the pseudo-random samples were generated by numerically solving $F(x|\alpha, \theta) = u$ using the Newton–Raphson method.

The resulting roots were stored in a matrix X of dimension $200 \times 1,000$, where each

column corresponds to a simulated sample of maximum size $n = 200$. For smaller sample sizes, the first n rows of each column were used. Specifically:

- The first 20 rows of a column represent a sample of size $n = 20$;
- The first 60 rows of a column represent a sample of size $n = 60$;
- The first 100 rows of a column represent a sample of size $n = 100$;
- All 200 rows of a column represent a sample of size $n = 200$.

By repeating this procedure across all 1,000 columns, we obtained $B = 1,000$ simulated samples for each sample size and parameter configuration.

4. Comparison of Estimation Methods

To evaluate and compare the performance of the different estimation methods, we computed the following metrics for each parameter and sample:

- **Bias:** measures the average difference between the estimated and true parameter values.
- **Root Mean-Squared Error (RMSE):** quantifies the overall estimation error.
- **Average Absolute Difference (D_{abs}):** measures the mean absolute discrepancy between the true and estimated cumulative distribution functions (CDFs).
- **Maximum Absolute Difference (D_{max}):** captures the largest pointwise deviation between the true and estimated CDFs.

These metrics are formally defined as:

$$\begin{aligned}
 \text{BIAS}(\hat{\alpha}) &= \frac{1}{B} \sum_{i=1}^B (\hat{\alpha}_i - \alpha), & \text{BIAS}(\hat{\theta}) &= \frac{1}{B} \sum_{i=1}^B (\hat{\theta}_i - \theta), \\
 \text{RMSE}(\hat{\alpha}) &= \sqrt{\frac{1}{B} \sum_{i=1}^B (\hat{\alpha}_i - \alpha)^2}, & \text{RMSE}(\hat{\theta}) &= \sqrt{\frac{1}{B} \sum_{i=1}^B (\hat{\theta}_i - \theta)^2}, \\
 D_{\text{abs}} &= \frac{1}{Bn} \sum_{i=1}^B \sum_{j=1}^n |F(x_{ij}|\alpha, \theta) - F(x_{ij}|\hat{\alpha}, \hat{\theta})|, \\
 D_{\text{max}} &= \frac{1}{B} \sum_{i=1}^B \max_j |F(x_{ij}|\alpha, \theta) - F(x_{ij}|\hat{\alpha}, \hat{\theta})|.
 \end{aligned}
 \tag{34}$$

The results of the comparisons between the different estimation methods are summarised in the Tables 1 to 7.

Table 1: Simulation results for the parameters $\alpha=0.5$ and $\theta=0.8$.

	n	<i>BIAS</i> α	<i>BIAS</i> θ	<i>RMSE</i> α	<i>RMSE</i> θ	<i>Dabs</i>	<i>Dmax</i>
MLE	20	0.0705(1)	0.1261(10)	0.547(1)	0.3594(4)	0.0576(2)	0.0975(3)
LSE		1.5758(9)	-0.0272(2)	17.4137(8)	0.3857(6)	0.0598(6)	0.1001(6)
WLSE		1.3019(8)	-0.0012(1)	25.0857(9)	0.3602(5)	0.0585(5)	0.0977(5)
CME		0.5846(6)	0.1163(8)	8.7437(6)	0.441(8)	0.0598(7)	0.1033(7)
KE		0.2557(5)	0.1251(9)	1.1527(5)	0.4418(9)	0.0608(8)	0.1052(8)
ADE		0.9622(7)	0.0405(3)	13.28(7)	0.3494(2)	0.058(3)	0.0969(2)
ADERE		3.1530(10)	0.0968(6)	30.7816(10)	0.5169(10)	0.0621(9)	0.1081(10)
ADELE		0.1941(4)	0.0691(5)	1.0381(4)	0.3542(3)	0.0581(4)	0.0976(4)
ADE2LE		0.1870(3)	0.0999(7)	0.7722(3)	0.4249(7)	0.0622(10)	0.1063(9)
BAYES		0.1556(2)	0.0585(4)	0.5647(2)	0.3236(1)	0.0574(1)	0.0951(1)
MLE	60	0.0238(1)	0.0409(10)	0.2397(1)	0.1847(2)	0.0338(2)	0.057(2)
LSE		0.1146(9)	-0.0072(1)	0.374(8)	0.2186(6)	0.0352(7)	0.0599(6)
WLSE		0.0759(7)	0.0098(2)	0.3067(5)	0.1996(5)	0.0343(4)	0.0582(5)
CME		0.0567(4)	0.0371(9)	0.321(6)	0.228(8)	0.0352(6)	0.0605(7)
KE		0.0720(6)	0.0347(8)	0.3671(7)	0.2337(9)	0.0361(8)	0.0619(8)
ADE		0.0681(5)	0.0143(5)	0.2997(4)	0.1924(4)	0.0342(3)	0.0577(3)
ADERE		0.1253(10)	0.0273(7)	0.5745(10)	0.262(10)	0.0364(9)	0.0636(9)
ADELE		0.0474(2)	0.0241(6)	0.2679(3)	0.1899(3)	0.0345(5)	0.058(4)
ADE2LE		0.1089(8)	0.0117(4)	0.4105(9)	0.2257(7)	0.0387(10)	0.0659(10)
BAYES		0.0531(3)	0.0113(3)	0.2598(2)	0.1725(1)	0.0328(1)	0.0544(1)
MLE	100	0.0130(1)	0.0213(10)	0.1777(1)	0.1324(2)	0.0257(2)	0.0432(2)
LSE		0.0659(8)	-0.0099(5)	0.2522(8)	0.1586(6)	0.0267(6)	0.0455(6)
WLSE		0.0413(7)	0.0025(1)	0.211(4)	0.1424(5)	0.026(4)	0.0439(4)
CME		0.0341(4)	0.0162(8)	0.2305(6)	0.1618(7)	0.0268(7)	0.0458(7)
KE		0.0392(5)	0.0182(9)	0.2516(7)	0.172(9)	0.0275(8)	0.0473(8)
ADE		0.0402(6)	0.0039(2)	0.2121(5)	0.1404(4)	0.0259(3)	0.0438(3)
ADERE		0.0686(9)	0.0104(6)	0.3343(10)	0.1942(10)	0.0278(9)	0.0485(9)
ADELE		0.0272(2)	0.0105(7)	0.1939(3)	0.1369(3)	0.0262(5)	0.044(5)
ADE2LE		0.0834(10)	-0.0062(3)	0.3124(9)	0.1702(8)	0.0303(10)	0.0515(10)
BAYES		0.0310(3)	0.0068(4)	0.1855(2)	0.1310(1)	0.0247(1)	0.0413(1)
MLE	200	0.0056(1)	0.0096(9)	0.1185(1)	0.0905(1)	0.0176(1)	0.0294(1)
LSE		0.0329(9)	-0.0082(8)	0.1496(7)	0.1122(6)	0.0184(7)	0.0314(6)
WLSE		0.0189(5)	0.0003(2)	0.1305(4)	0.0996(5)	0.0178(4)	0.0301(4)
CME		0.0180(4)	0.0047(5)	0.1427(6)	0.1129(7)	0.0184(6)	0.0314(7)
KE		0.0192(7)	0.0065(7)	0.1509(8)	0.1203(9)	0.0191(8)	0.0327(8)
ADE		0.0191(6)	0.0003(1)	0.1308(5)	0.0985(4)	0.0178(3)	0.03(3)
ADERE		0.0307(8)	0.0026(3)	0.174(9)	0.1345(10)	0.0192(9)	0.0333(9)
ADELE		0.0125(2)	0.004(4)	0.1256(2)	0.095(3)	0.018(5)	0.0302(5)
ADE2LE		0.0585(10)	-0.0143(10)	0.2184(10)	0.1187(8)	0.022(10)	0.0371(10)
BAYES		0.0142(3)	0.0059(6)	0.1275(3)	0.0936(2)	0.0178(2)	0.0299(2)

Table 2: Simulation results for the parameters $\alpha=0.5$ and $\theta=1.5$.

	n	<i>BIAS</i> α	<i>BIAS</i> θ	<i>RMSE</i> α	<i>RMSE</i> θ	<i>Dabs</i>	<i>Dmax</i>
MLE	20	0.0704(1)	0.2366(9)	0.5466(1)	0.6738(4)	0.0576(1)	0.0975(2)
LSE		1.6419(9)	-0.0511(2)	19.8484(9)	0.7231(6)	0.0598(6)	0.1001(6)
WLSE		0.9884(8)	-0.0023(1)	14.4253(8)	0.6754(5)	0.0585(4)	0.0977(5)
CME		0.7981(6)	0.2181(8)	13.229(7)	0.827(8)	0.0598(7)	0.1033(7)
KE		0.2557(5)	0.237(10)	1.1858(5)	0.8315(9)	0.0608(8)	0.1053(8)
ADE		0.8309(7)	0.0759(3)	10.9628(6)	0.6551(2)	0.058(2)	0.0969(1)
ADERE		3.2545(10)	0.1816(6)	33.1077(10)	0.9692(10)	0.0621(9)	0.1081(10)
ADELE		0.1941(3)	0.1296(5)	1.0369(4)	0.6642(3)	0.0581(3)	0.0976(3)
ADE2LE		0.187(2)	0.1874(7)	0.7719(3)	0.7963(7)	0.0622(10)	0.1063(9)
BAYES		0.1965(4)	0.0990(4)	0.6123(2)	0.6099(1)	0.0589(5)	0.0976(4)
MLE	60	0.0238(1)	0.0767(10)	0.2397(2)	0.3464(2)	0.0338(2)	0.057(2)
LSE		0.1147(9)	-0.0135(1)	0.374(8)	0.4098(6)	0.0352(7)	0.0599(6)
WLSE		0.076(6)	0.0185(2)	0.3067(5)	0.3743(5)	0.0343(4)	0.0582(5)
CME		0.0568(3)	0.0695(9)	0.321(6)	0.4275(8)	0.0352(6)	0.0605(7)
KE		0.0721(5)	0.0649(8)	0.367(7)	0.4381(9)	0.0361(8)	0.0619(8)
ADE		0.0682(4)	0.0268(4)	0.2997(4)	0.3608(4)	0.0342(3)	0.0577(3)
ADERE		0.1253(10)	0.0513(6)	0.5744(10)	0.4912(10)	0.0364(9)	0.0636(9)
ADELE		0.0475(2)	0.0452(5)	0.2679(3)	0.3561(3)	0.0345(5)	0.058(4)
ADE2LE		0.1089(8)	0.0219(3)	0.4106(9)	0.4232(7)	0.0387(10)	0.0659(10)
BAYES		0.0942(7)	-0.0626(7)	0.2250(1)	0.2663(1)	0.0317(1)	0.0518(1)
MLE	100	0.013(1)	0.0399(9)	0.1777(2)	0.2482(2)	0.0257(2)	0.0432(2)
LSE		0.066(8)	-0.0186(4)	0.2522(8)	0.2974(6)	0.0267(6)	0.0455(6)
WLSE		0.0413(6)	0.0048(1)	0.211(4)	0.2671(5)	0.026(4)	0.0439(4)
CME		0.0342(3)	0.0305(7)	0.2305(6)	0.3034(7)	0.0268(7)	0.0458(7)
KE		0.0394(4)	0.0334(8)	0.2514(7)	0.3214(9)	0.0275(8)	0.0472(8)
ADE		0.0403(5)	0.0073(2)	0.2121(5)	0.2632(4)	0.0259(3)	0.0438(3)
ADERE		0.0687(9)	0.0196(5)	0.3342(10)	0.3642(10)	0.0278(9)	0.0485(9)
ADELE		0.0272(2)	0.0196(6)	0.1939(3)	0.2567(3)	0.0262(5)	0.044(5)
ADE2LE		0.0835(10)	-0.0116(3)	0.3123(9)	0.3192(8)	0.0303(10)	0.0515(10)
BAYES		0.0630(7)	-0.0480(10)	0.1765(1)	0.2141(1)	0.0247(1)	0.0408(1)
MLE	200	0.0056(1)	0.0181(9)	0.1184(1)	0.1697(2)	0.0176(2)	0.0294(2)
LSE		0.0329(9)	-0.0154(8)	0.1496(7)	0.2103(6)	0.0184(7)	0.0314(6)
WLSE		0.0189(5)	0.0006(2)	0.1305(4)	0.1867(5)	0.0178(4)	0.0301(4)
CME		0.018(3)	0.0089(6)	0.1427(6)	0.2117(7)	0.0184(6)	0.0314(7)
KE		0.0192(7)	0.0122(7)	0.1509(8)	0.2256(9)	0.0191(8)	0.0327(8)
ADE		0.0192(6)	0.0005(1)	0.1308(5)	0.1847(4)	0.0178(3)	0.03(3)
ADERE		0.0307(8)	0.0048(4)	0.1739(9)	0.2523(10)	0.0192(9)	0.0333(9)
ADELE		0.0126(2)	0.0074(5)	0.1255(3)	0.1782(3)	0.018(5)	0.0302(5)
ADE2LE		0.0586(10)	-0.0269(10)	0.2184(10)	0.2225(8)	0.022(10)	0.0371(10)
BAYES		0.0182(4)	0.0044(3)	0.1187(2)	0.1684(1)	0.0171(1)	0.0286(1)

Table 3: Simulation results for the parameters $\alpha=0.5$ and $\theta=5$.

	n	<i>BIAS</i> α	<i>BIAS</i> θ	<i>RMSE</i> α	<i>RMSE</i> θ	<i>Dabs</i>	<i>Dmax</i>
MLE	20	0.0705(1)	0.7886(9)	0.5472(1)	2.246(4)	0.0576(1)	0.0975(3)
LSE		1.2337(8)	-0.1702(2)	12.2539(7)	2.4101(6)	0.0598(6)	0.1001(6)
WLSE		1.5106(9)	-0.0078(1)	27.5602(9)	2.2513(5)	0.0585(5)	0.0977(5)
CME		0.7937(6)	0.7273(8)	13.014(8)	2.7557(8)	0.0598(7)	0.1033(7)
KE		0.2555(5)	0.7902(10)	1.1647(5)	2.7722(9)	0.0608(8)	0.1054(8)
ADE		0.8155(7)	0.2535(4)	11.1846(6)	2.1836(2)	0.058(2)	0.0969(2)
ADERE		3.4999(10)	0.6066(6)	38.3057(10)	3.2284(10)	0.0621(9)	0.1081(10)
ADELE		0.1941(3)	0.432(5)	1.0376(4)	2.2138(3)	0.0581(3)	0.0976(4)
ADE2LE		0.187(2)	0.6248(7)	0.7719(3)	2.6559(7)	0.0622(10)	0.1063(9)
BAYES		0.2376(4)	-0.2328(3)	0.6478(2)	1.9231(1)	0.0584(4)	0.0961(1)
MLE	60	0.0238(1)	0.2554(10)	0.2397(1)	1.1546(2)	0.0338(2)	0.057(2)
LSE		0.1147(9)	-0.045(1)	0.374(8)	1.3661(6)	0.0352(7)	0.0599(6)
WLSE		0.076(7)	0.0615(2)	0.3067(5)	1.2475(5)	0.0343(4)	0.0582(5)
CME		0.0568(3)	0.2317(9)	0.321(6)	1.4249(8)	0.0352(6)	0.0605(7)
KE		0.0721(6)	0.216(8)	0.3671(7)	1.4599(9)	0.0361(8)	0.0619(8)
ADE		0.0682(4)	0.0892(5)	0.2997(4)	1.2025(4)	0.0342(3)	0.0577(3)
ADERE		0.1253(10)	0.1708(7)	0.5744(10)	1.6373(10)	0.0364(9)	0.0636(9)
ADELE		0.0475(2)	0.1506(6)	0.2679(2)	1.1871(3)	0.0345(5)	0.058(4)
ADE2LE		0.109(8)	0.073(3)	0.4106(9)	1.4107(7)	0.0387(10)	0.0659(10)
BAYES		0.0716(5)	-0.0887(4)	0.2706(3)	1.0708(1)	0.0335(1)	0.0555(1)
MLE	100	0.013(1)	0.1329(10)	0.1777(1)	0.8272(2)	0.0257(2)	0.0432(2)
LSE		0.066(8)	-0.0621(5)	0.2522(8)	0.9913(6)	0.0267(6)	0.0455(6)
WLSE		0.0413(6)	0.0159(1)	0.211(4)	0.8903(5)	0.026(4)	0.0439(4)
CME		0.0342(3)	0.1015(8)	0.2305(6)	1.0113(7)	0.0268(7)	0.0458(7)
KE		0.0391(4)	0.1143(9)	0.2516(7)	1.0756(9)	0.0275(8)	0.0473(8)
ADE		0.0403(5)	0.0242(2)	0.2121(5)	0.8773(4)	0.0259(3)	0.0438(3)
ADERE		0.0687(9)	0.0651(6)	0.3342(10)	1.214(10)	0.0278(9)	0.0485(9)
ADELE		0.0272(2)	0.0654(7)	0.1939(3)	0.8556(3)	0.0262(5)	0.044(5)
ADE2LE		0.0835(10)	-0.0386(3)	0.3123(9)	1.0638(8)	0.0303(10)	0.0515(10)
BAYES		0.0422(7)	-0.0580(4)	0.1904(2)	0.8162(1)	0.0251(1)	0.0418(1)
MLE	200	0.0057(1)	0.0602(9)	0.1185(1)	0.5654(1)	0.0176(1)	0.0294(1)
LSE		0.0329(9)	-0.0514(8)	0.1496(7)	0.7009(6)	0.0184(7)	0.0314(6)
WLSE		0.0189(4)	0.0019(2)	0.1305(4)	0.6222(5)	0.0178(3)	0.0301(4)
CME		0.018(3)	0.0296(6)	0.1427(6)	0.7056(7)	0.0184(6)	0.0314(7)
KE		0.0192(6)	0.0405(7)	0.1509(8)	0.7521(9)	0.0191(8)	0.0327(8)
ADE		0.0192(5)	0.0018(1)	0.1308(5)	0.6157(4)	0.0178(2)	0.03(3)
ADERE		0.0307(8)	0.0161(4)	0.174(9)	0.8409(10)	0.0192(9)	0.0333(9)
ADELE		0.0126(2)	0.0248(5)	0.1255(2)	0.5939(3)	0.018(5)	0.0302(5)
ADE2LE		0.0586(10)	-0.0896(10)	0.2184(10)	0.7418(8)	0.022(10)	0.0371(10)
BAYES		0.0196(7)	-0.0148(3)	0.1291(3)	0.5768(2)	0.0179(4)	0.0299(2)

Table 4: Simulation results for the parameters $\alpha=1.5$ and $\theta=0.8$.

	n	BIAS α	BIAS θ	RMSE α	RMSE θ	Dabs	Dmax
MLE	20	0.2008(1)	0.2446(8)	1.8891(2)	0.5854(4)	0.0568(2)	0.0951(4)
LSE		19.8476(9)	-.0054(1)	83.2247(9)	0.644(7)	0.059(6)	0.099(6)
WLSE		11.4262(8)	0.0271(2)	80.6739(8)	0.5988(5)	0.0577(5)	0.0959(5)
CME		9.7177(7)	0.2362(7)	75.328(7)	0.7792(9)	0.0595(7)	0.1029(8)
KE		1.1258(4)	0.2463(10)	3.8142(4)	0.7489(8)	0.0604(9)	0.1043(9)
ADE		8.7759(6)	0.0874(4)	72.7305(6)	0.5651(3)	0.0573(4)	0.0948(2)
ADERE		24.2016(10)	0.2449(9)	102.0726(10)	0.9779(10)	0.0621(10)	0.1093(10)
ADELE		3.5506(5)	0.1255(5)	44.1342(5)	0.5573(2)	0.0572(3)	0.0948(3)
ADE2LE		0.7994(3)	0.1755(6)	3.7174(3)	0.6237(6)	0.0601(8)	0.1013(7)
BAYES		0.2910(2)	0.0721(3)	1.3272(1)	0.4628(1)	0.0523(1)	0.0851(1)
MLE	60	0.0708(1)	0.0784(10)	0.7947(1)	0.28(2)	0.0333(2)	0.055(2)
LSE		0.8946(9)	0.0004(1)	5.2236(9)	0.3625(7)	0.035(6)	0.0593(6)
WLSE		0.3635(6)	0.0275(3)	1.3324(5)	0.3194(6)	0.034(4)	0.0569(5)
CME		0.4416(7)	0.0748(9)	2.6908(8)	0.3848(8)	0.035(7)	0.0601(7)
KE		0.4911(8)	0.0699(8)	1.886(7)	0.3982(9)	0.0362(8)	0.0621(9)
ADE		0.3472(5)	0.0306(4)	1.4944(6)	0.2998(4)	0.0337(3)	0.0562(4)
ADERE		2.6405(10)	0.068(7)	21.2351(10)	0.4605(10)	0.0365(9)	0.0641(10)
ADELE		0.2144(3)	0.043(6)	1.0168(3)	0.2866(3)	0.034(5)	0.0562(3)
ADE2LE		0.3238(4)	0.0314(5)	1.2519(4)	0.3062(5)	0.0373(10)	0.0618(8)
BAYES		0.1856(2)	0.0167(2)	0.8547(2)	0.2513(1)	0.0314(1)	0.0514(1)
MLE	100	0.0354(1)	0.0424(10)	0.579(1)	0.1952(2)	0.0253(2)	0.0417(2)
LSE		0.3625(9)	-0.0098(2)	1.3434(9)	0.2579(7)	0.0266(6)	0.045(6)
WLSE		0.182(4)	0.0104(3)	0.8064(4)	0.2218(6)	0.0257(4)	0.0429(5)
CME		0.2086(6)	0.0338(8)	1.1042(7)	0.2659(8)	0.0266(7)	0.0454(7)
KE		0.2849(8)	0.0339(9)	1.2704(8)	0.29(9)	0.0276(8)	0.0474(8)
ADE		0.1833(5)	0.0108(4)	0.8322(5)	0.2149(4)	0.0256(3)	0.0427(4)
ADERE		1.0208(10)	0.0288(7)	12.7823(10)	0.3323(10)	0.0278(9)	0.0488(10)
ADELE		0.1168(3)	0.0197(6)	0.6879(3)	0.2032(3)	0.0258(5)	0.0427(3)
ADE2LE		0.2357(7)	0.002(1)	0.9088(6)	0.2216(5)	0.0289(10)	0.0478(9)
BAYES		0.1134(2)	0.0111(5)	0.6235(2)	0.1891(1)	0.0239(1)	0.0394(1)
MLE	200	0.014(1)	0.0202(10)	0.3899(1)	0.1305(1)	0.0172(1)	0.0282(1)
LSE		0.1566(8)	-0.0097(6)	0.6062(7)	0.1807(7)	0.0183(6)	0.031(6)
WLSE		0.0782(4)	0.0044(2)	0.4663(4)	0.1524(6)	0.0176(4)	0.0293(4)
CME		0.0946(6)	0.0118(7)	0.5638(6)	0.183(8)	0.0183(7)	0.0311(7)
KE		0.1207(7)	0.0139(9)	0.6449(9)	0.2031(9)	0.0192(8)	0.0329(8)
ADE		0.0822(5)	0.0032(1)	0.4711(5)	0.1496(4)	0.0176(3)	0.0292(3)
ADERE		0.185(10)	0.0095(5)	0.8148(10)	0.2276(10)	0.0192(9)	0.0334(9)
ADELE		0.0517(2)	0.0084(3)	0.4274(3)	0.1401(3)	0.0177(5)	0.0293(5)
ADE2LE		0.1627(9)	-0.0127(8)	0.6307(8)	0.1522(5)	0.021(10)	0.0345(10)
BAYES		0.0522(3)	0.0093(4)	0.4170(2)	0.1336(2)	0.0174(2)	0.0287(2)

Table 5: Simulation results for the parameters $\alpha=1.5$ and $\theta=1.5$.

	n	<i>BIAS</i> α	<i>BIAS</i> θ	<i>RMSE</i> α	<i>RMSE</i> θ	<i>Dabs</i>	<i>Dmax</i>
MLE	20	0.2006(1)	0.4588(9)	1.8878(2)	1.0978(4)	0.0568(2)	0.0951(4)
LSE		22.7754(9)	-0.0102(1)	96.1041(9)	1.2075(7)	0.059(6)	0.0989(6)
WLSE		12.3541(8)	0.0509(2)	90.0936(8)	1.1231(5)	0.0577(5)	0.0959(5)
CME		10.34(7)	0.4428(7)	74.8016(7)	1.461(9)	0.0595(7)	0.1029(8)
KE		1.1381(4)	0.4571(8)	3.8393(4)	1.3891(8)	0.0604(9)	0.1042(9)
ADE		8.797(6)	0.1637(3)	67.9383(6)	1.0596(3)	0.0573(4)	0.0948(3)
ADERE		25.9719(10)	0.459(10)	117.8518(10)	1.8337(10)	0.0621(10)	0.1093(10)
ADELE		3.795(5)	0.2354(4)	49.7045(5)	1.0448(2)	0.0572(3)	0.0948(2)
ADE2LE		0.8012(3)	0.329(6)	3.7381(3)	1.1694(6)	0.0601(8)	0.1013(7)
BAYES		0.3341(2)	0.2433(5)	1.3569(1)	0.8656(1)	0.0559(1)	0.0903(1)
MLE	60	0.0708(1)	0.1471(10)	0.7947(1)	0.525(2)	0.0333(2)	0.055(2)
LSE		1.0408(9)	0.0008(1)	9.4165(9)	0.6797(7)	0.035(6)	0.0593(6)
WLSE		0.3635(6)	0.0516(3)	1.3323(5)	0.5989(6)	0.034(4)	0.0569(5)
CME		0.4697(7)	0.1402(9)	3.4054(8)	0.7215(8)	0.035(7)	0.0601(7)
KE		0.4984(8)	0.1328(8)	1.9199(7)	0.7509(9)	0.0361(8)	0.0621(9)
ADE		0.3471(5)	0.0573(4)	1.4945(6)	0.5622(4)	0.0337(3)	0.0562(4)
ADERE		3.028(10)	0.1276(7)	25.5184(10)	0.8634(10)	0.0365(9)	0.0641(10)
ADELE		0.2144(3)	0.0807(6)	1.0167(3)	0.5374(3)	0.034(5)	0.0562(3)
ADE2LE		0.3238(4)	0.0587(5)	1.2515(4)	0.574(5)	0.0373(10)	0.0618(8)
BAYES		0.1915(2)	0.0473(2)	0.8518(2)	0.4675(1)	0.0321(1)	0.0522(1)
MLE	100	0.0356(1)	0.0796(10)	0.5795(1)	0.3662(2)	0.0253(2)	0.0417(2)
LSE		0.3625(9)	-0.0184(2)	1.3436(9)	0.4835(7)	0.0266(6)	0.045(6)
WLSE		0.1815(4)	0.0196(3)	0.8059(4)	0.4156(6)	0.0257(4)	0.0429(5)
CME		0.2086(6)	0.0634(8)	1.1044(7)	0.4986(8)	0.0266(7)	0.0454(7)
KE		0.2854(8)	0.0638(9)	1.2727(8)	0.5441(9)	0.0276(8)	0.0474(8)
ADE		0.1833(5)	0.0202(4)	0.8321(5)	0.4029(4)	0.0256(3)	0.0427(4)
ADERE		0.8657(10)	0.0541(7)	8.4605(10)	0.6231(10)	0.0278(9)	0.0488(10)
ADELE		0.1168(2)	0.037(6)	0.6879(3)	0.3809(3)	0.0258(5)	0.0427(3)
ADE2LE		0.2355(7)	0.0039(1)	0.9087(6)	0.4156(5)	0.0289(10)	0.0478(9)
BAYES		0.1227(3)	0.0244(5)	0.6273(2)	0.3545(1)	0.0241(1)	0.0396(1)
MLE	200	0.014(1)	0.0379(10)	0.3898(1)	0.2446(1)	0.0172(1)	0.0282(1)
LSE		0.1566(8)	-0.0182(6)	0.6062(7)	0.3388(7)	0.0183(6)	0.031(6)
WLSE		0.0782(4)	0.0083(2)	0.4663(4)	0.2858(6)	0.0176(4)	0.0293(4)
CME		0.0947(6)	0.0221(7)	0.5638(6)	0.3429(8)	0.0183(7)	0.0311(7)
KE		0.1204(7)	0.0262(9)	0.6443(9)	0.3808(9)	0.0192(8)	0.0329(8)
ADE		0.0822(5)	0.006(1)	0.4711(5)	0.2806(4)	0.0176(3)	0.0292(3)
ADERE		0.185(10)	0.0179(5)	0.8148(10)	0.4267(10)	0.0192(9)	0.0334(9)
ADELE		0.0516(2)	0.0158(3)	0.4274(3)	0.2626(3)	0.0177(5)	0.0293(5)
ADE2LE		0.1626(9)	-0.0237(8)	0.6306(8)	0.2854(5)	0.021(10)	0.0345(10)
BAYES		0.0584(3)	0.0170(4)	0.4205(2)	0.2504(2)	0.0175(2)	0.0287(2)

Table 6: Simulation results for the parameters $\alpha=2$ and $\theta=1.5$.

	n	<i>BIAS</i> α	<i>BIAS</i> θ	<i>RMSE</i> α	<i>RMSE</i> θ	<i>Dabs</i>	<i>Dmax</i>
MLE	20	0.2662(1)	0.5636(8)	2.6704(2)	1.2898(4)	0.0565(1)	0.0944(3)
LSE		37.4401(9)	0.0278(1)	129.6031(8)	1.4181(7)	0.0586(5)	0.0982(5)
WLSE		25.7858(8)	0.0887(2)	149.61(9)	1.3267(5)	0.0575(4)	0.0952(4)
CME		18.2977(7)	0.5521(7)	101.8159(6)	1.7443(9)	0.0593(6)	0.1025(7)
KE		1.4975(3)	0.567(9)	4.7768(3)	1.6296(8)	0.0599(8)	0.1029(8)
ADE		17.184(6)	0.21(3)	107.5207(7)	1.2354(3)	0.057(3)	0.094(2)
ADERE		39.3189(10)	0.6077(10)	168.0948(10)	2.2416(10)	0.062(9)	0.1091(9)
ADELE		8.3735(5)	0.2869(5)	86.1526(5)	1.2155(2)	0.0569(2)	0.094(1)
ADE2LE		1.7689(4)	0.3971(6)	12.092(4)	1.3453(6)	0.0597(7)	0.1001(6)
BAYES		-0.6444(2)	0.2867(4)	1.1438(1)	1.0079(1)	0.0896(10)	0.1400(10)
MLE	60	0.0968(1)	0.1799(10)	1.116(2)	0.6(2)	0.0331(1)	0.0544(1)
LSE		2.5983(9)	0.0113(1)	17.3432(9)	0.7993(7)	0.0349(5)	0.0591(5)
WLSE		0.6225(5)	0.0691(2)	2.3326(5)	0.6977(6)	0.0338(3)	0.0565(4)
CME		1.2985(8)	0.1757(8)	10.4285(8)	0.8546(8)	0.035(6)	0.0599(6)
KE		0.7681(7)	0.1702(7)	2.654(6)	0.8758(9)	0.036(7)	0.0618(8)
ADE		0.6803(6)	0.0719(3)	3.894(7)	0.6471(5)	0.0336(2)	0.0557(3)
ADERE		5.9195(10)	0.1696(6)	40.5557(10)	1.0311(10)	0.0365(8)	0.0641(10)
ADELE		0.3532(2)	0.0974(5)	1.5779(3)	0.6149(3)	0.0338(4)	0.0557(2)
ADE2LE		0.4536(3)	0.0766(4)	1.748(4)	0.6389(4)	0.037(9)	0.0608(7)
BAYES		-0.4915(4)	0.1785(9)	0.6250(1)	0.3838(1)	0.0394(10)	0.0624(9)
MLE	100	0.0475(1)	0.098(9)	0.8092(2)	0.4141(2)	0.0251(1)	0.0413(1)
LSE		0.7414(9)	-0.0164(2)	4.0347(9)	0.5651(7)	0.0266(5)	0.0448(5)
WLSE		0.292(3)	0.0275(4)	1.2286(4)	0.4793(6)	0.0256(3)	0.0426(4)
CME		0.4212(7)	0.0801(7)	2.4695(8)	0.5852(8)	0.0266(6)	0.0452(6)
KE		0.4787(8)	0.0819(8)	1.8974(7)	0.6356(9)	0.0275(7)	0.0473(9)
ADE		0.3(4)	0.0267(3)	1.3006(6)	0.4608(5)	0.0255(2)	0.0423(3)
ADERE		2.1496(10)	0.0733(6)	20.2339(10)	0.737(10)	0.0278(8)	0.0488(10)
ADELE		0.186(2)	0.0454(5)	1.0119(3)	0.4329(3)	0.0257(4)	0.0423(2)
ADE2LE		0.3219(5)	0.0117(1)	1.2407(5)	0.455(4)	0.0286(9)	0.0469(8)
BAYES		-0.3649(6)	0.1300(10)	0.5274(1)	0.3101(1)	0.0293(10)	0.0466(7)
MLE	200	0.0183(1)	0.047(10)	0.5455(2)	0.2739(2)	0.0171(1)	0.0278(1)
LSE		0.2597(9)	-0.0183(4)	0.9499(8)	0.3945(7)	0.0183(5)	0.0309(6)
WLSE		0.1216(4)	0.0126(2)	0.6785(4)	0.3271(6)	0.0175(3)	0.029(4)
CME		0.1623(6)	0.0293(8)	0.8694(7)	0.4003(8)	0.0183(6)	0.031(7)
KE		0.2183(7)	0.0337(9)	1.033(9)	0.4492(9)	0.0192(9)	0.0329(8)
ADE		0.1293(5)	0.0089(1)	0.6898(5)	0.3199(5)	0.0175(2)	0.0289(2)
ADERE		0.3444(10)	0.0254(7)	1.4647(10)	0.5035(10)	0.0192(8)	0.0334(9)
ADELE		0.0808(3)	0.0199(5)	0.6121(3)	0.2975(3)	0.0177(4)	0.029(3)
ADE2LE		0.2192(8)	-0.0214(6)	0.8541(6)	0.3103(4)	0.0207(10)	0.0338(10)
BAYES		-0.0636(2)	0.0154(3)	0.4968(1)	0.2674(1)	0.0189(7)	0.0304(5)

Table 7: Simulation results for the parameters $\alpha=5$ and $\theta=5$.

	n	<i>BIAS</i> α	<i>BIAS</i> θ	<i>RMSE</i> α	<i>RMSE</i> θ	<i>Dabs</i>	<i>Dmax</i>
MLE	20	0.662(1)	3.897(8)	8.5017(3)	7.9157(5)	0.0558(2)	0.0944(4)
LSE		137.7753(10)	1.2904(2)	313.2926(7)	8.6794(8)	0.0571(5)	0.0982(6)
WLSE		135.215(9)	1.3454(3)	394.8529(9)	8.2513(7)	0.0562(4)	0.0952(3)
CME		76.2086(5)	4.3965(9)	266.2909(4)	11.6518(9)	0.0586(8)	0.1025(8)
KE		2.5287(3)	2.9715(7)	7.9618(2)	7.9591(6)	0.0573(6)	0.1029(5)
ADE		86.1423(7)	1.7243(4)	295.6509(6)	7.2827(3)	0.0558(1)	0.094(1)
ADERE		102.9408(8)	5.3604(10)	277.8809(5)	15.7864(10)	0.0614(9)	0.1091(9)
ADELE		77.7715(6)	2.0429(5)	328.7321(8)	7.1767(2)	0.0559(3)	0.094(2)
ADE2LE		62.9742(4)	2.6679(6)	403.064(10)	7.8382(4)	0.0585(7)	0.1001(7)
BAYES		-2.1239(2)	0.9880(1)	2.8359(1)	3.2176(1)	0.1112(10)	0.1665(10)
MLE	60	0.2842(1)	1.2237(7)	3.5503(2)	3.3063(3)	0.0326(1)	0.0544(1)
LSE		29.4129(9)	0.4024(2)	92.1025(9)	4.7756(8)	0.0344(5)	0.0591(5)
WLSE		10.7627(7)	0.6108(5)	54.174(6)	4.1314(6)	0.0334(3)	0.0565(4)
CME		17.4751(8)	1.3699(9)	68.2378(8)	5.2922(9)	0.0347(6)	0.0599(8)
KE		1.8329(3)	1.2628(8)	5.7987(3)	4.5075(7)	0.0348(7)	0.0618(6)
ADE		10.7433(6)	0.5384(3)	57.1575(7)	3.6014(5)	0.0331(2)	0.0557(2)
ADERE		32.5566(10)	1.6004(10)	97.8538(10)	6.5201(10)	0.036(8)	0.0641(9)
ADELE		5.6667(5)	0.6448(6)	40.6521(5)	3.3949(4)	0.0334(4)	0.0557(3)
ADE2LE		1.8595(4)	0.5978(4)	8.0566(4)	3.2766(2)	0.0363(9)	0.0608(7)
BAYES		-0.9673(2)	0.3620(1)	2.1741(1)	2.2877(1)	0.0567(10)	0.0865(10)
MLE	100	0.135(1)	0.6755(8)	2.5368(2)	2.1738(2)	0.0247(1)	0.0413(1)
LSE		12.8024(9)	0.0604(1)	55.8416(9)	3.299(7)	0.0263(5)	0.0448(5)
WLSE		2.3525(6)	0.2747(5)	13.3374(6)	2.7212(6)	0.0253(3)	0.0426(4)
CME		7.8347(8)	0.6364(7)	39.6913(8)	3.4952(9)	0.0264(6)	0.0452(7)
KE		1.6123(5)	0.6844(9)	5.2205(4)	3.3265(8)	0.0266(7)	0.0473(8)
ADE		3.5319(7)	0.2226(4)	21.4179(7)	2.5065(5)	0.0252(2)	0.0423(3)
ADERE		16.9984(10)	0.7487(10)	59.0158(10)	4.4635(10)	0.0275(8)	0.0488(9)
ADELE		1.2626(4)	0.3114(6)	5.5096(5)	2.3242(4)	0.0254(4)	0.0423(2)
ADE2LE		1.0491(3)	0.1939(2)	3.8795(3)	2.1777(3)	0.0278(9)	0.0469(6)
BAYES		-0.6395(2)	0.1673(3)	1.8248(1)	1.7832(1)	0.0407(10)	0.0621(10)
MLE	200	0.0503(1)	0.3265(10)	1.713(2)	1.3721(2)	0.0168(1)	0.0278(1)
LSE		2.8013(9)	-0.0353(2)	13.1723(9)	2.3005(7)	0.0181(5)	0.0309(5)
WLSE		0.6517(5)	0.1395(5)	2.7648(5)	1.7942(6)	0.0173(3)	0.029(4)
CME		1.88(8)	0.2518(7)	9.5622(8)	2.3598(8)	0.0181(6)	0.031(6)
KE		1.1194(7)	0.2917(9)	3.7878(7)	2.4316(9)	0.0188(7)	0.0329(7)
ADE		0.7488(6)	0.0911(4)	3.1147(6)	1.7195(5)	0.0172(2)	0.0289(2)
ADERE		6.0125(10)	0.2742(8)	25.9292(10)	3.0263(10)	0.019(8)	0.0334(9)
ADELE		0.4469(3)	0.1449(6)	2.3202(3)	1.5725(4)	0.0174(4)	0.029(3)
ADE2LE		0.6464(4)	-0.0172(1)	2.4708(4)	1.4348(3)	0.0201(9)	0.0338(8)
BAYES		-0.3214(2)	0.0884(3)	1.4849(1)	1.3293(1)	0.0253(10)	0.0389(10)

It can be observed that the Minimum Squares (Weighted) and Anderson-Darling Right Tail estimators for θ exhibit a negative bias. In contrast, all other estimators, as well as the estimates for both θ and α , show a positive bias. However, the property of asymptotic unbiasedness is maintained, since the bias tends to zero as the sample size n increases.

Regarding the RMSE, all estimation methods demonstrate consistency: as n grows, the variance of the estimators approaches zero, confirming their reliability in larger samples.

For the parameter α , the Maximum Likelihood estimator achieves the lowest bias, while the Anderson-Darling Right Tail estimator exhibits the highest bias. In the case of θ , the Ordinary Least Squares, Weighted Least Squares, and Anderson-Darling estimators show the lowest error rates, whereas the Maximum Likelihood estimator presents the largest errors.

When examining the RMSE, for α the Bayesian estimator performs best for larger parameter values, whereas the Maximum Likelihood estimator is more accurate for smaller values. Conversely, the Anderson-Darling Right Tail estimator consistently shows the poorest performance. For θ , the Bayesian estimator attains the lowest RMSE overall. For small sample sizes, the Anderson-Darling and Anderson-Darling Left Tail estimators provide comparable results, while for larger samples, the Maximum Likelihood estimator approaches similar performance levels. Again, the Anderson-Darling Right Tail estimator displays the worst performance.

The patterns observed in maximum and absolute differences are consistent with the RMSE results. For $n = 20$, the Anderson-Darling estimator is the most efficient, while for larger sample sizes, the Bayesian estimator demonstrates superior performance.

5. Application

An application was conducted to evaluate the fit of the new MOEEG distribution using a specific dataset and to compare its performance against established distributions commonly used in reliability analysis. The dataset, obtained from Box and Cox 1964, consists of the lifetimes of 48 animals exposed to various types of poisons, followed by specific treatments, providing a practical example for modeling survival times under stress conditions.

The analysis aimed to assess whether the MOEEG distribution could capture the underlying risk pattern more accurately than traditional distributions. Goodness-of-fit measures and graphical assessments were employed to evaluate the adequacy of the proposed model relative to existing alternatives. As illustrated in Figure (3), the hazard function appears to increase over time, suggesting a growing risk of failure as the exposure duration extends. This trend highlights the relevance of using flexible distributions, such as MOEEG, that can accommodate non-constant risk behaviors observed in real-world survival data.

Overall, the application demonstrates the practical usefulness of the MOEEG distribution for modeling lifetime data and provides a comparative perspective on its performance relative to classical approaches.

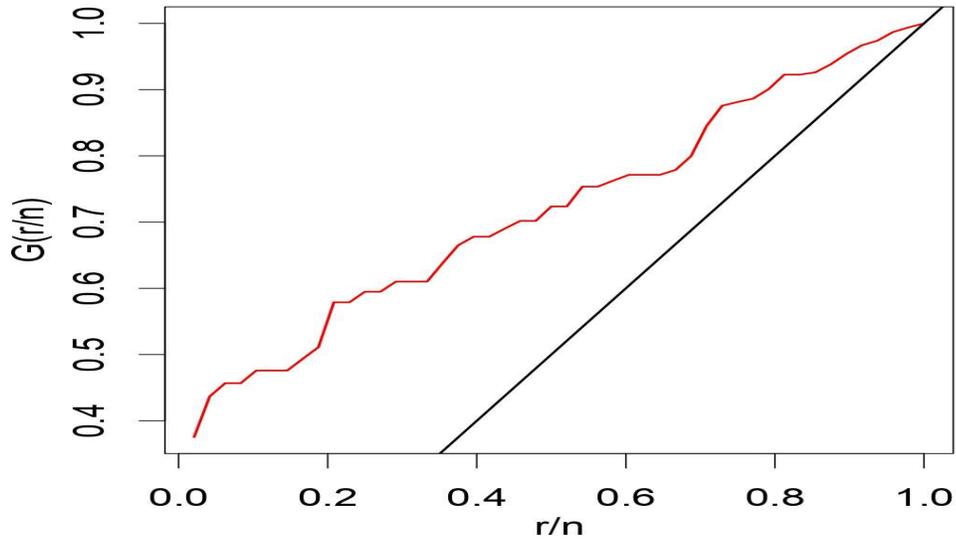


Figure 3: TTT Plot.

The distributions considered for comparison, along with their parameters estimated via the maximum likelihood method, are summarized as follows. The MOEEG distribution was fitted with $\hat{\alpha} = 0.0036$ and $\hat{\theta} = 2.0509$, providing the baseline for assessing the performance of the new model. The Exponentiated Gamma (EG) distribution was estimated with $\hat{\theta} = 0.3681$, while the Nadarajah-Haghighi Exponential (NHE) had $\hat{\alpha} = 0.0090$ and $\hat{\beta} = 153.0002$. For the Weibull distribution, the estimates were $\hat{\alpha} = 2.0606$ and $\hat{\beta} = 0.5445$, and for the Exponential Power (EP), $\hat{\lambda} = 0.7603$ and $\hat{\theta} = 1.3798$. Finally, the Gamma distribution was fitted with $\hat{\alpha} = 4.3079$ and $\hat{\beta} = 0.1112$.

These parameter estimates allow a comprehensive comparison of the MOEEG distribution with other well-known lifetime distributions. By evaluating the fit of each model, it is possible to analyze the ability of the MOEEG distribution to capture the underlying behavior of the observed lifetimes, providing insights into its applicability in modeling real-world survival data.

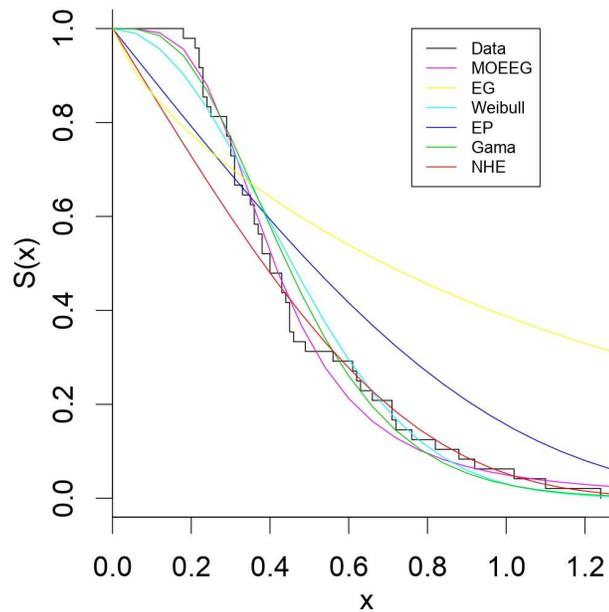


Figure 4: Adjusted distributions.

In the context of data analysis under uncertainty, the use of model selection criteria is essential for comparing and selecting parametric models. Such criteria account for the complexity of the models by penalizing the likelihood function according to the number of parameters and, in some cases, the sample size. Specifically, the penalty subtracts an amount from the likelihood function that reflects model complexity.

The Kullback-Leibler information was introduced by Akaike (1974) as a basis for model selection. This led to the development of the Akaike Information Criterion (AIC), which establishes a direct relationship between the maximized likelihood function and the Kullback-Leibler divergence. An improved version, the CAIC, was proposed by Liang and Zou (2008), incorporating a correction that balances the number of parameters with the sample size.

Another widely used criterion is the Bayesian Information Criterion (BIC), also known as Schwarz criterion (Schwarz 1978). Defined in terms of the posterior probability, BIC is grounded in a Bayesian argument and provides a measure for model selection. According to these criteria, the preferred model is the one minimizing the values of AIC, BIC, and CAIC, defined as follows:

$$\begin{aligned}
 AIC &= -2l(\theta) + 2d, \\
 BIC &= -2l(\theta) + d\log(n), \\
 CAIC &= AIC + \frac{2(d+2)(d+3)}{n-d-3},
 \end{aligned} \tag{35}$$

where d denotes the number of model parameters and $l(\theta)$ is the log-likelihood function.

Table 8: Model selection criteria for the competing distributions.

Model	AIC	BIC	CAIC
MOEEG	-9.709	-5.967	-8.779
NHE	12.762	16.504	13.692
Gamma	-8.340	-4.598	-7.409
EG	61.362	63.234	61.908
EP	5.672	9.414	6.602
Weibull	-2.506	1.236	-1.575

As shown in Table 8, the MOEEG distribution provides the best fit for the dataset, outperforming its special case, the EG distribution, as well as other two-parameter models. It also exhibits superior performance compared to well-established distributions and more recently proposed models. These results highlight the flexibility and robustness of the MOEEG distribution, demonstrating its suitability as a reliable alternative for modeling lifetime data.

6. Concluding Remarks

This study investigated several estimation methods for the MOEEG distribution, focusing on the shape and scale parameters, and evaluated their performance. Across all methods, the property of unbiasedness was confirmed, with the BIAS approaching zero as the sample size n increases. Regarding the RMSE, Bayesian estimators exhibited superior performance for small samples, while for larger sample sizes, the differences among estimators became negligible. All methods demonstrated consistency, with variance diminishing as n grows.

The application to the lifetime data clearly highlighted the flexibility and effectiveness of the MOEEG distribution. Compared to its competing distributions, including both classical and recently proposed models, the MOEEG provided a better fit, confirming its potential as a robust alternative for analyzing lifetime and reliability data. These findings reinforce the relevance of the MOEEG distribution in practical applications and underscore its adaptability across different parameter settings and sample sizes.

Availability of Data and Material

The dataset used in this study was obtained from Box and Cox 1964, comprising the lifetimes of 48 animals exposed to various poisons and subsequently subjected to specific treatments.

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