

Selecting without Replacement from a Population of Bands of Serially Connected Objects

James E. Marengo

School of Mathematics and Statistics
Rochester Institute of Technology, USA

Dominick Banasik

Department of Computer Science
Rochester Institute of Technology, USA

Joseph G. Voelkel

School of Mathematics and Statistics
Rochester Institute of Technology, USA

David L. Farnsworth

School of Mathematics and Statistics
Rochester Institute of Technology, USA

ABSTRACT

The sampling procedure from a finite population of objects that are serially attached into bands is described and analyzed. One object is randomly selected and removed at a time, which results in that object's band being broken into two bands or shortened by one object. The main result gives the probability of choosing an object that is part of a band of serially connected objects of any specified size at each stage of the selection process.

Keywords: Sampling without replacement; Ordered objects in bands; Random sampling; The Bandages Problem; Discrete probability

1. Introduction

Consider sampling from a population that has been organized into serially connected bands. The members of the population are essentially identical objects. It is assumed that, initially, each band has the same number of the objects. As the sampling proceeds, by selecting one object at a time and removing it, the sizes of the bands will not remain equal. The equality of the sizes at the outset is taken solely for convenience and simplicity of exposition. Sampling is performed by randomly selecting one item at a time from the population of remaining objects and removing it permanently. At the time of its selection and its removal from its band and the population, a break is created in that band at the selected object's position. Thus, when an item is selected, the band either becomes two bands or just one band, if the selected item is at either end of the band. At each draw, all remaining items are equally likely to be selected for deletion, regardless of their being connected to other items or not.

The probability that the selected individual item at each draw is taken from a band of each possible size is the main result and appears in Theorem 1. An extreme example is that the probability that the first individual drawn is from a band of current size one is zero, if each original band has more than one item. Another is that, if the sampling is to be continued until the population is exhausted, then the last selection must be from a band that has only one member, and the probability is one that the last selected item's band is of size one. Theorem 1 supplies the probabilities for each possible band size at each selection time.

This is a generalization of The Bandages Problem in [4], where sampling from a box of bandages that are initially in pairs is analyzed. An example of such a pair of bandages is in Figure 1. One bandage is removed from the box daily to obtain one that is used. Each unused bandage is

- Received November 1, 2025, in final form February 2026.
- David L. Farnsworth (corresponding author) is affiliated with School of Mathematics and Statistics, Rochester Institute of Technology, USA
dlfsma@rit.edu

equally likely to be chosen on each day, whether currently it is in a pair or it is a single. The probability is found that each day's selected bandage is a single one.

Besides bandages that are sold in boxes of pairs, other serially ordered products are candy in mini snacks that is packaged as connected pairs, restaurant packets of soy sauce that are in pairs, sticks of cheese that are wrapped in plastic and sold in bags of bands with three sticks connected side-by-side, chocolate-covered wafers that are typically sold with four wafers side-by-side, and sticks of cheese that are sold in attached sets of 12 side-by-side. Figures 2–6 contain photographs of examples.



Figure 1. Bandages in attached pairs.



Figure 2. Candy mini snacks.



Figure 3. Packets of soy sauce.



Figure 4. Sticks of cheese sold in triples.



Figure 5. Chocolate-covered wafers.



Figure 6. Sticks of cheese sold in attached strips of 12.

2. Probabilities of band sizes

Initially, there are B bands, each consisting of M items that are serially connected. The MB items are identical for all practical purposes. The items in each band are numbered from 1 to M with those numbered 1 and M end items. In [4], the main example is a box that contains 60 bandages in $B = 30$ pairs or bands of bandages with $M = 2$, like the example in Figure 1. For the candy in Figure 2 and the soy sauce packets in Figure 3, $M = 2$. In Figures 4–6, $M = 3, 4,$ and 12 , respectively.

On each of MB consecutive occurrences, one individual item is selected at random, i.e., each remaining item is equally likely to be chosen. For uniformity, take the occurrences of selections to be daily, as they are in The Bandages Problem in [4].

Definition 1. At each selection, the bands are named for their number of items before the selection. For example, initially, all bands are M -bands.

If on the first day an item that is numbered 1 or M is selected and removed, then the band containing the item becomes a $M - 1$ -band and its items are numbered 1 to $M - 1$ with the items that are numbered 1 and $M - 1$ being at the new band's ends. If on the first day the selected item is numbered j with $2 \leq j \leq M - 1$, then that band is broken into two bands with one a $j - 1$ -band and one a $M - j$ -band.

Definition 2. An item is called type k at the beginning of a given day if the band to which it belonged immediately following the previous day's selection contained k items.

There are two mutually exclusive classes of k -bands.

Definition 3. A k -band is called Class I if it does not contain an item that was at either end of a M -band. The remainder of the k -bands are called Class II, i.e., they contain at least one item that was at either end of an M -band.

For $k = 1, 2, \dots, M - 2$, there are $(M - k - 1)B$ Class I k -bands. Every k -band is Class II for $k = M - 1$ or M . The number of Class II k -bands is either $2B$ or B according as $k < M$ or $k = M$.

Definition 4. The probability that a type k item is chosen on day n for an initial B bands each containing M items is designated $P(n, k; M, B)$.

Theorem 1 supplies a formula for $P(n, k; M, B)$.

Theorem 1. Positive integers M and B are predetermined and fixed, and $n = 1, 2, \dots, MB$. For $k = 1, 2, \dots, M - 1$,

$$P(n, k; M, B) = \frac{k(MB-k-2)!}{M(MB-1)!} ((M - k - 1)n + 2M(B - 1))(n - 1) \prod_{j=MB-(k-2)}^{MB} (j - n), \quad (1)$$

and for $k = M$,

$$P(n, M; M, B) = \frac{(M(B-1))!}{(MB-1)!} \prod_{j=MB-(M-2)}^{MB} (j - n). \quad (2)$$

Proof. The proof proceeds by conditioning on which set of k serially connected items will form the k -band from which the item is to be selected on day n . The orderings that ensure that the item chosen on day n is part of any specific Class I k -band can be enumerated. Take $1 \leq k < M - 2$. The items on either side of a designated k -band must be chosen to be removed during the first $n - 1$ days in order to create the k -band, and there are $(n - 1)(n - 2)$ ways to choose those days. One must also choose $n - 3$ other items which are not in the designated k -band on those days, and there are

$$(MB - k - 2)(MB - k - 3) \cdots (MB - k - (n - 2))$$

ways to make this choice. On day n , one must choose a single item in the designated k -band, and there are k ways to do this. There are $(MB - n)!$ ways to choose the items on days $n + 1, n + 2, \dots, MB$. There are $(M - k - 1)B$ Class I k -bands. Multiplying these counts gives the total number of orderings which ensure that an item is chosen on day n which is part of a specific Class I k -band [1, pp. 63–64]. Because, on each day, each of the remaining items has the same probability of being chosen, $(MB)!$ possible orderings for these choices are equally likely. Thus, the probability of choosing an item on day n which is part of a Class I k -band is

$$\begin{aligned}
 P_{\text{Class I}}(n, k; M, B) &= \frac{(n-1)(n-2)(MB-k-2)(MB-k-3)\cdots(MB-k-(n-2))k(MB-n)!(M-k-1)B}{(MB)!} \\
 &= \frac{(M-k-1)Bk(MB-k-1)!}{(MB)!} (n-1)(n-2) \prod_{j=MB-(k-2)}^{MB} (j-n). \tag{3}
 \end{aligned}$$

This expression is zero for $k = M - 1$.

The method for enumerating the orderings which ensure that an item is chosen on day n that is part of any Class II k -band is similar. Because, for $k < M$, there are $2B$ such k -bands, and each of them has only one adjacent neighboring item that must be previously selected and deleted, the probability of choosing an item on day n that is part of a Class II k -band is

$$\begin{aligned}
 P_{\text{Class II}}(n, k; M, B) &= \frac{(n-1)(MB-k-1)(MB-k-2)\cdots(MB-k-(n-2))k(MB-n)!2B}{(MB)!} \\
 &= \frac{2Bk(MB-k-1)!}{(MB)!} (n-1) \prod_{j=MB-(k-2)}^{MB} (j-n). \tag{4}
 \end{aligned}$$

Adding the probabilities in (3) and (4), which are from mutually exclusive events [1, p. 58, 2, p. 12, 6, p. 4], for $k = 1, 2, \dots, M - 1$ and $n = 1, 2, \dots, MB$, the probability of choosing a type k item on day n is

$$\begin{aligned}
 P(n, k; M, B) &= P_{\text{Class I}}(n, k; M, B) + P_{\text{Class II}}(n, k; M, B) \\
 &= \frac{k(MB-k-2)!}{M(MB-1)!} ((M-k-1)n + 2M(B-1))(n-1) \prod_{j=MB-(k-2)}^{MB} (j-n),
 \end{aligned}$$

which is (1).

A similar argument gives (2). ■

Theorem 1 shows that for $M \geq 3$ and $k = 1, 2, \dots, M - 2$, $P(n, k; M, B)$ is a polynomial of degree $k + 1$ in n and the degrees for $P(n, M - 1; M, B)$ and $P(n, M; M, B)$ are both $M - 1$.

Example 1. For $M = 2$ items per band, e.g., the examples in Figures 1–3, (1) and (2) show that, for $n = 1, 2, \dots, 2B$,

$$P(n, 1; 2, B) = \frac{n-1}{2B-1} \text{ and } P(n, 2; 2, B) = \frac{2B-n}{2B-1},$$

respectively. These probabilities change linearly with n . In [4], these formulas are derived using different methods than appear here.

3. Concluding comments

This population and sampling procedure have other applications. Besides those in Figures 2 and 5, there are additional types of candy that are sold in serially connected bands. For example, there are snack-size milk-chocolate bars that are scored so that each of the four side-by-side pieces can be selected, and twisted licorice and other flavor sticks that are sold in various numbers of attached sticks. Sometimes, construction nails for a nail gun are serially connected in strips of paper. Some molecules may have that structure, as well. Generalizations of this sampling scheme are mentioned in [4]. More dynamic situations, such as the possibility of bands joining into larger bands, might be considered.

In the sampling procedure for the bands with ordered elements, all items are considered to be equivalent, and all remaining objects are equally likely to be chosen at each selection. A band

is likely to be broken into two bands at a selection. This is different from standard stratified sampling in which the original strata may be of unequal sizes, because the strata had been created according to some criterium, so that the membership of each stratum is homogeneous by that criterium. Also, each stratum is not ordered internally. In that scheme, a stratum is randomly selected first, independently of its size, then, an individual from the chosen stratum is randomly selected. This causes the phenomenon that individuals in a large stratum at a given time are less likely to be selected at that draw. The strata remain intact. That is called length-based sampling or size-based sampling [3, 5, 7].

References

- [1]. Devore, J. L. (2015). *Introduction to Probability and Statistics for Engineering and the Sciences* (9th ed.). Boston, MA, USA: Cengage Learning.
- [2]. Hogg, R. V., McKean, J. W., and Craig, A. T. (2005). *Introduction to Mathematical Statistics* (6th ed.). Upper Saddle River, NJ, USA: Pearson Prentice Hall.
- [3]. Marengo, J. E., and Farnsworth, D. L. (2025). The Useful and not so Paradoxical Inspection Paradox. *International Journal of Mathematical Education in Science and Technology*, 55 (6, June), 1196–1208, published online March 27, 2024.
- [4]. Marengo, J. E., Voelkel, J. G., and Farnsworth, D. L. (2025). The Bandages Problem. *Journal of Probability and Statistical Science*, 23(1, August), 77–81.
- [5]. Ross, S. M. (2003). The Inspection Paradox. *Probability in the Engineering and Informational Sciences*, 17(1), 47–51.
- [6]. Ross, S. M. (2014). *Introduction to Probability Models* (11th ed.). Cambridge, MA, USA: Academic Press.
- [7]. Wagner, C. H. (2009). Average Perceived Class Size and Average Perceived Population Density. *College Mathematics Journal*, 40(4, September), 284–292.