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Power Quasi Rama Distribution with Properties and Applications in Reliability Engineering

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ABSTRACT

Three-parameter power quasi Rama distribution for modelling over-dispersed and under-dispersed engineering data having increasing or decreasing hazard rates has been introduced and it includes quasi Rama distribution, power Rama distribution, Rama distribution, generalized gamma distribution and one parameter gamma distribution as particular cases. Its statistical properties including behaviour of its probability density function, cumulative density function for varying values of parameters, moments, survival function, hazard function, mean residual life function and stochastic ordering has been discussed. The estimation of the parameters of the distribution has been discussed using maximum likelihood estimation and maximum product spacing estimation. A simulation study has been carried out to know the consistency of the estimators. As the proposed distribution is suitable for over-dispersed and under-dispersed engineering data having increasing and deceasing hazard rates, the applications of the proposed distribution has been tested with three dataset from engineering of various natures and the goodness of fit shows that power quasi Rama distribution provides better fit as compared to the generalized power Sujatha distribution, three-parameter power Sujatha distribution, power quasi Lindley distribution, generalized gamma distribution, three-parameter Sujatha distribution and three-parameter generalized Lindley distribution.

Keywords: Quasi Rama distribution, Descriptive measures, Reliability properties, Estimation of parameters, Applications

1. Introduction

The Rama distribution (RD) has been proposed by Shanker (2017) with probability density function (pdf) and cumulative density function (cdf) as

$$f(x,\theta)=\frac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x};x>0,\theta>0 \tag{1}$$

$$F(x,\theta)=1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\theta^3+6}\right]e^{-\theta x};x>0,\theta>0 \tag{2}$$

This distribution is a convex combination of exponential (θ) and gamma $(4,\theta)$ distributions with proportion value $p=\frac{\theta^3}{\theta^3+6}$. Shanker (2017) has studied its statistical properties and application in engineering field and observed that RD is unimodal, decreasing and positively skewed distribution. One of the striking features of RD is that it is useful for modeling lifetime data under varying hazard functions and this makes it an important distribution in medical science, engineering and industry. However, there are some situations in which the RD may not be suitable

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from applied and theoretical point of views. Recently, RD has been extended by incorporating additional parameter by different researchers. Tesfalem and Shanker (2019) proposed two-parameter weighted Rama distribution (WRD) and discussed its characterization and applications. Subramanian and Shenbagaraja (2019) proposed new version of RD and studied some of its statistical properties and applications. Edith et al. (2019) introduced the two-parameter RD and studied some of its statistical properties and applications. An inverted power Rama distribution with applications to lifetime data has been proposed by Chrisogonus et al (2020). Generalized weighted Rama distribution has been suggested by Samuel et al (2020). The exponentiated Rama distribution has been suggested by Chrisogonus et al (2021). Mohiuddin and Kannan (2021) proposed new generalization of the RD with application to model machinery data. Khaldoon et al (2021) extended the RD using a suitable transformation and discussed its properties and applications. Singh et al (2025) proposed neutrosophic Rama distribution (NRD) and discussed its properties and application.

Although several extensions of RD have been proposed using different methods by introducing an additional parameter in RD, but there are several datasets where these generalizations of RD failed to provide better fit either because of the theoretical nature of distributions or stochastic nature of data. Even the well-known two- parameter distributions including gamma and Weibull distribution did not provide satisfactory fit over data having bi-modal and tri-modal behaviours. Keeping these points of the distributions and the nature of data, Shanker et al (2023) introduced quasi Rama distribution (QRD) having pdf and cdf as

$$f(x;\theta,\alpha)=\frac{\theta^3}{\alpha\theta^2+6}\left(\alpha+\theta x^3\right)e^{-\theta x};\ x>0,\theta,\alpha>0 \tag{3}$$

$$F(x,\theta,\alpha)=1-\left[1+\frac{\theta^3x^3+3\theta^2x^2+6\theta x}{\alpha\theta^2+6}\right]e^{-\theta x};\ x>0,\theta>0,\alpha>0 \tag{4}$$

QRD is a convex combination of exponential (θ) and gamma $(4,\theta)$ distributions with proportion value $p=\frac{\alpha\theta^2}{\alpha\theta^2+6}$ and it reduces to RD at $\alpha=\theta$. Various statistical properties, estimation of parameters and applications of QRD are available in Shanker et al (2023). However, the additional shape parameter α increases the flexibility and tractability of the distribution but unable to deal with the situations of bimodal natures, V and J shape natures of hazard function.

Power versions of lifetime distributions have been derived by several researchers using the power function $X=Y^{1/\beta}$. For examples, Weibull (1951) introduced Weibull distribution from exponential distribution, Ghitany et al (2013) introduced power Lindley distribution (PLD) from Lindley distribution of Lindley (1958), Abebe et al (2019) introduced power Rama distribution (PRD) from RD of Shanker (2017), Shanker et al (2023) introduced power Komal distribution from Komal distribution of Shanker (2023a), Prodhani and Shanker (2024a) introduced power Pratibha distribution from Pratibha distribution of Shanker (2023b), Stacy (1962) introduced generalized gamma distribution (GGD) from gamma distribution (GD), Alkarni (2015) introduced power quasi Lindley distribution from quasi Lindley distribution of Shanker and Mishra (2013), Prodhani and Shanker (2024b) proposed three-parameter power Sujatha distribution (TPPSD) from two-parameter Sujatha distribution of Mussie and Shanker (2018), Prodhani and Shanker (2025) suggested generalized power Sujatha distribution (GPSD) introduced from generalization of Sujatha distribution of Shanker et al (2017), are some among others. A three-parameter Sujatha distribution (ATPSD) proposed by Nwikpe and Iwok (2020) and later on its statistical properties and applications have been thoroughly studied by Prodhani and Shanker (2023). Nosakhare and Festus (2018) proposed three-parameter generalized Lindley distribution (TPGLD) and studied its properties and applications. It has been noted that, depending on a conceptual or applied angle, these power version and non-power version distributions did not provide suitable fit in certain datasets. This necessitates the search for another power version of the lifetime distribution and we are interested in the power version of QRD because QRD has several interesting and useful applications where Weibull and gamma distributions failed.

The main motivations for proposing the power quasi Rama distribution are due to the fact that

i) It demonstrates enhanced flexibility as its pdf can exhibit various shapes, such as being unimodal, bimodal and positively skewed. Its hazard function has increasing, unimodal, J shape and V shape natures.
ii) This distribution provides consistently better fit than other considered models under the same underlying method.
iii) The proposed model retains mathematical tractability and includes PRD, QRD, GGD, GD and RD as special cases.
iv) The proposed distribution can be considered as an important lifetime distribution for modelling lifetime data from engineering of over –dispersed and under- dispersed nature having increasing or decreasing hazard rates.

2. Power Quasi Rama Distribution

Assuming random variable Y to be QRD and using the power transformation $X = Y^{1/\beta}$ in (3), the pdf of the random variable X can be obtained as

$$f\left(x;\theta,\alpha,\beta\right) = \frac{\beta\theta^3}{\alpha\theta^2+6} x^{\beta-1}\left(\alpha+\theta x^{3\beta}\right)e^{-\theta x^{\beta}}, \quad x>0,\theta,\alpha,\beta>0$$

We would call this as power quasi Rama distribution (PQRD) with parameters θ,α and β because it reduce to QRD at $\beta=1$. PQRD is also a convex combination of Weibull (θ,β) distribution and generalized gamma $(\theta,\beta,4)$ distribution. For, we have

$$f\left(x;\theta,\alpha,\beta\right) = p\, g_1\left(x;\theta,\beta\right) + \left(1-p\right) g_2\left(x;\theta,\beta,4\right),$$

where $p_1 = \dfrac{\alpha\theta^2}{\alpha\theta^2+6}$, $g_1\left(x;\theta,\beta\right) = \beta\theta x^{\beta-1}e^{-\theta x^{\beta}}$ and $g_2\left(x;\theta,\beta,4\right) = \dfrac{\beta\theta^4}{\Gamma\left(4\right)} x^{4\beta-1}e^{-\theta x^{\beta}}$ for $x>0,\theta>0,\alpha>0,\beta>0$. The corresponding cdf of PQRD can be obtained as

$$F\left(x;\theta,\alpha,\beta\right) = 1-\left[1+\frac{\theta^3x^{3\beta}+3\theta^2x^{2\beta}+6\theta x^{\beta}}{\alpha\theta^2+6}\right]e^{-\theta x^{\beta}}; \quad x>0,\theta>0,\alpha>0,\beta>0.$$

The pdf and cdf of PQRD for various values of its parameters are represented in figure 1 and figure 2, respectively. From the figure 1, it is observed that PQRD has positively skewed unimodal, bimodal and leptokurtic natures for different values of the parameters. The particular distributions of PQRD are presented in the table 1.

Table 1: Particular distributions of PQRD

Values of Parameters	Distributions	Introducer (year)
$\alpha=\theta$	$f(x;\theta,\beta)=\dfrac{\beta\theta^4}{\theta^3+6}x^{\beta-1}\left(1+x^{3\beta}\right)e^{-\theta x^{\beta}}$	PRD [Abebe et al (2019)]
$\alpha=0$	$f(x;\theta)=\dfrac{\beta\theta^4}{\Gamma(4)}x^{4\beta-1}e^{-\theta x^{\beta}}$	GGD [Stacy (1962)]
$\alpha=0\,,\beta=1$	$f(x;\theta)=\dfrac{\theta^4}{\Gamma(4)}x^{4-1}e^{-\theta x}$	Gamma distribution
$\beta=1$	$f(x;\theta,\alpha)=\dfrac{\theta^3}{\alpha\theta^2+6}\left(\alpha+\theta x^3\right)e^{-\theta x}$	QRD [Shanker et al (2023)]
$\alpha=\theta,\beta=1$	$f(x;\theta)=\dfrac{\theta^4}{\theta^3+6}\left(1+x^3\right)e^{-\theta x}$	RD [Shanker (2017)]

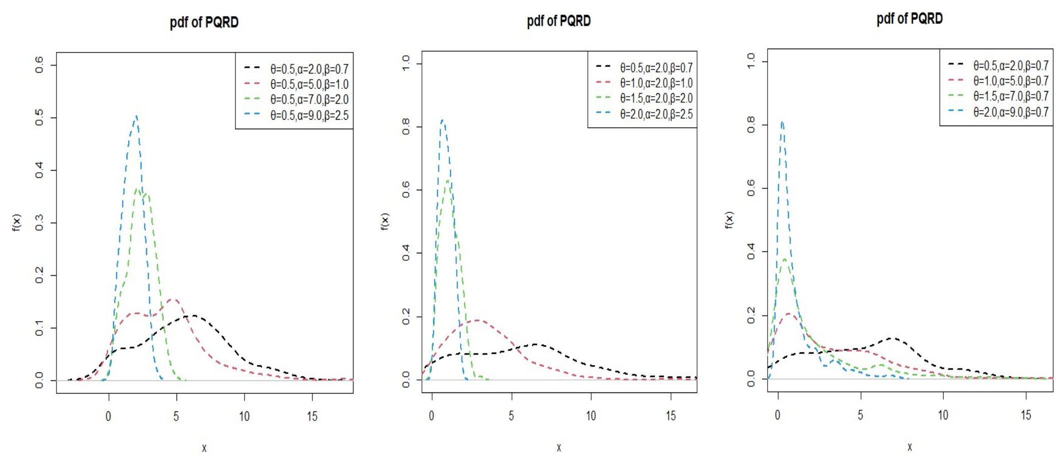

Figure 1: pdf of PQRD for varying values of parameters

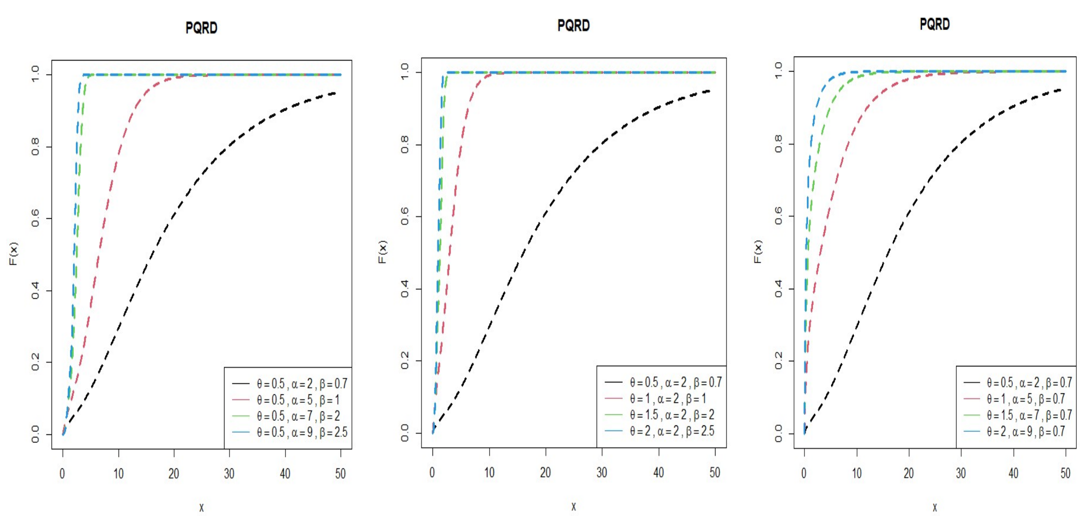

Figure 2: cdf of PQRD for varying values of parameters

3. Reliability Properties of Power Quasi Rama Distribution

3.1. Survival Function

The survival function of PQRD can be obtained as

$$S(x;\theta,\alpha,\beta)=1-F(x;\theta,\alpha,\beta)=\left[\frac{(\alpha\theta^2+6)+\theta^3x^{3\beta}+3\theta^2x^{2\beta}+6\theta x^{\beta}}{\alpha\theta^2+6}\right]e^{-\theta x^{\beta}}$$

3.2. Hazard Function

The hazard function $h(x;\theta,\alpha,\beta)$ of PQRD are given as

$$h(x;\theta,\alpha,\beta)=\frac{\beta\theta^3x^{\beta-1}(\alpha+\theta x^{3\beta})}{(\alpha\theta^2+6)+\theta^3x^{3\beta}+3\theta^2x^{2\beta}+6\theta x^{\beta}};\ x>0,\theta,\alpha,\beta>0\,.$$

3.3. Mean Residual Life Function

The mean residual life function $m(x;\theta,\alpha,\beta)$ of PQRD are obtained as

$$m(x;\theta,\alpha,\beta)=E(X-x\mid X\geq x)=\frac{1}{S(x;\theta,\alpha,\beta)}\int_x^{\infty}t\,f(t;\theta,\alpha,\beta)\,dt-x$$

$$=\frac{\alpha\theta^3\Gamma\left(\frac{1}{\beta}+1,\theta x^{\beta}\right)+\beta\Gamma\left(\frac{1}{\beta}+4,\theta x^{\beta}\right)}{\theta^{\frac{1}{\beta}}\left[\alpha\theta^2\Gamma(1,\theta x^{\beta})+\Gamma(4,\theta x^{\beta})\right]}-x$$

The survival, hazard and mean residual life functions of PQRD for various values of its parameters are presented in figure 3, 4 and 5 respectively.

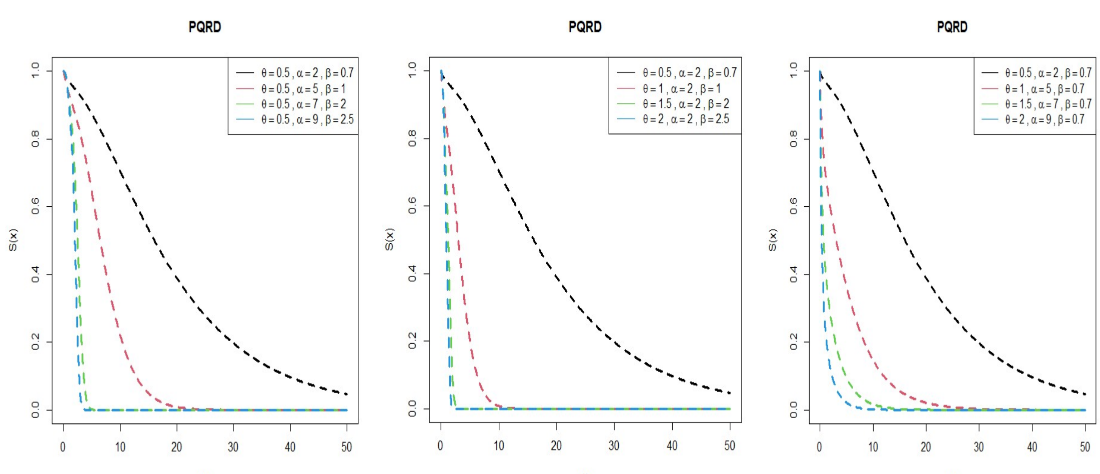

Figure 3: Survival function of PQRD for varying values of parameters

From the figure 4, it observed that when $\beta \le 1$, lower values of (θ,α), PQRD has J shape hazard nature and for higher values of (θ,α), it has V Shape hazard nature. On the other hand, for $\beta > 1$, it exhibits unimodal hazard nature. Similarly, the shapes of mean residual life function of PQRD are both increasing and decreasing.

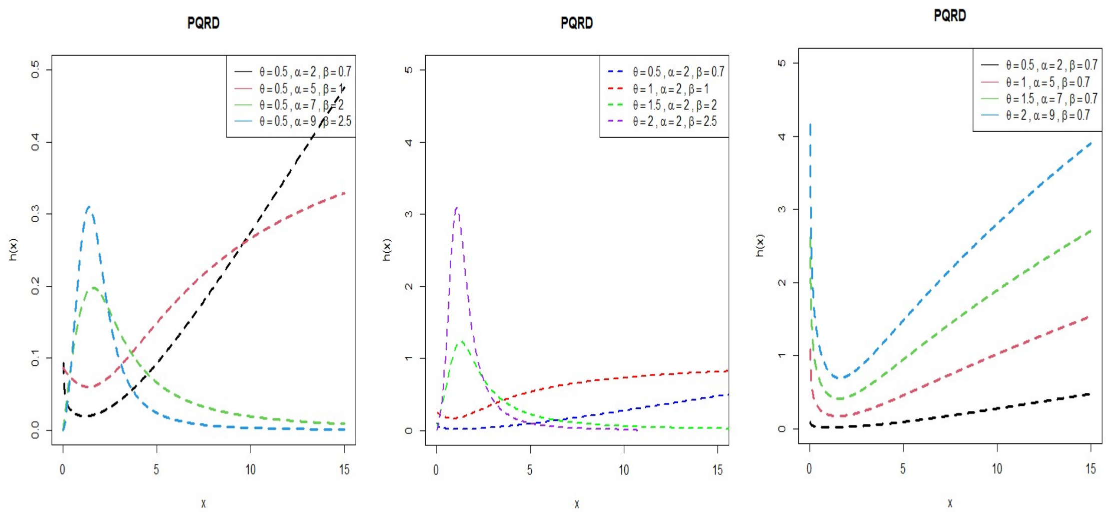

Figure 4: Hazard function of PQRD for varying values of parameters

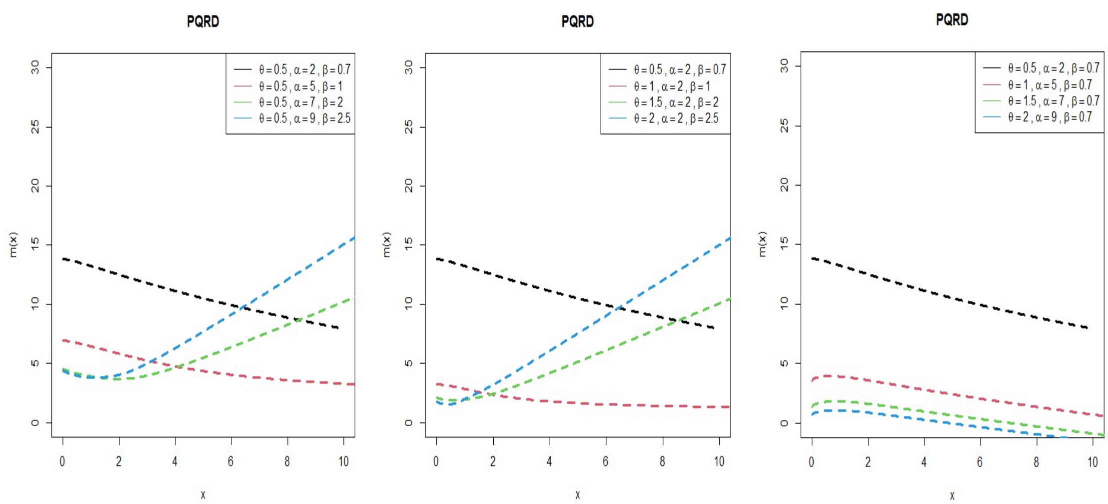

Figure 5: Mean residual life function of PQRD for varying values of parameters

4. Moments of Power Quasi Rama Distribution

The rth moment about origin denoted by μ_r' of PQRD can be obtained as

$$\mu_r' = E\left(X^r\right) = \frac{\alpha\theta^2\Gamma\left(\frac{r}{\beta}+1\right)+\Gamma\left(\frac{r}{\beta}+4\right)}{\theta^{\frac{r}{\beta}}\left(\alpha\theta^2+6\right)};r=1,2,3,...$$

Thus, the first four moments about origin of the PQRD are obtained as

$$\mu_1' = \frac{\alpha\theta^2\Gamma\left(\frac{1}{\beta}+1\right)+\Gamma\left(\frac{1}{\beta}+4\right)}{\theta^{\frac{1}{\beta}}\left(\alpha\theta^2+6\right)}, \qquad \mu_2' = \frac{\alpha\theta^2\Gamma\left(\frac{2}{\beta}+1\right)+\Gamma\left(\frac{2}{\beta}+4\right)}{\theta^{\frac{2}{\beta}}\left(\alpha\theta^2+6\right)}$$

$$\mu_3' = \frac{\alpha\theta^2\Gamma\left(\frac{3}{\beta}+1\right)+\Gamma\left(\frac{3}{\beta}+4\right)}{\theta^{\frac{3}{\beta}}\left(\alpha\theta^2+6\right)} \qquad \mu_4' = \frac{\alpha\theta^2\Gamma\left(\frac{4}{\beta}+1\right)+\Gamma\left(\frac{4}{\beta}+4\right)}{\theta^{\frac{4}{\beta}}\left(\alpha\theta^2+6\right)}$$

Therefore, the variance of PQRD can be obtained as

$$\mu_2 = \mu_2' - \left(\mu_1'\right)^2 = \frac{\left[\alpha\theta^2\Gamma\left(\frac{2}{\beta}+1\right)+\Gamma\left(\frac{2}{\beta}+4\right)\right]\left(\alpha\theta^2+6\right)-\left[\alpha\theta^2\Gamma\left(\frac{1}{\beta}+1\right)+\Gamma\left(\frac{1}{\beta}+4\right)\right]^2}{\theta^{\frac{2}{\beta}}\left(\alpha\theta^2+6\right)^2}$$

The third and fourth central moments are not being given here due to its complicated forms. However, if required, central moments μ_3 and μ_4 can be obtained using the formula

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3 \text{ and } \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4$$

The moments based descriptive measures such as coefficient of variation (CV), skewness $\left(\sqrt{\beta_1}\right)$, kurtosis (β_2) and index of dispersion (γ) of PQRD can be obtained using formula

$$CV = \frac{\sigma}{\mu_1'} \qquad \sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} \qquad \gamma = \frac{\mu_2}{\mu_1'}$$

Table 2: Moments based measures of PQRD for various values of α and $\theta = 0.5, \beta = 0.7$

α	μ_1'	μ_2	μ_3	μ_4	CV	CS	CK	ID
2.0	19.5920	234.1304	5819.134	406142.2	0.7809	1.6243	7.4090	11.9502
2.3	19.4074	234.6969	5825.575	406837.1	0.7894	1.6202	7.3859	12.0931
2.6	19.2269	235.1846	5832.441	407433.2	0.7976	1.6171	7.3661	12.2320
2.9	19.0505	235.5985	5839.631	407937.1	0.8057	1.6148	7.3493	12.3670
3.2	18.8780	235.9431	5847.055	408354.9	0.8136	1.6133	7.3353	12.4983
3.5	18.7092	236.2226	5854.632	408692.2	0.8215	1.6125	7.3240	12.6259
3.8	18.5441	236.4409	5862.293	408954.1	0.8292	1.6124	7.3152	12.7501
4.1	18.3825	236.6017	5869.973	409145.4	0.8367	1.6129	7.3087	12.8710
4.4	18.2243	236.7086	5877.618	409270.6	0.8442	1.6139	7.3043	12.9886
4.7	18.0694	236.7647	5885.179	409333.7	0.8515	1.6154	7.3020	13.1030
5.0	17.9177	236.7732	5892.612	409338.6	0.8587	1.6173	7.3015	13.2144
5.3	17.7691	236.7368	5899.881	409288.7	0.8658	1.6197	7.3029	13.3228
5.6	17.6236	236.6585	5906.952	409187.4	0.8729	1.6224	7.3059	13.4284
5.9	17.4809	236.5405	5913.798	409037.7	0.8798	1.6255	7.3105	13.531
6.2	17.3411	236.3855	5920.392	408842.4	0.8866	1.6289	7.3166	13.6314
6.5	17.2041	236.1955	5926.715	408604.3	0.8933	1.6327	7.3241	13.7290
6.8	17.0697	235.9728	5932.748	408325.7	0.8999	1.6366	7.3330	13.8240
7.1	16.9379	235.7192	5938.477	408009.1	0.9064	1.6409	7.3431	13.9166
7.4	16.8086	235.4368	5943.888	407656.5	0.9128	1.6453	7.3543	14.0068
7.7	16.6818	235.1272	5948.971	407270.1	0.9191	1.6500	7.3667	14.0947
8.0	16.55741	234.7922	5953.718	406851.6	0.9254	1.6548	7.3802	14.1804

Table 3: Moments based measures of PQRD for various values of θ and $\alpha = 2, \beta = 0.7$

θ	μ_1'	μ_2	μ_3	μ_4	CV	CS	CK	ID
0.2	76.6773	3142.4560	294215.0	75172136	0.7310	1.6701	7.6123	40.9828
0.3	42.3835	992.8996	51779.97	7444043	0.7434	1.6550	7.5508	23.4265
0.4	27.5817	439.7002	15108.27	1446044	0.7602	1.6386	7.4794	15.9417
0.5	19.5920	234.1304	5819.134	406142.2	0.7809	1.6243	7.4090	11.9502
0.6	14.6911	139.9245	2673.012	143911.7	0.8051	1.6149	7.3503	9.5243
0.7	11.4268	90.43663	1386.858	59806.33	0.8322	1.6125	7.3123	7.9144
0.8	9.12498	61.81807	786.5454	27904.12	0.8616	1.6182	7.3019	6.7745
0.9	7.4333	44.05009	477.2641	14210.89	0.8928	1.6324	7.3236	5.9259
1.0	6.1510	32.40395	305.2529	7749.339	0.9254	1.6548	7.3802	5.2680
1.1	5.1553	24.4399	203.5776	4463.504	0.9589	1.6849	7.4726	4.7406
1.2	4.3674	18.80703	140.4319	2688.506	0.9929	1.7218	7.6009	4.3061
1.3	3.7343	14.71301	99.58708	1680.736	1.0271	1.7646	7.7641	3.9399
1.4	3.2189	11.67036	72.25886	1084.242	1.0612	1.8124	7.9608	3.6255
1.5	2.7948	9.366843	53.44872	718.4902	1.0950	1.8644	8.1890	3.3514
1.6	2.4426	7.59554	40.18793	487.3226	1.1282	1.9198	8.4469	3.1095
1.7	2.1477	6.215255	30.64674	337.3263	1.1607	1.9778	8.7323	2.8938
1.8	1.8989	5.127222	23.66051	237.7338	1.1924	2.0379	9.0432	2.6999
1.9	1.6877	4.260863	18.46681	170.2508	1.2230	2.0996	9.3776	2.5245
2.0	1.5073	3.5648	14.55406	123.6907	1.2526	2.1623	9.7334	2.3650

Table 4: Moments based measures of PQRD for various values of β and $\theta = 0.5, \alpha = 2$

β	μ_1'	μ_2	μ_3	μ_4	CV	CS	CK	ID
0.7	19.5920	234.1304	5819.1340	406142.2	0.7809	1.6243	7.4090	11.9502
0.8	13.0915	80.5993	937.2826	37454.53	0.6857	1.2953	5.7655	6.1565
0.9	9.6163	34.8306	214.9678	5822.113	0.6137	1.0457	4.7990	3.6220
1.0	7.5384	17.6331	62.6672	1304.188	0.5570	0.8463	4.1945	2.3390
1.1	6.1913	10.0129	21.5744	381.3224	0.5110	0.6809	3.8034	1.6172
1.2	5.2633	6.1969	8.3282	136.2368	0.4729	0.5398	3.5476	1.1773
1.3	4.5932	4.0983	3.4599	56.8113	0.4407	0.4170	3.3823	0.8922
1.4	4.0909	2.8558	1.4879	26.7544	0.4130	0.3082	3.2803	0.6980
1.5	3.7029	2.0753	0.6303	13.8866	0.3890	0.2108	3.2240	0.5604
1.6	3.3956	1.5608	0.2389	7.7992	0.3679	0.1225	3.2013	0.4596
1.7	3.1471	1.2076	0.0555	4.6733	0.3491	0.0418	3.2041	0.3837
1.8	2.9426	0.9569	-0.0302	2.9548	0.3324	-0.0322	3.2264	0.3252
1.9	2.7717	0.7737	-0.0686	1.9542	0.3173	-0.1009	3.2639	0.2791
2.0	2.6271	0.6366	-0.0836	1.3426	0.3037	-0.1647	3.3131	0.2423
2.1	2.5032	0.5316	-0.0869	0.9528	0.2912	-0.2243	3.3716	0.2123
2.2	2.3961	0.4497	-0.0845	0.6953	0.2798	-0.2801	3.4374	0.1877
2.3	2.3026	0.3849	-0.0794	0.5198	0.2694	-0.3326	3.5089	0.1671
2.4	2.2205	0.3327	-0.0733	0.3968	0.2597	-0.3821	3.5850	0.1498
2.5	2.1476	0.2902	-0.0670	0.3086	0.2508	-0.4289	3.6645	0.1351
2.6	2.0828	0.2551	-0.0610	0.2439	0.2425	-0.4732	3.7468	0.1225
2.7	2.0246	0.2259	-0.0553	0.1956	0.2347	-0.5153	3.8311	0.1116
2.8	1.9722	0.2014	-0.0501	0.1588	0.2275	-0.5553	3.9168	0.1021
2.9	1.9247	0.1805	-0.0455	0.1305	0.2207	-0.5934	4.0036	0.0938
3.0	1.8815	0.1627	-0.0413	0.1083	0.2144	-0.6298	4.0911	0.0865

5. Stochastic Ordering

The stochastic ordering of positive continuous random variables is an important tool for judging their comparative behaviours. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X \le_{st} Y\right)$ if $F_X(x) \ge F_Y(x)$ for all x

(ii) hazard rate order $\left(X \le_{hr} Y\right)$ if $h_X(x) \ge h_Y(x)$ for all x

(iii) mean residual life order $\left(X \le_{mrl} Y\right)$ if $m_X(x) \le m_Y(x)$ for all x

(iv) likelihood ratio order $\left(X \le_{lr} Y\right)$ if $\dfrac{f_X(x)}{f_Y(x)}$ decreases in x.

The following interrelationship due to Shaked and Shanthikumar (1994) are well- known for establishing stochastic ordering of distributions

$$\begin{array}{c} X \le_{lr} Y \Rightarrow X \le_{hr} Y \Rightarrow X \le_{mrl} Y \\ \Downarrow \\ X \le_{st} Y \end{array}$$

It can be easily shown that PQRD is ordered with respect to the strongest 'likelihood ratio' ordering. The stochastic ordering of PQRD has been explained in the following theorem:

***Theore*m**: Suppose $X \sim$ PQRD$\left(\theta_1,\alpha_1,\beta_1\right)$ and $Y \sim$ PQRD$\left(\theta_2,\alpha_2,\beta_2\right)$. If $\alpha_1=\alpha_2, \beta_1=\beta_2$ and $\theta_1>\theta_2$ (or $\alpha_1<\alpha_2, \beta_1=\beta_2$ and $\theta_1=\theta_2$ or $\alpha_1=\alpha_2, \beta_1<\beta_2$ and $\theta_1=\theta_2$), then $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

Proof: We have

$$\frac{f_X\left(x;\theta_1,\alpha_1,\beta_1\right)}{f_Y\left(x;\theta_2,\alpha_2,\beta_1\right)}=\left[\frac{\beta_1\theta_1^3\left(\alpha_2\theta_2^2+6\right)}{\alpha_2\theta_2^4\left(\alpha_1\theta_1^2+6\right)}\right]\left(\frac{\alpha_1+\theta_1x^{3\beta_1}}{\alpha_2+\theta_2x^{3\beta_2}}\right)x^{\beta_1-\beta_2}\; e^{-\left(\theta_1x^{x^{\beta_1}}-\theta_2x^{\beta_2}\right)};x>0$$

Now, taking logarithm both sides, we get

$$\log\frac{f_X\left(x;\theta_1,\alpha_1,\beta_1\right)}{f_Y\left(x;\theta_2,\alpha_2,\beta_1\right)}=\log\left[\frac{\beta_1\theta_1^3\left(\alpha_2\theta_2^2+6\right)}{\alpha_2\theta_2^4\left(\alpha_1\theta_1^2+6\right)}\right]+\log\left(\frac{\alpha_1+\theta_1x^{3\beta_1}}{\alpha_2+\theta_2x^{3\beta_2}}\right)+\left(\beta_1-\beta_2\right)\log x-\left(\theta_1x^{x^{\beta_1}}-\theta_2x^{\beta_2}\right)$$

Now, differentiating both side w.r.t x, we get

$$\frac{d}{dx}\left[\log\frac{f_X\left(x;\theta_1,\alpha_1,\beta_1\right)}{f_Y\left(x;\theta_2,\alpha_2,\beta_1\right)}\right]=\frac{3\theta_1\beta_1x^{3\beta_1-1}}{\alpha_1+\theta_1x^{3\beta_1}}-\frac{3\theta_2\beta_2x^{3\beta_2-1}}{\alpha_2+\theta_2x^{3\beta_2}}+\frac{\beta_1-\beta_2}{x}-\left(\theta_1\beta_1x^{\beta_{11}-1}-\theta_2\beta_2x^{\beta_2-1}\right).$$

Thus, for $\alpha_1=\alpha_2, \beta_1=\beta_2$ and $\theta_1>\theta_2$ (or $\alpha_1<\alpha_2, \beta_1=\beta_2$ and $\theta_1=\theta_2$ or $\alpha_1=\alpha_2, \beta_1<\beta_2$ and $\theta_1=\theta_2$), $\dfrac{d}{dx}\left[\log\dfrac{f_X\left(x;\theta_1,\alpha_1,\beta_1\right)}{f_Y\left(x;\theta_2,\alpha_2,\beta_1\right)}\right]<0$. This means that $X \le_{lr} Y$ and hence $X \le_{hr} Y$, $X \le_{mrl} Y$ and $X \le_{st} Y$.

6. Estimation of Parameters of PQRD

6.1.Maximum Likelihood Estimation (MLE)

Suppose $(x_1, x_2, ..., x_n)$ be a random sample of size n from PQRD (θ, α, β). The log-likelihood function of PQRD can be expressed as

$$\log L = \sum_{i=1}^{n} \log f(x_i; \theta, \alpha, \beta)$$

$$= n\log\beta + 3n\log\theta - n\log(\alpha\theta^2 + 6) + \sum_{i=1}^{n}\log(\alpha + \theta x_i^{3\beta}) + (\beta - 1)\sum_{i=1}^{n}\log x_i - \theta\sum_{i=1}^{n} x_i^{\beta}$$

The maximum likelihood estimates $(\hat{\theta}, \hat{\alpha}, \hat{\beta})$ of parameters (θ, α, β) of PQRD are the solution of the following log-likelihood equations

$$\frac{\partial \log L}{\partial \theta} = \frac{3n}{\theta} - \frac{2n\alpha\theta}{\alpha\theta^2 + 6} + \sum_{i=1}^{n}\frac{x_i^{3\beta}}{\alpha + \theta x_i^{3\beta}} - \sum_{i=1}^{n} x_i^{\beta} = 0$$

$$\frac{\partial \log L}{\partial \alpha} = -\frac{n\theta^2}{\alpha\theta^2 + 6} + \sum_{i=1}^{n}\frac{1}{\alpha + \theta x_i^{3\beta}} = 0$$

$$\frac{\partial \log L}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\frac{3\theta x_i^{3\beta}\log x_i}{\alpha + \theta x_i^{3\beta}} + \sum_{i=1}^{n}\log(x_i) - \theta\sum_{i=1}^{n} x_i^{\beta}\log(x_i) = 0$$

These three natural log-likelihood equations do not seem to be solved directly, because they cannot be expressed in closed forms. The maximum likelihood estimates $(\hat{\theta}, \hat{\alpha}, \hat{\beta})$ of (θ, α, β) can be computed directly by solving these natural log-likelihood equations using Newton-Raphson iteration available in R-software till sufficiently close values $(\hat{\theta}, \hat{\alpha}, \hat{\beta})$ are obtained.

For Fisher's Information matrix, we have

$$\frac{\partial^2 \log L}{\partial \theta^2} = -\frac{3n}{\theta^2} - \frac{12n\alpha - 2n\alpha^2\theta^2}{(\alpha\theta^2 + 6)^2} - \sum_{i=1}^{n}\frac{x_i^{6\beta}}{(\alpha + \theta x_i^{3\beta})^2}$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{n\theta^4}{(\alpha\theta^2 + 6)^2} - \sum_{i=1}^{n}\frac{1}{(\alpha + \theta x_i^{3\beta})^2}$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{n}{\beta^2} + \sum_{i=1}^{n}\frac{9\theta\alpha x_i^{3\beta}(\log x_i)^2}{(\alpha + \theta x_i^{3\beta})^2} - \theta\sum_{i=1}^{n} x_i^{\beta}\left[\log(x_i)\right]^2$$

$$\frac{\partial^2 \log L}{\partial \theta\,\partial \alpha} = -\frac{12n\theta}{(\alpha\theta^2 + 6)^2} - \sum_{i=1}^{n}\frac{x_i^{3\beta}}{(\alpha + \theta x_i^{3\beta})^2} = \frac{\partial^2 \log L}{\partial \alpha\,\partial \theta}$$

$$\frac{\partial^2 \log L}{\partial \theta\,\partial \beta} = \sum_{i=1}^{n}\frac{3\alpha x_i^{3\beta}\log(x_i)}{(\alpha + \theta x_i^{3\beta})^2} - \sum_{i=1}^{n} x_i^{\beta}\left[\log(x_i)\right]^2 = \frac{\partial^2 \log L}{\partial \beta\,\partial \theta}$$

$$\frac{\partial^2 \log L}{\partial \alpha\,\partial \beta} = -\sum_{i=1}^{n}\frac{3\theta x_i^{3\beta}\log(x_i)}{(\alpha + \theta x_i^{3\beta})^2} = \frac{\partial^2 \log L}{\partial \beta\,\partial \alpha}$$

The Fisher's Information matrix of PQRD can be obtained as

$$I(\theta,\alpha,\beta) = -E\begin{pmatrix} \frac{\partial^2 \log L}{\partial\theta^2} & \frac{\partial^2 \log L}{\partial\theta\,\partial\alpha} & \frac{\partial^2 \log L}{\partial\theta\,\partial\beta} \\ \frac{\partial^2 \log L}{\partial\alpha\,\partial\theta^2} & \frac{\partial^2 \log L}{\partial\alpha^2} & \frac{\partial^2 \log L}{\partial\alpha\,\partial\beta} \\ \frac{\partial^2 \log L}{\partial\beta\,\partial\theta} & \frac{\partial^2 \log L}{\partial\beta\,\partial\alpha} & \frac{\partial^2 \log L}{\partial\beta^2} \end{pmatrix} = \begin{pmatrix} I_{\theta\theta} & I_{\theta\alpha} & I_{\theta\beta} \\ I_{\alpha\theta} & I_{\alpha\alpha} & I_{\alpha\beta} \\ I_{\beta\theta} & I_{\beta\alpha} & I_{\beta\beta} \end{pmatrix}$$

The following equation can be solved for maximum likelihood estimates $(\hat{\theta},\hat{\alpha},\hat{\beta})$ of parameters (θ,α,β) of PQRD

$$\begin{pmatrix} \frac{\partial^2 \log L}{\partial\theta^2} & \frac{\partial^2 \log L}{\partial\theta\,\partial\alpha} & \frac{\partial^2 \log L}{\partial\theta\,\partial\beta} \\ \frac{\partial^2 \log L}{\partial\alpha\,\partial\theta^2} & \frac{\partial^2 \log L}{\partial\alpha^2} & \frac{\partial^2 \log L}{\partial\alpha\,\partial\beta} \\ \frac{\partial^2 \log L}{\partial\beta\,\partial\theta} & \frac{\partial^2 \log L}{\partial\beta\,\partial\alpha} & \frac{\partial^2 \log L}{\partial\beta^2} \end{pmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0\\ \hat{\beta}=\beta_0}} = \begin{pmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \\ \hat{\beta}-\beta_0 \end{pmatrix} = \begin{pmatrix} \frac{\partial \log L}{\partial\theta} \\ \frac{\partial \log L}{\partial\alpha} \\ \frac{\partial \log L}{\partial\beta} \end{pmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0\\ \hat{\beta}=\beta_0}},$$

where θ_0, α_0 and β_0 are initial value of θ, α and β respectively.

The solution of the Fisher's information matrix will provide asymptotic variance and covariance of the maximum likelihood estimator for $(\hat{\theta},\hat{\alpha},\hat{\beta})$. The approximate $100(1-\gamma)\%$ confidence intervals for (θ,α,β) respectively are $\hat{\theta}\pm Z_{\frac{\gamma}{2}}\sqrt{\frac{I_{\theta\theta}^{-1}}{n}}$, $\hat{\alpha}\pm Z_{\frac{\gamma}{2}}\sqrt{\frac{I_{\alpha\alpha}^{-1}}{n}}$ and $\hat{\beta}\pm Z_{\frac{\gamma}{2}}\sqrt{\frac{I_{\beta\beta}^{-1}}{n}}$ where Z_{γ} is the upper $100\,\gamma^{\text{th}}$ percentile of the standard normal distribution.

Theorem 2. Maximum likelihood estimators of PQRD are consistent estimators. That is as $n\to\infty$, $P\{|\hat{\theta}-\theta|>\varepsilon\}\to 0$, $P\{|\hat{\alpha}-\alpha|>\varepsilon\}\to 0$, $P\{|\hat{\beta}-\beta|>\varepsilon\}\to 0$

Proof: The asymptotic variance of the MLEs is given by
$V(\hat{\theta},\hat{\alpha},\hat{\beta}) = \frac{1}{n}I^{-1}(\hat{\theta},\hat{\alpha},\hat{\beta})$. Therefore, $V(\hat{\theta},\hat{\alpha},\hat{\beta})\to 0$ as $n\to\infty$

By Chebyshev's inequality, we have

$P\{|\hat{\theta}-\theta|>\varepsilon\} \le \frac{V(\hat{\theta})}{\varepsilon^2} = \frac{1}{n}\frac{I_{\theta\theta}^{-1}}{\varepsilon^2}$. As $n\to\infty$, $V(\hat{\theta}) = \frac{I_{\theta\theta}^{-1}}{n}\to 0$.

Thus, $P\{|\hat{\theta}-\theta|>\varepsilon\}\to 0$ as $n\to\infty$. Hence, $\hat{\theta}\xrightarrow{P}\theta$. Similarly, it can be shown that $\hat{\alpha}\xrightarrow{P}\alpha$ and $\hat{\beta}\xrightarrow{P}\beta$.

6.2.Maximum Product Spacing Estimation (MPSE)

The maximum product spacing estimates $(\hat{\theta},\hat{\alpha},\hat{\beta})$ for (θ,α,β) of PQRD can be obtained numerically by maximizing the following function with respect to θ, α and β.

$$MPSE = \frac{1}{n+1}\sum_{i=1}^{n+1}\log\left[F\left(x_{(i)},\theta,\alpha,\beta\right) - F\left(x_{(i-1)},\theta,\alpha,\beta\right)\right]$$

7. A Simulation Study

This section contains a simulation study to examine the consistency of maximum likelihood estimators of the PQRD. The mean, bias (B), MSE and variance of the Maximum likelihood estimators are computed using the formulas

$$\text{Mean} = \frac{1}{n}\sum_{i=1}^{n}\hat{H}_i\ ,\ B = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{H}_i - H\right),\ \text{MSE} = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{H}_i - H\right)^2,\ \text{Variance} = \text{MSE} - B^2,$$

where $H = (\theta, \alpha, \beta)$ and $\hat{H} = \left(\hat{\theta}_i, \hat{\alpha}_i, \hat{\beta}_i\right)$.

The simulation results for different parameter values of PQRD have been presented in tables 5 and 6 respectively using acceptance-rejection method:

Acceptance -rejection method for generating random samples from the PQRD consists of following steps.

i. Generate a random variable Y from exponential (θ) and U from Uniform $(0,1)$

ii. If $U \le \frac{f(y)}{M\,g(y)}$, then set $X = Y$ ("accept the sample"); otherwise ("reject the sample") and if rejected, then repeat the whole process until we get the required samples, where M is a constant.

where each sample size is replicated 10000 times.

Tables 5 and 6 reveal that for increasing sample size, the values of the biases, MSE and variances of the Maximum likelihood estimators of the parameters of PQRD becoming smaller, mean value of PQRD close to the true parameter values and certify the first-order asymptotic theory of maximum likelihood estimators.

Table 5: The Mean, Biases, MSE and Variances of PQRD for $\theta = 0.5, \alpha = 6.0, \beta = 3.0$

Parameters	Sample size	Mean	Bias	MSE	Variance	95% CI	
						Lower	Upper
θ	20	0.51862	0.01862	0.00217	0.00182	0.49992	0.53731
	40	0.51579	0.01579	0.00171	0.00146	0.50394	0.52763
	60	0.51138	0.01138	0.00134	0.00121	0.50257	0.52018
	80	0.50702	0.00702	0.00108	0.00103	0.49998	0.51405
	100	0.50510	0.00510	0.00089	0.00086	0.49935	0.51084
	200	0.50194	0.00194	0.00062	0.00062	0.49848	0.50539
	300	0.50089	0.00089	0.00053	0.00053	0.49828	0.50349
α	20	6.28235	0.28235	0.21542	0.13569	6.12091	6.44379
	40	6.21887	0.21887	0.15682	0.10891	6.11660	6.32114
	60	6.15970	0.15970	0.10512	0.07961	6.08830	6.23109
	80	6.12983	0.12983	0.07924	0.06238	6.05087	6.20878
	100	6.11132	0.11132	0.06367	0.05127	6.06694	6.15570
	200	6.07474	0.07474	0.03256	0.02698	6.05197	6.09750
	300	6.06253	0.06253	0.02219	0.01828	6.04723	6.07783
β	20	3.08299	0.08299	0.01307	0.00618	3.04853	3.11744
	40	3.05812	0.05812	0.00744	0.00406	3.03837	3.07786
	60	3.04374	0.04373	0.00520	0.00329	3.02922	3.05825
	80	3.03450	0.03450	0.00400	0.00281	3.02288	3.04611
	100	3.02495	0.02495	0.00327	0.00264	3.01487	3.03502
	200	3.00905	0.00905	0.00173	0.00165	3.00342	3.01468
	300	3.00124	0.00124	0.00134	0.00134	2.99709	3.00538

Table 6: The Mean, Biases, MSE and Variances of PQRD for $\theta = 2.0, \alpha = 4.0, \beta = 0.7$

Parameters	Sample size	Mean	Bias	MSE	Variance	95% CI	
						Lower	Upper
θ	20	2.06032	0.06032	0.05012	0.04648	1.96583	2.15480
	40	2.03521	0.03521	0.02534	0.02410	1.98710	2.08332
	60	2.02543	0.02543	0.01821	0.01757	1.99189	2.05897
	80	2.01567	0.01567	0.01234	0.01209	1.99157	2.03976
	100	2.01089	0.01089	0.00912	0.00900	1.99229	2.02948
	200	2.00545	0.00545	0.00423	0.00420	1.99646	2.01443
	300	2.00234	0.00234	0.00212	0.00211	1.99714	2.00753
α	20	4.12045	0.12045	0.09023	0.07574	3.99983	4.24106
	40	4.07032	0.07032	0.04512	0.04018	4.00820	4.13244
	60	4.05021	0.05021	0.03045	0.02793	4.00792	4.09249
	80	4.03012	0.03012	0.02023	0.01932	3.99966	4.06057
	100	4.02034	0.02034	0.01512	0.01471	3.99656	4.04411
	200	4.01023	0.01023	0.00834	0.00824	3.99764	4.02281
	300	4.00512	0.00512	0.00523	0.00520	3.99704	4.01336
β	20	0.73015	0.03015	0.01532	0.01441	0.67753	0.78276
	40	0.71522	0.01522	0.00745	0.00692	0.68944	0.74099
	60	0.71033	0.01033	0.00412	0.00401	0.69430	0.72635
	80	0.70544	0.00544	0.00256	0.00253	0.69441	0.71646
	100	0.70311	0.00311	0.00178	0.00176	0.69488	0.71133
	200	0.70122	0.00122	0.00089	0.00088	0.69711	0.70533
	300	0.70056	0.00056	0.00045	0.00045	0.69815	0.70296

8. Goodness of Fit

In this section, the goodness of fit of PQRD using both the maximum likelihood estimates and maximum product spacing estimates of parameters has been discussed with three real life datasets from engineering. Since PQRD exhibits both over-dispersion and under-dispersion and have both increasing and decreasing hazard rates, it would be a suitable model in reliability engineering to model data of both increasing or decreasing hazard rates. Considering these features of PQRD, the following three datasets have been considered for testing the goodness of fit of PQRD over other three-parameter lifetime distributions including, generalized power Sujatha distribution (GPSD), three-parameter power Sujatha distribution (TPPSD), power quasi Lindley distribution (PQLD), generalized gamma distribution (GGD), a three-parameter Sujatha distribution (ATPSD) and three-parameter generalized Lindley distribution (TPGLD).

Dataset-1: The following extreme skewed to right data, discussed by Murthy et al (2004), presents the failure times of 50 components and the observations are:

0.036, 0.058, 0.061, 0.074, 0.078, 0.086, 0.102, 0.103, 0.114, 0.116, 0.148, 0.183, 0.192,0.254, 0.262, 0.379, 0.381, 0.538, 0.570, 0.574, 0.590, 0.618, 0.645, 0.961, 1.228, 1.600,2.006, 2.054, 2.804, 3.058, 3.076, 3.147, 3.625, 3.704, 3.931, 4.073, 4.393, 4.534, 4.893,6.274, 6.816, 7.896, 7.904, 8.022, 9.337, 10.940, 11.020, 13.880, 14.730, 15.080.

Dataset-2: The moderately right-skewed dataset, as discussed by Barlow et al. (1984), elucidates the fatigue fracture of Kevlar 373/epoxy under constant pressure at a 90% stress level until all specimens failed. The recorded observations are as follows:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960.

Dataset-3: The following set of complete left skewed data, discussed by Murthy et al (2004), reports the failure times of 20 components. The values are:

0.481, 1.196, 1.438, 1.797, 1.811, 1.831, 1.885, 2.104, 2.133, 2.144, 2.282, 2.322, 2.334,2.341, 2.428, 2.447, 2.511, 2.593, 2.715, 3.218.

The summary and total time on test (TTT) plots of the three-datasets is presented in the table 7 and figure 6 respectively.

Table 7: Summary of the datasets 1, 2 and 3

Data sets	Minimum	1st Quartile	Median	Mean	Variance	3rd Quartile	Maximum
1	0.0360	0.2075	1.4140	3.3430	17.4847	4.4988	15.0800
2	0.0251	0.9048	1.7362	1.9592	2.4774	2.2959	9.0960
3	0.4810	1.8260	2.2130	2.1010	0.3493	2.4330	3.2180

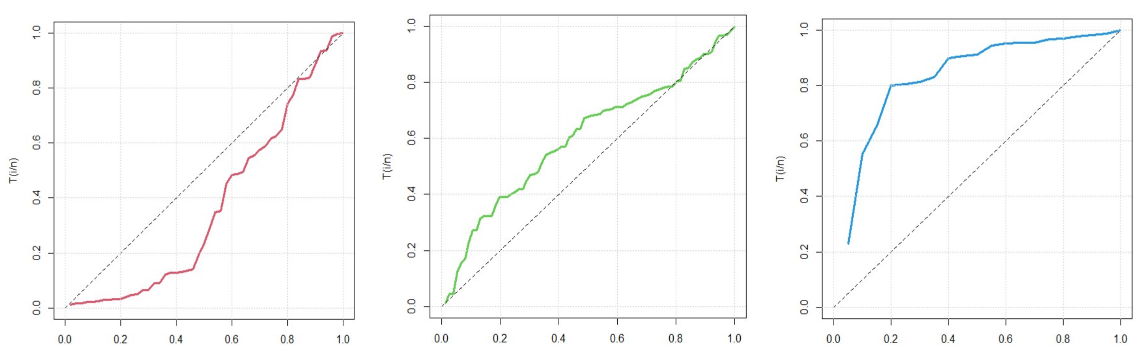

Figure 6: TTT plot of datasets 1, 2 and 3 respectively

From the figure 6, it observed that dataset 1 has decreasing hazard rate and datasets 2 and 3 has increasing hazard rate. From the table 7, it observed that for datasets 1 and 2, variance is greater than mean and for dataset 3, variance is less than mean which means that datasets 1 and 2 are over- dispersed and dataset 3 is under- dispersed.

In order to compare the considered distributions, values of MLE $\left(\hat{\theta}, \hat{\alpha}, \hat{\beta}\right)$ along with their standard errors, $-2\log L$, AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), CAIC (Consistent Akaike Information Criterion), HQIC (Hannan-Quinn Information Criterion), K-S (Kolmogorov-Smirnov) Statistic and p-values for the real lifetime dataset 1, 2 and 3 has been computed and presented in table 8, 9,10, 11, 12 and 13 respectively. The formulas for computing AIC, BIC, CAIC, HQIC and K-S are as follows:

$$AIC=-2\log L+2p,\quad BIC=-2\log L+p\log\left(n\right),\quad CAIC=-2\log L+2\frac{pn}{n-p-1},$$

$$HQIC=-2\log L+2p\log\left(\log\left(n\right)\right),\quad \text{K-S}=\underset{x}{Sup}\,|\,G_n\left(x\right)-G_0\left(x\right)|,$$

where, p =number of parameters, n =size of the sample,

$G_n\left(x\right)=$ empirical cdf of considered distribution and $G_0\left(x\right)=$ cdf of considered distribution.

The best distribution is the distribution corresponding to lower values of $-2\log L$, AIC, BIC, CAIC, HQIC and K-S.

Table 8: MLE and MPSE of the parameters of distributions for the dataset-1

Distributions	**MLE**			**MPSE**		
	$\hat{\eta}$ SE$(\hat{\eta})$	$\hat{\omega}$ SE$(\hat{\omega})$	$\hat{\tau}$ SE$(\hat{\tau})$	$\hat{\eta}$ SE$(\hat{\eta})$	$\hat{\omega}$ SE$(\hat{\omega})$	$\hat{\tau}$ SE$(\hat{\tau})$
PQRD	1.0989 (0.2438)	7.7671 (6.1727)	0.7029 (0.0760)	1.1270 (0.2536)	7.4013 (6.0513)	0.6711 (0.0741)
GPSD	1.0914 (0.3160)	0.2827 (0.6527)	0.5789 0.0654	1.3268 (0.1460)	0.7146 (0.0524)	1.1811 (0.1520)
TPPSD	0.8512 (0.2399)	7.4947 (9.2187)	0.6663 (0.0829)	0.8708 (0.2601)	6.3291 (6.7633)	0.6326 (0.0812)
PQLD	0.6468 (0.3000)	4.6050 (13.4643)	0.6521 (0.0861)	0.6560 (0.3495)	4.8827 (17.3594)	0.6198 (0.0855)
GGD	0.5090 (0.1070)	1.4804 (3.1953)	0.5186 (0.6743)	1.0475 (2.7280)	1.5283 (2.8508)	0.4835 (0.5402)
ATPSD	0.4403 (0.0412)	0.0551 (0.0493)	0.0100 (…)	0.5862 (0.0774)	0.3508 (0.1406)	0.4333 (0.1645)

TPGLD	0.6468 (0.2999)	7.11874 (23.8745)	0.6521 (0.0861)	0.6565 (0.3464)	7.3909 (29.5818)	0.6197 (0.0853)

Table 9: MLE and MPSE of the parameters of distributions for the dataset-2

Distributions	**MLE**			**MPSE**		
	$\hat{\eta}$ SE$(\hat{\eta})$	$\hat{\omega}$ SE$(\hat{\omega})$	$\hat{\tau}$ SE$(\hat{\tau})$	$\hat{\eta}$ SE$(\hat{\eta})$	$\hat{\omega}$ SE$(\hat{\omega})$	$\hat{\tau}$ SE$(\hat{\tau})$
PQRD	2.3890 (0.2860)	0.1000 (0.0993)	0.7482 (0.0855)	2.5279 (0.2397)	0.0633 (0.0520)	0.6858 (0.0708)
GPSD	1.6742 (0.2301)	17.5316 (19.7920)	0.8307 (0.0932)	1.0978 (0.0864)	0.9706 (0.0785)	0.9296 (0.1086)
TPPSD	1.3795 (0.1905)	0.1382 (0.2026)	0.9050 (0.0992)	1.3993 (0.1814)	0.2162 (0.2480)	0.8624 (0.0917)
PQLD	0.9633 (0.1852)	0.1424 (0.1891)	0.9931 (0.1197)	0.9873 (0.1751)	0.1521 (0.1770)	0.9411 (0.1090)
GGD	0.7083 (0.6152)	1.4774 (0.8027)	1.0653 (0.3368)	0.6746 (0.5709)	1.3990 (0.7390)	1.0336 (0.3202)
ATPSD	0.96196 (0.2600)	0.1000 (3.2261)	6.1346 (6.8389)	0.9075 (0.1815)	0.0002 (1.2395)	3.6648 (3.6292)
TPGLD	0.9633 (0.1852)	0.1478 (0.2190)	0.9931 (0.1197)	0.9867 (0.1752)	0.1546 (0.2009)	0.9415 (0.1091)

Table 10: MLE and MPSE of the parameters of distributions for the dataset-3

Distribution	**MLE**			**MPSE**		
	$\hat{\eta}$ SE($\hat{\eta}$)	$\hat{\omega}$ SE($\hat{\omega}$)	$\hat{\tau}$ SE($\hat{\tau}$)	$\hat{\eta}$ SE($\hat{\eta}$)	$\hat{\omega}$ SE($\hat{\omega}$)	$\hat{\tau}$ SE($\hat{\tau}$)
PQRD	0.3965 (0.2491)	6.1752 (12.6334)	2.7544 (0.6426)	0.5646 (0.2844)	3.0135 (4.9621)	2.3169 (0.5194)
GPSD	0.2587 (0.1693)	0.3236 (0.7717)	2.8755 (0.6075)	0.6730 (0.2074)	2.1222 (0.2454)	2.5968 (0.2615)
TPPSD	0.1965 (0.1316)	12.4203 (27.2864)	3.1767 (0.6778)	0.5646 (0.2844)	3.0135 (4.9621)	2.3169 (0.5194)
PQLD	0.1000 (0.0833)	0.3464 (0.5134)	3.5053 (0.7994)	0.1487 (0.1083)	0.3673 (0.5107)	2.9899 (0.6900)
GGD	0.1000 (0.0222)	0.7347 (0.4103)	5.0870 (1.8342)	0.2428 (113.842)	3.2361 (625.009)	3.8255 (374.072)1
ATPSD	1.4280 (0.1844)	14347.6730 (7718.7931)	0.1000 (2.3974)	1.9235 (0.2741)	12.9328 (2.1816)	12.1801 (2.9433)
TPGLD	0.1000 (0.0832)	3.4641 (7.3513)	3.5053 (0.7987)	0.1486 (0.1079)	2.4699 (4.7624)	2.9906 (0.6887)

Table 11: Goodness of fit measures of distributions for dataset-1

Distributions	$-2\log L$	AIC	BIC	CAIC	HQIC	MLE		MPSE	
						K-S	**p-value**	**K-S**	**p-value**
PQRD	**200.00**	**206.00**	**211.73**	**206.52**	**208.18**	**0.12**	**0.52**	**0.13**	**0.35**
GPSD	205.28	211.28	217.01	211.80	213.46	0.15	0.22	0.37	0.00
TPPSD	203.49	209.49	215.22	210.01	211.67	0.13	0.34	0.16	0.19
PQLD	204.66	210.66	216.39	211.18	212.84	0.18	0.07	027	0.00
GGD	204.69	210.69	216.42	211.21	212.87	0.25	0.00	0.15	0.22
ATPSD	217.88	223.88	229.61	224.40	226.06	0.24	0.02	0.34	0.00
TPGLD	204.66	210.66	216.39	211.18	212.84	0.16	0.15	0.15	0.20

Table 12: Goodness of fit measures of distributions for dataset-2

Distributions	$-2\log L$	**AIC**	**BIC**	**CAIC**	**HQIC**	**MLE**		**MPSE**	
						K-S	**p-value**	**K-S**	**p-value**
PQRD	241.65	247.65	254.64	247.98	250.44	0.07	**0.85**	**0.08**	**0.71**
GPSD	242.10	248.10	255.09	248.43	250.89	0.07	0.87	0.15	0.08
TPPSD	243.16	249.16	256.15	249.49	251.95	0.07	0.81	0.13	0.18
PQLD	243.29	249.29	256.28	249.62	252.08	0.08	0.71	0.11	0.36
GGD	244.46	250.46	257.45	250.79	253.25	0.28	0.00	0.10	0.47

ATPSD	243.30	249.30	256.29	249.63	252.09	0.09	0.55	0.12	0.16
TPGLD	243.29	249.29	256.28	249.62	252.08	0.10	0.33	0.09	0.60

Table 13: Goodness of fit measures of distributions for dataset-3

Distributions	$-2\log L$	**AIC**	**BIC**	**CAIC**	**HQIC**	**MLE**		**MPSE**	
						K-S	**p-value**	**K-S**	**p-value**
PQRD	**31.34**	**37.34**	**40.32**	**38.84**	**37.92**	**0.20**	**0.36**	**0.19**	**0.43**
GPSD	33.40	39.40	42.38	40.90	39.98	0.22	0.25	0.53	0.00
TPPSD	32.33	38.33	41.31	39.83	38.91	0.23	0.21	0.43	0.00
PQLD	33.49	39.49	42.47	40.99	40.07	0.57	0.00	0.57	0.00
GGD	33.95	39.95	42.93	41.45	40.53	0.72	0.00	0.34	0.01
ATPSD	50.23	56.23	59.21	57.73	56.81	0.33	0.02	0.64	0.00
TPGLD	33.49	39.49	42.47	40.99	40.07	0.23	0.20	0.25	0.13

Table 11, 12 and 13 show that, in comparison to GPSD, TPPSD, PQLD, GGD, ATPSD and TPGLD, PQRD has the least $-2\log L$, AIC, BIC, CAIC, HQIC and K-S values. As a result, we conclude that PQRD provides better fit as compared to GPSD, TPPSD, PQLD, GGD, ATPSD and TPGLD. In case of MLE, dataset 1 and 2 have least K-S value as compared to MPSE but for dataset 3, MLE has greater value as compared to the MPSE. Thus, we conclude that, for PQRD, MLE is better than MPSE for dataset 1 and 2 but for dataset 3, MPSE is better than MLE. Further, PQRD fits better than the GPSD, TPPSD, PQLD, GGD, ATPSD and TPGLD as shown by fitted plot of the distributions, quantile-quantile (Q-Q) plot, probability-probability (P-P) plot and empirical cdf (ECDF) plot in figures 7 and 8 respectively.

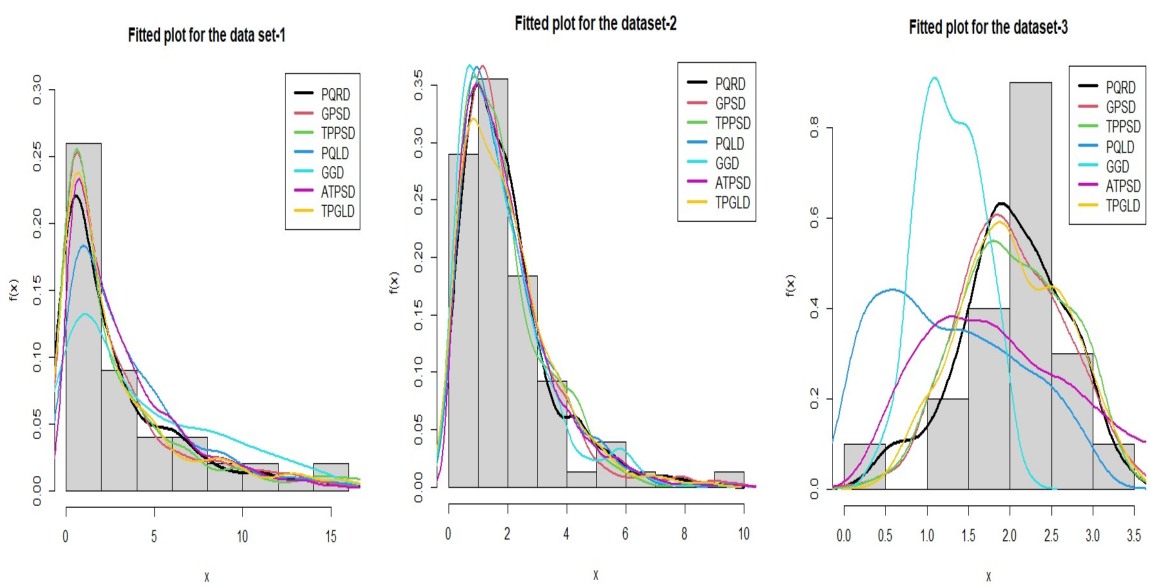

Figure 7: Fitted plot of the considered distributions

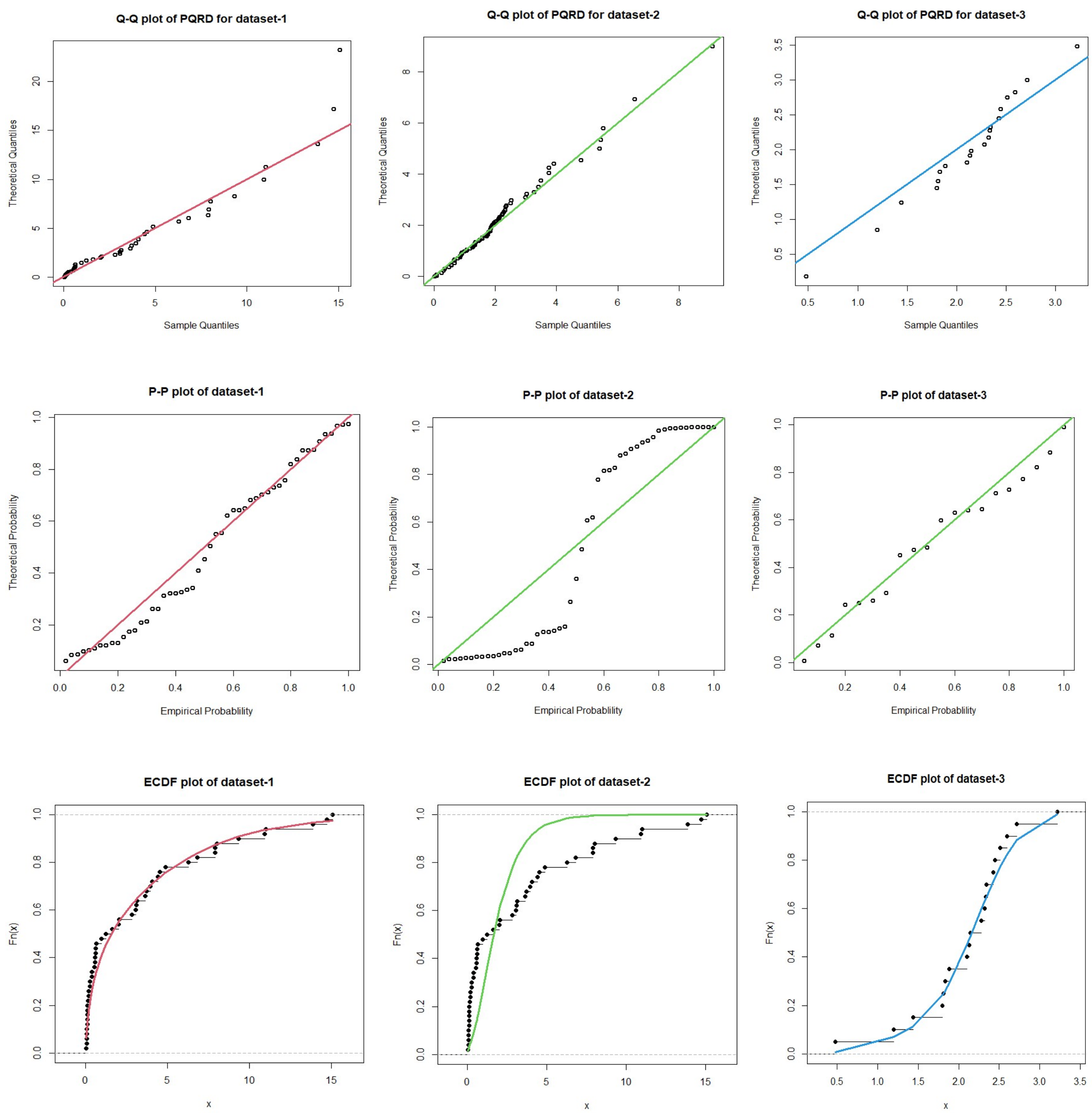

Figure 8: Q-Q plot, P-P plot and ECDF plot of the datasets 1,2 and 3 respectively

9. Concluding Remarks

Three-parameter continuous power quasi Rama distribution (PQRD) has been introduced using power transformation approach on Quasi Rama distribution (QRD) and it includes Rama distribution (RD), quasi Rama distribution (QRD), power Rama distribution (PRD) introduced, gamma distribution and generalized gamma distribution as particular cases. The statistical and reliability properties including moments about the origin, the variance, survival function, hazard function, mean residual function, stochastic ordering of PQRD have been discussed. The method of MLE and MPSE for estimating the parameters has been discussed. Finally, the goodness of fit of PQRD has been discussed with three real lifetime dataset and the fit has been found quite satisfactory as compared with GPSD, TPPSD, PQLD, GGD, ATPSD and TPGLD where it has been observed that MLE is better than MPSE for dataset 1 and 2 but MPSE is better than MLE for dataset 3 in terms of fit. Therefore, PQRD can be considered as an important lifetime distribution for modelling lifetime data from engineering of over-dispersed, under- dispersed, increasing hazard rate and decreasing hazard rate natures.

Declarations

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