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A New Abbas Distribution and Its Applications

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ABSTRACT

In this paper, an attempt has been made to propose a new one parameter continuous probability distribution named "Abbas Distribution" for modeling the lifetime data in biomedical science and in the field of engineering. The distribution is derived from combining a logarithmic function with the hazard function of the exponential distribution. In addition, structural properties such as reliability components, moments, and ordered statistics of the distribution were derived, and the unknown parameter was estimated using maximum likelihood estimation. The utility and adaptability of the proposed distribution are demonstrated by real-world datasets. Also, the various diagnostic tools such as -2LogL, AIC, BIC and AICC show that the proposed distribution provides better fit than other distributions for the considered datasets.

Keywords: Lifetime distributions, Gamma distribution, Weibull distribution, Reliability components, Maximum Likelihood Estimation (MLE), Goodness of fit.

1. Introduction

Lifetime distributions are widely applied in fields such as reliability engineering, biomedical sciences, actuarial science, and survival analysis. Classical models such as exponential, Weibull, Gamma, and Gompertz distributions have been extensively used due to their mathematical simplicity and interpretability which were discussed by *Lawless (2003); Lai, Murthy and Xie (2006); Johnson Kotz and Balakrishnan (1995)*. However, these traditional distributions often fail to capture the full range of behaviors observed in practice, such as logarithmic hazard rates. This limitation has motivated the development of many new families of lifetime distributions in recent years.

In survival analysis, many probability distributions arise from specific hazard shapes. For example, the exponential distribution has a constant hazard rate studied by *Lawless (2003)*, the Weibull distribution has a polynomial hazard rate studied by *Lai, Murthy and Xie (2006)*, and the Gompertz distribution has an exponential form of hazard rate studied by *Johnson Kotz and Balakrishnan (1995)*. These distributions have been widely used to model lifetime data under different hazard rate behaviors.

A key drawback of the exponential distribution is its ever-constant hazard function across all observations, which may not reflect real-world situations where risk changes over time. To overcome this limitation, several generalized lifetime distributions have been proposed. Notable examples include the Weibull distribution by *Lai, Murthy and Xie (2006)*, which models faster

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hazard growth, and the Gompertz distribution by *Johnson Kotz and Balakrishnan (1995)*, which models slower hazard growth. Other models such as the gamma distribution by *Johnson Kotz and Balakrishnan (1995)* have also been developed to provide more flexibility in hazard behavior.

To address the limitations of these classical models and to model the situation where the data have a logarithmic growth, a new lifetime distribution, known as the Abbas distribution, is proposed. This model is defined through a logarithmic structure that provides greater flexibility to model complex lifetime data compared to many existing models.

1.1. Probability Density and Distribution Functions

Let X be a non-negative random variable. the probability density function (pdf) of the Abbas distribution is given by

$$f(x) = \begin{cases} \theta \ln(x+1) e^{(-\theta[(x+1)\ln(x+1)-x])} & x \geq 0, \theta > 0 \\ 0 & otherwise \end{cases} \quad (1.1)$$

The corresponding cumulative distribution function (cdf) is

$$F(x) = 1 - \exp(-\theta[(x+1)\ln(x+1) - x]), \quad x \geq 0 \quad (1.2)$$

This formulation introduces a new class of lifetime models with flexible structural properties. The subsequent sections examine its reliability characteristics, statistical properties, parameter estimation and applications to real life datasets.

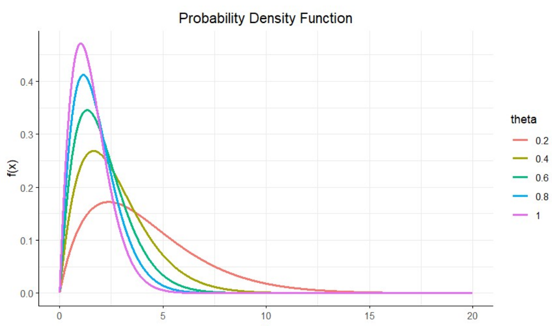

Figure 1: pdf plot of Abbas Distribution

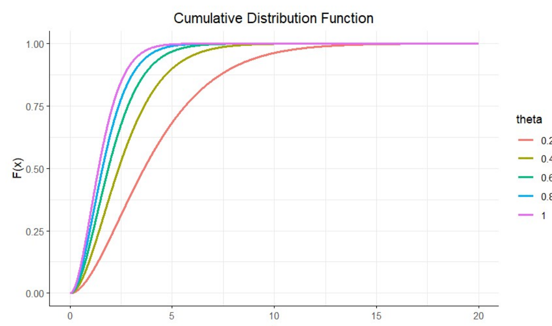

Figure 2: Cdf plot of Abbas Distribution

2. Reliability Measures

In this section, various reliability measures including, hazard rate, survival function (or reliability function), cumulative hazard function, reverse hazard rate, and Mill's ratio functions are all discussed for the newly proposed Abbas distribution.

2.1. Hazard function

The hazard function, also known as hazard rate or instantaneous failure rate of a random variable following Abbas distribution is given by

$$h(x) = \theta \ln(x+1), \quad x \geq 0, \ \theta > 0 \quad (2.1)$$

where, θ is the parameter.

The behavior or the shape of the hazard function of Abbas distribution is shown in Figure (3). It is obvious that the hazard function of Abbas distribution is monotonically increasing in nature.

Theorem 2.1. The hazard function of the Abbas distribution,

$$h(x) = \theta \ln(x+1), \quad x \geq 0,\ \theta > 0,$$

is strictly increasing (monotonic) function of x.

Proof:

Let X be a non-negative random variable follows Abbas distribution, then its hazard function is given by,

$$h(x) = \theta \ln(x+1)$$

Differentiating the h(x) with respect to x,

$$h'(x) = \frac{d[h(x)]}{dx} = \frac{\theta}{(x+1)} > 0 \quad \text{for all } x \geq 0$$

Therefore, by the criterion of the monotonicity studied by *Thomson, Brian. (1980)*, the Abbas distribution belongs to the Increasing Failure Rate (IFR) class due to its monotonic hazard function.

2.2. Survival function

The Survival function of the random variable following Abbas distribution is given by

$$S(x) = exp(-\theta[(x+1)\ln(x+1) - x]) \tag{2.2}$$

And the shape of the survival function is shown in the Figure (4).

2.3. Cumulative Hazard function

The Cumulative hazard function is,

$$H(x) = \theta[(x+1)\ln(x+1) - x] \tag{2.3}$$

2.4. Reverse Hazard function

The reverse hazard function of the newly proposed Abbas distribution is given by

$$h_r(x) = \frac{\theta \ln(x+1)}{\exp(\theta[(x+1)\ln(x+1)-x])-1} \tag{2.4}$$

2.5. Mills Ratio

The Mill's ratio is the reciprocal of the hazard function of a probability distribution explained by Fan (2012). The Mill's ratio of the Abbas distribution is given by

$$M.R. = \frac{1}{\theta \ln(x+1)} \tag{2.5}$$

which is the reciprocal of the hazard function h(x).

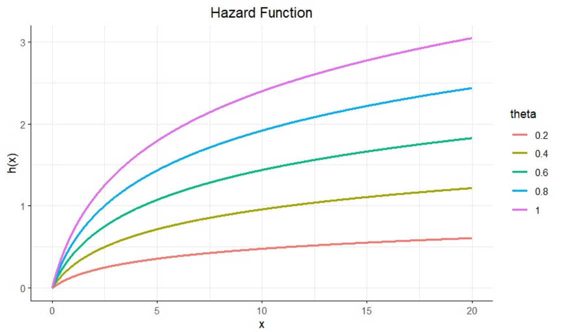

Figure 3: Hazard function of Abbas Distribution

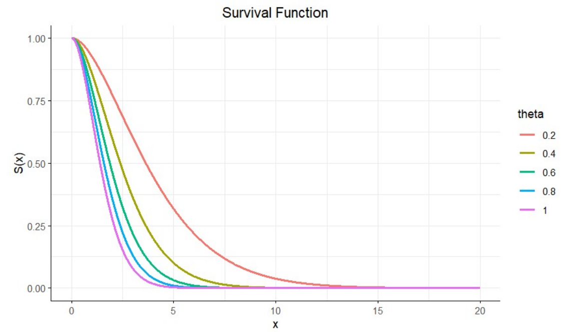

Figure 4: Survival function of Abbas Distribution

3. Statistical Properties

Let $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n from the newly proposed Abbas distribution. The statistical properties of the distribution are as follows.

3.1. Moments

The rth order moment of the Abbas distribution is given by,

$$E(x^r) = \int_0^\infty x^r\, \theta \ln(x+1) exp(-\theta[(x+1)\ln(x+1) - x])dx$$

Using the Maclaurin series expansion from Bender and Orszag (1999), the rth order moment of the Abbas distribution is given,

$$E(x^r) = \sum_{i=0}^{\infty} \sum_{k=0}^{i+r} \frac{\theta^{(i+1)}(-1)^k}{i!} \binom{i+r}{k} J_r(i,k) \tag{3.1}$$

where,

$$J_r(i,k) = \int_0^\infty (x+1)^{i+r-k-\theta(x+1)} \ln(x+1)dx$$

This integral function doesn't have a closed form due to its complex combination of logarithmic and exponential terms. Therefore, the numerical integration is used to obtain the rth-order moment of the distribution. Applying the r=1 in equation (3.1), the mean of the Abbas distribution can be obtained and is given by

$$E(x) = \sum_{i=0}^{\infty} \sum_{k=0}^{i+1} \frac{\theta^{(i+1)}(-1)^k}{i!} \binom{i+1}{k} J_1(i,k) \tag{3.2}$$

In a similar manner, the variance of the Abbas distribution is given by

$$V(x) = \sum_{i=0}^{\infty} \sum_{k=0}^{i+2} \frac{\theta^{(i+1)}(-1)^k}{i!} \binom{i+2}{k} J_2(i,k) - \left(\sum_{i=0}^{\infty} \sum_{k=0}^{i+1} \frac{\theta^{(i+1)}(-1)^k}{i!} \binom{i+1}{k} J_1(i,k) \right)^2 \tag{3.3}$$

The Abbas distribution requires the numerical integration method to compute value of both the mean (3.2) and variance (3.3). It is common for some of the well-known lifetime distributions including the Exponential-Logarithmic distribution by Adamis and Loukas (1998), Generalized Inverse Gaussian (GIG) distribution by Gupta and Viles (2011), and the Log-

Logistic distribution by Johnson, Kotz and Balakrishnan (1994) at some particular case of shape parameter to have its mean and variance in an unclosed elementary form. In such cases, the mean and variances can be computed through numerical integration method using any of the programming languages.

3.2. Moment generating function and Characteristic function

The moment generating function (MGF) of the Abbas distribution is given by,

$$M_X(t) = \int_0^{\infty} \theta \ln(x+1) exp(-\theta[(x+1)\ln(x+1) - x] + tx)dx$$

Again, using the Maclaurin series expansion for the exponential term from Bender and Orszag (1999) as done in equation (3.1)

$$M_X(t) = \theta \sum_{i=0}^{\infty} \sum_{k=0}^{i} (-1)^k \binom{i}{k} \frac{(\theta+1)^i}{i!} J_0(i,k) \tag{3.4}$$

The Characteristic function (CF) of the Abbas distribution is given by

$$\Phi_X(t) = \int_0^{\infty} \theta \ln(x+1) exp(-\theta[(x+1)\ln(x+1) - x] + itx)dx$$

As performed in MGF (3.4), the characteristic function can be written as,

$$\Phi_X(t) = \theta \sum_{j=0}^{\infty} \sum_{k=0}^{j} (-1)^k \binom{j}{k} \frac{(\theta+it)^j}{j!} J_0(j,k) \tag{3.5}$$

The MGF (3.4) and CF (3.5) of the Abbas distribution aren't possible to obtain in a closed form due to the complexity of the integral. Many standard and well known lifetime distribution such as Weibull distribution (*Lai, Murthy and Xie (2006)*), Log-Normal distribution (Tellambura and Senaratne (2010)) have this same problem for the MGF and CF.So, the numerical integration approach is suggested.

4. Ordered Statistics

4.1. r^{th} Order Statistics

Let $X_1, X_2, \dots, X_n$ be random variables drawn from a continuous population with the pdf $f(x)$ (1.1) and the cdf $F(x)$ (1.2). Let $X_{(1)}, X_{(2)}, \dots, X_{(r)}, \dots, X_{(n)}$ be the ordered statistics of the random sample, where $X_{(r)}$ represents the r^{th} order statistic. Then, the pdf of $X_{(r)}$ is:

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!\,(n-r)!}\left(F(x)^{(r-1)} f(x)[1-F(x)]^{(n-r)}\right)$$

Applying the pdf (1.1) and cdf (1.2) of the distribution, the r^{th} order statistic of the Abbas distribution is given by,

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!}\left(1 - e^{-\theta[(x+1)\ln(x+1)-x]}\right)^{r-1} \theta \ln(x+1) e^{-\theta[(x+1)\ln(x+1)-x]} \times \left(e^{-\theta[(x+1)\ln(x+1)-x]}\right)^{n-r} \tag{4.1}$$

4.2. Minimum and Maximum of the Order Statistic

Let $X_{(1)}$ and $X_{(n)}$ denote the minimum and maximum order statistics, respectively; their pdf is given by,

$$f_{X_{(1)}}(x) = n[1 - F(x)]^{(n-1)} f(x)$$

and

$$f_{X_{(n)}}(x) = n[F(x)]^{(n-1)} f(x)$$

Applying the probability density function (1.1) and the cumulative distribution function (1.2) of the Abbas Distribution to the above, the minimum and maximum order statistics of the Abbas distribution are given by,

$$f_{X_{(1)}}(x) = n\left(e^{-\theta[(x+1)\ln(x+1)-x]}\right)^{(n-1)} \theta\ln(x+1) e^{-\theta[(x+1)\ln(x+1)-x]} \quad (4.2)$$

and

$$f_{X_{(n)}}(x) = n\left(1 - e^{-\theta[(x+1)\ln(x+1)-x]}\right)^{(n-1)} \theta\ln(x+1) e^{-\theta[(x+1)\ln(x+1)-x]} \quad (4.3)$$

5. Estimation of Parameter by MLE

The method of maximum likelihood estimation is one of the best method for estimating the parameter value of the probability distributions. Let $X_1, X_2, X_3, \dots, X_n$ be the random sample of size n from the Abbas distribution.

The likelihood function of the Abbas distribution is given by

$$L(x;\theta) = L = \theta^n \prod_{i=1}^{n} \ln(x_i + 1) e^{-\theta \sum_{i=1}^{n}((x_i+1)\ln(x_i+1)-x_i)} \quad (5.1)$$

The Log-likelihood function of the Abbas distribution is given by

$$\log(L) = n\log(\theta) + \sum_{i=1}^{n} \ln(x_i + 1) - \theta \sum_{i=1}^{n}\left((x_i + 1)\ln(x_i + 1) - x_i\right) \quad (5.2)$$

By differentiating the Log(L) with respect to θ and equating the equation with zero, the MLE of the parameter θ can be obtained, and is given by,

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n}((x_i+1)\ln(x_i+1)-x_i)} \quad (5.3)$$

6. Data Analysis

The Abbas Distribution shows promise for a wide range of applications, both in current research and in future studies. In this section, the Abbas distribution is applied to two different datasets and compare its performance with other well-known lifetime distributions, including the Gamma, Weibull, Gompertz, and Exponential distributions.

Dataset 1 (Biomedical Data)

This dataset contains lifetime measurements, specifically relief times (in minutes) for 20 patients who received an analgesic. The data were originally reported by Gross and Clark (1975, p.105) and are shown in Table (1).

Table 1: Relief times (minutes) for 20 patients reported by Gross and Clark

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7	2.7
4.1	1.8	1.5	1.2	1.4	3	1.7	2.3	1.6	2

Dataset 2 (Engineering Data)

This dataset presents the strength measurements of aircraft window glass, as reported by Fuller et al. (1994). The recorded values are shown in Table (2).

Table 2: Data reported by Fuller et al. on strength measurements of window glass of aircraft

18.83	20.8	21.657	23.03	23.23	24.05
24.321	25.5	25.52	25.8	26.69	26.77
26.78	27.05	27.67	29.9	31.11	33.2
33.73	33.76	33.89	34.76	35.75	35.91
36.98	37.08	37.09	39.58	44.045	45.29
45.381	-	-	-	-	-

In order to compare the goodness of fit of Abbas distribution with the Gamma, Weibull, Gompertz and Exponential distribution, the measures -2lnL, AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion) and AICC (Akaike Information Criterion Corrected) for the real lifetime datasets have been calculated and presented in the Table (3).

Table 3: Comparison of MLEs, Information Criteria among the Survival Distributions

Dataset	Distribution	MLE of Parameter(s)	−2lnL	AIC	BIC	AICC
1	**Abbas**	θ = 0.7930	7.6295	9.6295	10.6253	9.8518
	Gamma	k = 9.6712, θ = 5.0902	35.6372	39.6372	41.6287	40.3431
	Weibull	k = 2.7868, β = 2.1299	41.1728	45.1728	47.1643	45.8787
	Gompertz	k = 0.8944, λ = 0.1453	49.1802	53.1802	55.1717	53.8861
	Exponential	θ = 0.5263	65.6742	67.6742	68.6699	67.8964
2	**Abbas**	θ = 0.0125	120.7553	122.7553	124.1893	122.8993
	Gamma	k = 18.9252, θ = 0.6142	208.2312	212.2312	215.0992	212.6598
	Weibull	k = 4.6350, β = 33.6733	210.9778	214.9778	217.8458	215.4064
	Gompertz	k = 0.1336, λ = 0.0014	245.9238	219.9238	222.7917	220.3523
	Exponential	θ = 0.0324	274.5289	276.5289	277.9629	276.6668

where, k is the shape parameter of the respective distributions. From the Table (3), In both of the datasets, the Abbas distribution gives a better fitting than the Gamma, Weibull, Gompertz and Exponential distributions.

7. Conclusion

A new continuous one parameter lifetime distribution named, "Abbas distribution" has been suggested for modeling lifetime datasets from biomedical science and from the field of engineering, its important mathematical and statistical properties, including the shape, moments, reliability components, ordered statistics are all have been discussed. Since the moments and characteristic function were not having a closed form, the numerical integration have been suggested. In addition, the maximum likelihood estimation was derived for the estimation of the parameter θ. Finally, two real-life datasets have been used to fit and compare the Abbas distribution with the other lifetime distributions including Gamma, Weibull, Gompertz and the Exponential distributions, and the comparsion results have been presented.

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