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# Bayesian and Non-Bayesian Parametric Estimation of the Bivariate Generalized Burr Distribution using Ranked Set Sampling with Concomitant Variable

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## ABSTRACT

Ranked set sampling (RSS) has proven to be an efficient alternative to simple random sampling, particularly in situations where exact measurements are costly or difficult to obtain. Although extensive work has been devoted to parametric estimation under RSS for univariate models, relatively limited attention has been given to bivariate models, despite their importance in modeling dependence between random variables. In this paper, the likelihood function under ranked set sampling for the Marshal-Olkin bivariate class of distributions is derived in general and applies it on the bivariate generalized Burr distribution. Maximum likelihood estimation is considered for the model's unknown parameters. Bayesian estimation is also considered in both simple random sampling and ranked set sampling; moreover, the Bayes estimators are obtained explicitly with respect to the square error loss function in both cases.

***Keywords:*** Bivariate Ranked Set Samples; Marshall-Olkin Bivariate distributions; Generalized Burr distribution, Maximum likelihood Estimation; Bayesian Estimation.

## 1. Introduction

McIntyre (1952) was the first one to suggest the method for the estimation of pasture and forage yields. He proposed a method of sampling to estimate mean pasture yields with greater efficiency than simple random sampling (SRS). This method of sampling has come to be known as ranked set sampling (RSS) because it involves a preliminary ranking of randomly selected units from the population, after which only a certain few of these sampled units are quantified. McIntyre's goal was to maintain the unbiasedness of unbiased for the estimator with increased efficiency. The basic idea is that the collection of sampled units is randomly partitioned into small groups. A group size (called the set size) of two, three, or four units is typical. The members of each given group are then ranked relative to one another, and based on the ranking, exactly one member of each group is chosen for quantification. Even if there are ranking errors, the method is shown to be at least as efficient as SRS with the same number of quantifications.

The general distribution theory for concomitants of order statistics has been discussed by Yang (1977). The asymptotic theory of concomitants of order statistics has been investigated by David and Galambos (1974) under the assumption that $(X, Y)$ has a bivariate normal distribution. Qinying (2007) has introduced an expression for the joint pdf of $(X_{(j)}, Y_{[j]})$ as follows

$$f_{(X_{(j)},Y_{[j]})}(x,y) = \frac{n!}{(j-1)!\,(n-j)!} f(x,y)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j}.$$

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Then, the marginal density function of $Y_{[j]}$ is given by

$$f_{Y_{[j]}}(y) = \frac{n!}{(j-1)!\,(n-j)!}\int_{-\infty}^{\infty} f(x,y)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j}dx.$$

The procedure of RSS using a concomitant variable described by Stokes (1977) is as follows. Choose $n$ independent bivariate samples, each of size $n$, and observe the value of the auxiliary variable $X$ on each of these units. For the first sample, select that unit for which the measurement on the auxiliary variable $X$ is the smallest and measure the $Y$ variable associated with it. In the second sample, choose $Y$ associated with the second smallest $X$. This procedure is repeated until $Y$ associated with the largest $X$ in the last sample is measured. The resulting set of $n$ units is called a ranked set sample. Let $(X_{(i)i}, Y_{[i]i})$ $i = 1..n$ be the pair selected from the $i^{th}$ sample where $X_{(i)j}$ is the $i^{th}$ order statistic of $X$ in the $i^{th}$ sample and $Y_{[i]i}$ be its concomitant of $Y$. Stokes (1977) proposes RSS mean as an estimator for the mean of the study variate $Y$, when an auxiliary variable $X$ is used for ranking the sample units, under the assumption that $(X, Y)$ follows a bivariate normal distribution.

Dubey (1968) introduced the generalized Burr distribution by compounding the Weibull distribution with the gamma distribution. She derived the compound Weibull distribution by assuming a conditional random variable $X$ follows the Weibull distribution, and its scale parameter follows a gamma distribution. The resulting unconditional pdf is called the compound Weibull (CW) distribution. Because the Burr distribution, which was defined by Burr (1942) is resulted as a special case of the CW distribution, she renamed the CW distribution the generalized Burr (GB) distribution.

A random variable with the generalized Burr (GB) distribution has a pdf, a cdf, a survival function, and a hazard function for $x > 0$, in the following form.

$$f_{GB}(x;\alpha,\delta,\vartheta) = \frac{\alpha\,\vartheta}{\delta}\, x^{\alpha-1}\,(1+\frac{x^{\alpha}}{\delta})^{-\vartheta-1},\ F_{GB}(x;\alpha,\delta,\vartheta) = 1-(1+\frac{x^{\alpha}}{\delta})^{-\vartheta},$$

$$S_{GB}(x;\alpha,\vartheta,\delta) = (1+\frac{x^{\alpha}}{\delta})^{-\vartheta} \qquad \text{and} \qquad h_{GB}(x;\alpha,\vartheta,\delta) = \frac{\alpha\,\vartheta}{\delta}\, x^{\alpha-1}(1+\frac{x^{\alpha}}{\delta})^{-1}.$$

Respectively, where the quantities $\delta > 0$ is a scale parameter and $\alpha > 0$ and $\vartheta > 0$ are shape parameters, respectively. From now on, it will be denoted by $GB(\alpha,\delta,\vartheta)$. Figure 1 displays different graphs for the pdf, hazard function, cdf and survival function of the GB distribution for different parameters values.

The interrelations between particular cases of the GB distribution and other distributions will be considered clearly as follows:

**For $\boldsymbol{\delta = 1}$**, the GB distribution reduces to the Burr Type XII (Signh-Maddala) distribution.
**For $\boldsymbol{\alpha = 1}$**, the GB distribution reduces to the Lomax distribution.
**For $\boldsymbol{\vartheta = 1}$**, the GB distribution reduces to the Weibull – exponential distribution.
**For $\boldsymbol{\delta = 1}$ and $\boldsymbol{\alpha = 1}$** , the GB distribution reduces to the Pareto Type II distribution.

**For $\boldsymbol{Z = \frac{1}{X}}$ and $\boldsymbol{\lambda = \frac{1}{\delta}}$** , the GB distribution reduces to the inverted generalized Burr (Dagum) distribution.

**For** $\boldsymbol{Z = \frac{1}{X}}$ **and** $\boldsymbol{\delta = 1}$ the GB distribution reduces to the inverted Burr Type XII distribution.

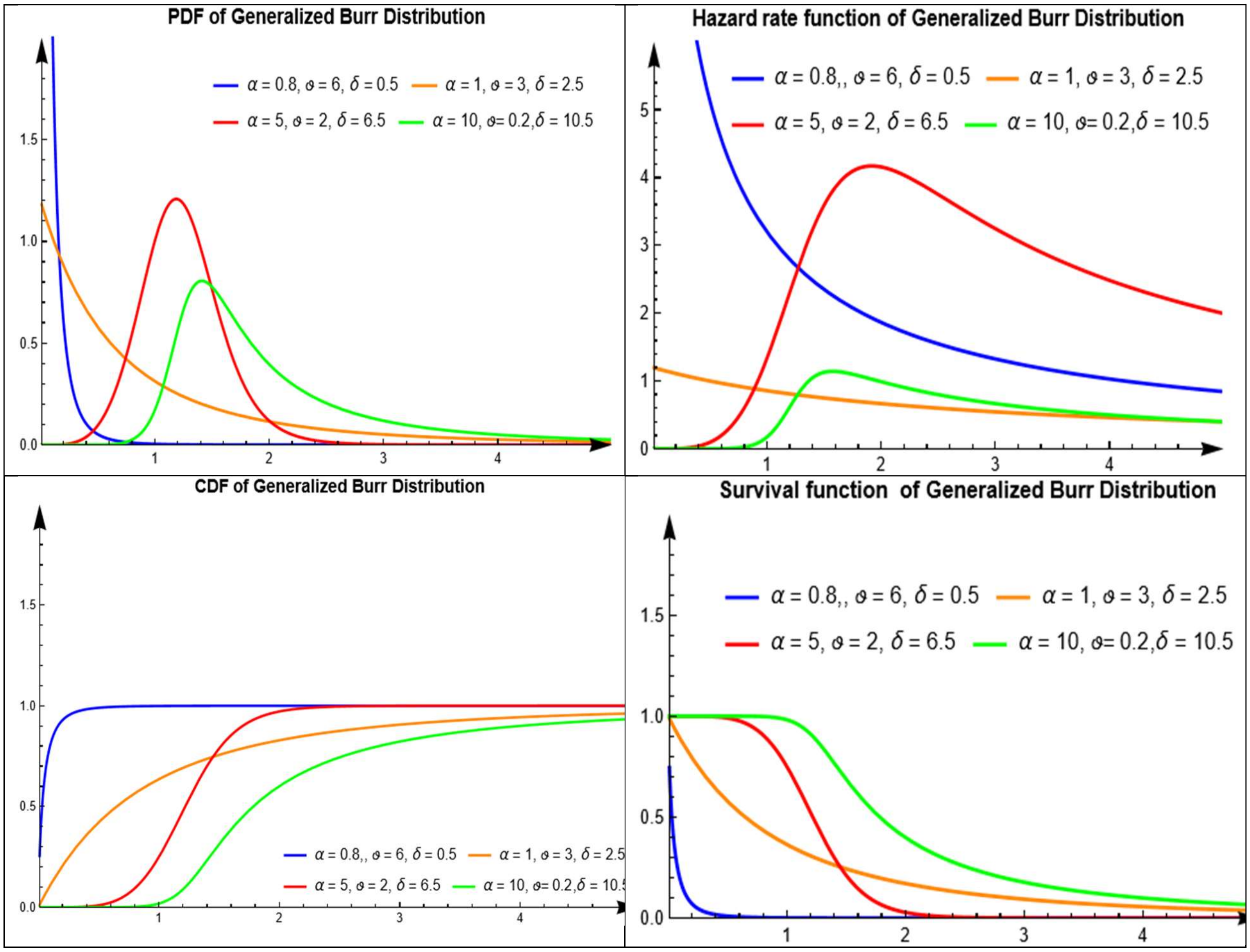



Figure1: Graphs for the pdf, hazard function, cdf and survival function of GB Distribution

The rest of the paper is organized as follows: In Section 2,Marshal–Olkin bivariate BGB distribution is introduced, and the data description is also provided in Section 2. MLE based on SRS and RSS for the BGB model are provided in Section 3. Bayesian estimation for the model parameters is proposed based on RSS and SRS in Section 4. Finally conclusion of the paper is given in Section 5.

## 2. Model Assumption and Data Description

### 2.1. Bivariate Generalized Burr Model

Muhammed (2019) defined the Bivariate generalized Burr distribution as a member of Marshal Olkin's class of bivariate distributions as follows:

Suppose $U_1, U_2$ and $U_3$ are three independent random variables, and $U_i \sim$ GB $(\vartheta_i, \alpha, \delta)$ for $i = 1,2,3$, where GB $(\vartheta_i, \alpha, \delta)$ denotes a GB distribution.

Define $X = \min(U_1, U_3)$ and $Y = \min(U_2, U_3)$ then it is said that the bivariate vector ($X$, $Y$) has a bivariate GB distribution with parameters $(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$ denoted by $BGB(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$.

It should be noted that the random variables $U_1, U_2$ and $U_3$ have common parameters. This ensures that the marginal distributions of $X$ and $Y$ are $GB(\vartheta_1 + \vartheta_3, \alpha, \delta)$ and $GB(\vartheta_2 + \vartheta_3, \alpha, \delta)$ respectively, furthermore, the distribution of $\min(X, Y)$ is $GB(\vartheta_1 + \vartheta_2 + \vartheta_3, \alpha, \delta)$. In addition, when $\vartheta_3 = 0$, the two random variables $X$ and $Y$ will be independent, hence $\vartheta_3$ can be considered as a correlation control parameter.

The joint survival function of $(X, Y)$ following the BGB distribution is given as follows

$$S_{BGB}(x, y) = S_{GB}(x; \vartheta_1, \alpha, \delta) S_{GB}(y; \vartheta_2, \alpha, \delta) S_{GB}(z; \vartheta_3, \alpha, \delta) .$$

where $z = \max(x, y)$ and $S_{GB}(x; \vartheta, \alpha, \delta) = 1 - F_{GB}(x; \vartheta, \alpha, \delta) = (1 + \frac{x^\alpha}{\delta})^{-\vartheta}$.

Then, $S_{BGB}(x, y)$ can be stretched in the following form

$$S_{BGB}(x, y) = \begin{cases} S_{GB}(x; \alpha, \delta, \vartheta_1) S_{GB}(y; \alpha, \delta, \vartheta_{23}), & x < y \\ S_{GB}(x; \alpha, \delta, \vartheta_{13}) S_{GB}(y; \alpha, \delta, \vartheta_2), & x > y \\ S_{GB}(x; \alpha, \delta, \vartheta_{123}), & x = y \end{cases}$$

And $\vartheta_{13} = \vartheta_1 + \vartheta_3$, $\vartheta_{23} = \vartheta_2 + \vartheta_3$ and $\vartheta_{123} = \vartheta_1 + \vartheta_2 + \vartheta_3$.

The joint pdf of $(X, Y) \sim BGB(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$ is given as follows

$$f_{BGB}(x, y) = \begin{cases} f_{GB}(x; \alpha, \delta, \vartheta_1)\, f_{GB}(y; \alpha, \delta, \vartheta_{23}) & x < y \\ f_{GB}(x; \alpha, \delta, \vartheta_{13})\, f_{GB}(y; \alpha, \delta, \vartheta_2), & x > y \\ \dfrac{\vartheta_3}{\vartheta_{123}} \; f_{GB}(x; \alpha, \delta, \vartheta_{123}), & x = y. \end{cases}$$

Figure 2 displays Surface plots of the absolutely continuous part of the joint PDF of the BGB distribution at different values for the model parameters

The joint CDF of the BGB distribution is given as

$$F_{BGB}(x, \mathrm{y}) = \begin{cases} F_{GB}(\mathrm{x}; \vartheta_{13}) - F_{GB}(\mathrm{y}; \vartheta_1)[1 - F_{GB}(\mathrm{y}; \vartheta_{23})], & x < \mathrm{y} \\ F_{GB}(\mathrm{y}; \vartheta_{23}) - F_{GB}(\mathrm{y}; \vartheta_2)[1 - F_{GB}(\mathrm{x}; \vartheta_{13})], & \mathrm{x} > \mathrm{y} \\ [1 - F_{GB}(x; \vartheta_{123})], & \mathrm{x} = \mathrm{y} \end{cases}$$

The joint hazard function of the BGB distribution is given as

$$h_{BGB}(x,\mathrm{y}) = \begin{cases} (\frac{\alpha}{\delta})^2(\vartheta_{23})\vartheta_1\ \mathrm{x}^{\alpha-1}\mathrm{y}^{\alpha-1}(1+\frac{\mathrm{x}^{\alpha}}{\delta})^{-1}(1+\frac{\mathrm{y}^{\alpha}}{\delta})^{-1}, & \mathrm{x} < \mathrm{y} \\ (\frac{\alpha}{\delta})^2(\vartheta_{13})\vartheta_2\ \mathrm{x}^{\alpha-1}\mathrm{y}^{\alpha-1}(1+\frac{\mathrm{x}^{\alpha}}{\delta})^{-1}(1+\frac{\mathrm{y}^{\alpha}}{\delta})^{-1}, & \mathrm{x} > \mathrm{y} \\ \frac{\alpha}{\delta}\ \vartheta_3\ x^{\alpha-1}(1+\frac{x^{\alpha}}{\delta})^{-1}, & \mathrm{x} = \mathrm{y}\ . \end{cases}$$

Figure 3 displays a surface plots for the joint hazard function for the BGB distributions with different values of $\alpha, \delta, \vartheta_1, \vartheta_2, \vartheta_3$.

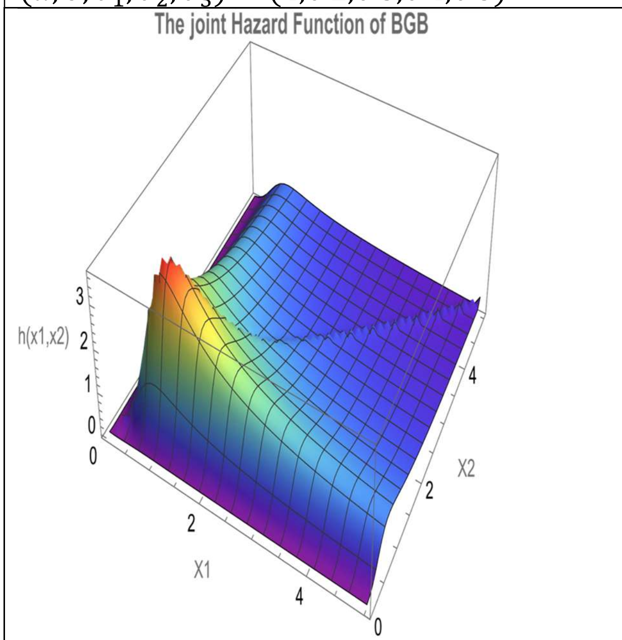
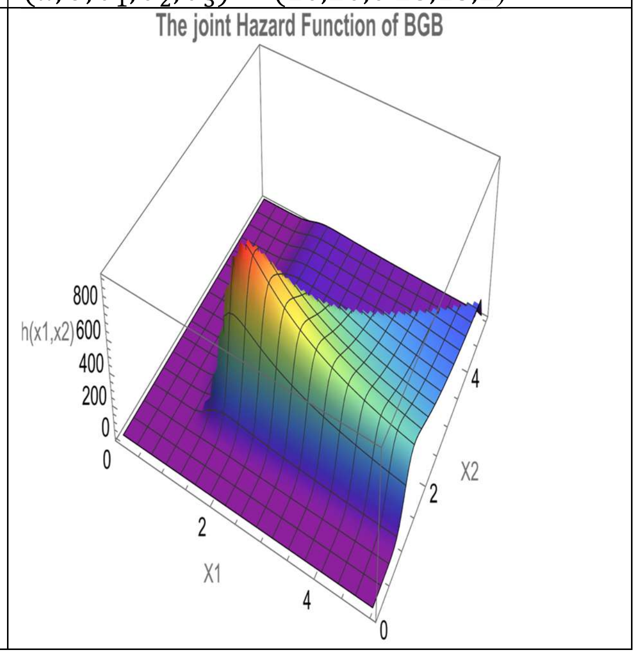
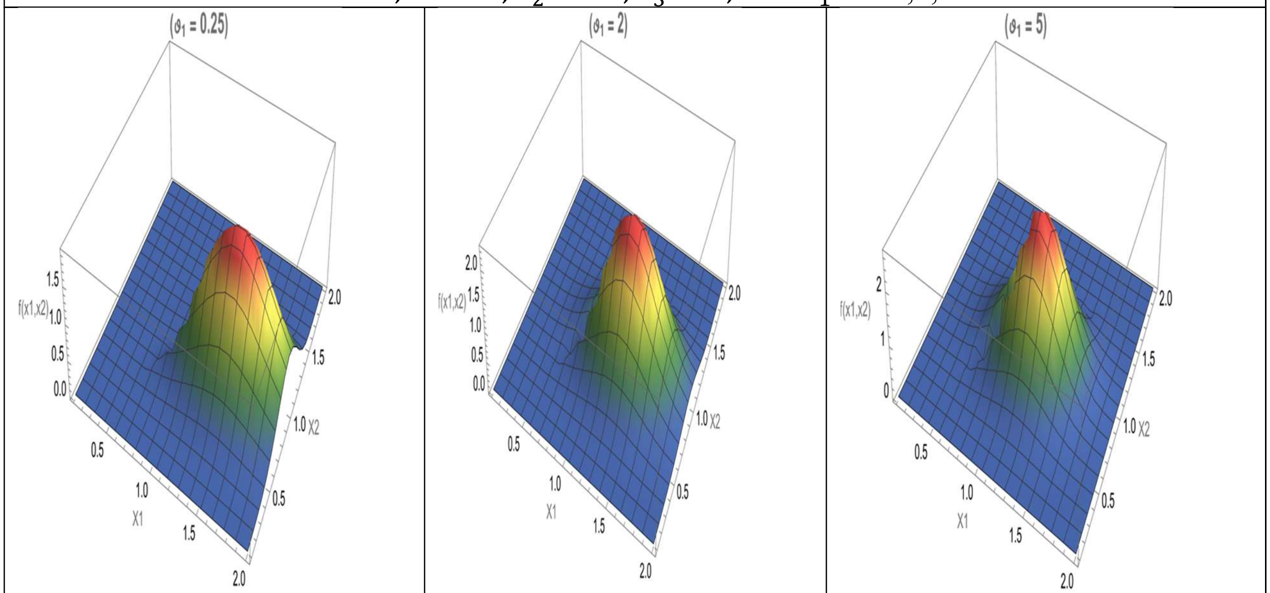
| $(\alpha,\delta,\vartheta_1,\vartheta_2,\vartheta_3) = (4,0.2,0.3,0.4,0.5)$ | $(\alpha,\delta,\vartheta_1,\vartheta_2,\vartheta_3) = (10,20,0.25,15,2)$ |
| --- | --- |
|   |   |

Figure 3: The joint Hazard function of the BGB distribution

$\alpha = 5, \delta = 20, \vartheta_2 = 15, \vartheta_3 = 2, and\ \vartheta_1$=0.25,2,5




$\alpha = 5, \delta = 20, \vartheta_1 = 1, \vartheta_3 = 2, and\ \vartheta_2$=5,10,15

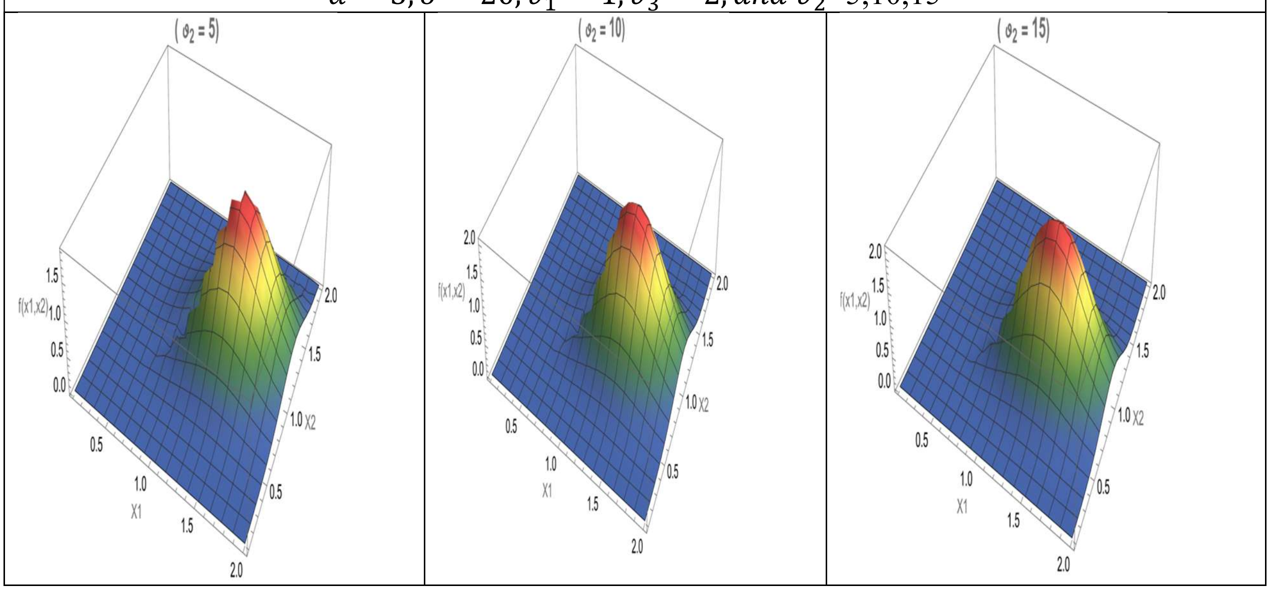



Figure 2: 3D plots for the joint pdf of BGB distribution

### 2.2. Data Description and Likelihood Function

Ranked Set Sampling (RSS) technique, introduced by McIntyre (1952), is a useful procedure when quantification of all sampling units is costly, but a small set of units can be easily ranked, according to the characteristics under investigation, without actual quantification. The procedure for obtaining the RSS can be summarized as follows:

**Step 1:** Randomly select $m^2$ units from the target population.
**Step 2:** Allocate the $m^2$ selected units as randomly as possible into $m$ sets, each of size $m$.

**Step 3:** Without yet knowing any values for the variable of interest, rank the units within each set with respect to the variable of interest. This may be based on personal professional judgment or done with a concomitant variable correlated with the variable of interest.

**Step 4:** Choose a sample for actual quantification by including the smallest ranked unit in the first set, the second smallest ranked unit in the second set, the process continues in this way, until the largest ranked unit is selected from the last set.

**Step 5:** Repeat steps 1 through 4 for $r$ cycles to obtain a sample of size $mr$.

Where $X_{(ii)j}$ is the $i^{th}$order statistic from the $i^{th}$set of the $j^{th}$ cycle, RSS uses only one observation, namely, $X_{(11)j}$ the lowest observation in the $j^{th}$ cycle, from this set, then $X_{(22)j}$ the second lowest from another independent set of $m$ observations, and finally $X_{(mm)j}$ the largest observation from the last set of $m$ observations. This process can be described in Figure 4

| | | | | | |
|---|---|---|---|---|---|
| $x_{(1\,1)j}$ | $x_{(1\,2)j}$ | $\cdots$ | $x_{(1\,(m-1))j}$ | $x_{(1\,m)j}$ | $x_{(1\,1)j}$ |
| $x_{(2\,1)j}$ | $x_{(2\,2)j}$ | $\cdots$ | $x_{(2\,(m-1))j}$ | $x_{(2\,m)j}$ | $x_{(2\,2)j}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $x_{(m\,1)j}$ | $x_{(m\,2)j}$ | $\cdots$ | $x_{(m\,(m-1))}$ | $x_{(m\,m)j}$ | $x_{(m\,m)j}$ |

Figure 4: Display of $\mathrm{m}^2$ observations in the $\mathrm{j}^{\mathrm{th}}$ set cycle sets of size m

Then the CDF and the PDF of $X_{(ii)j}$ is given by

$$F_m\left(x_{(ii)j};\theta\right)=\sum_{t=i}^{m}\binom{m}{t}\left[F\left(x_{(ii)j};\theta\right)\right]^{t}\times\left[1-F\left(x_{(ii)j};\theta\right)\right]^{m-t},$$

and

$$f_m(x_{(ii)j};\theta)=\frac{m!}{(i-1)!(m-i)!}f\left(x_{(ii)j};\theta\right)\left[F\left(x_{(ii)j};\theta\right)\right]^{i-1}.\left[1-F\left(x_{(ii)j};\theta\right)\right]^{m-i}$$

respectively where,$-\infty<x_{(ii)j}<\infty$ $x_{(ii)j}$, $F(x;\theta)$ and $f(x;\theta)$ are respectively the CDF and PDF of a random sample selected in step 1, and for simplification $x_{(i)j}$ is replaced by $x_{(ii)j}$ . The joint PDF of $X_{(ii)j}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,r$ is then given by

$$L(\theta;X)\propto\prod_{j=1}^{r}\prod_{i=1}^{m}f_m\left(x_{(ii)j};\theta\right),$$

$$L(\theta; x) \propto \prod_{j=1}^{r}\prod_{i=1}^{m} f\big(x_{(i)j};\theta\big)[F\big(x_{(i)j};\theta\big)]^{i-1}\cdot\big[1-F\big(x_{(i)j};\theta\big)\big]^{m-i}.$$

The Ranked Set sampling (RSS) procedure can be applied to a bivariate population as follows.

Consider the symbols $m$ and $r$ such that $m$: is the set size and $r$: is the number of cycles.

**Step1:** Select $m^2$ pairs from the population for the $j^{th}$ cycle

**Step2:** Divide the pairs into the $m$ sets at random.

**Step3:** Select the $i^{th}$ order statistics and its concomitant from $i^{th}$ set, where $i = 1,2,\dots,m$.

**Step4:** Repeat steps 1-3 $r$ cycle, $j = 1,2,\dots,r$.

Based on the RSS scheme, we have the following observations

For one cycle $r = 1$: $\big[(x_{(1)1}, y_{[1]1}), (x_{(2)1}, y_{[2]1}), \dots, (x_{(m)1}, y_{[m]1})\big]$

For $r$ cycle $r > 1$: $[(x_{(i)j}, y_{[i]j}), i = 1,2,\dots,m\ , j = 1,2,\dots,r]$.

Where $X_{(i)j}$ is the $i^{th}$ order statistic of $X$ in the $j^{th}$ cycle and $Y_{[i]j}$ be its concomitant of $Y$.

The corresponding joint pdf of $(X_{(i)j}, Y_{[i]j})$ can be written as

$$f_{i:m}(x_{(i)j}, y_{[i]j}) = \frac{m!}{(i-1)!(m-i)!}\,[F(x_{(i)j})]^{i-1}[1-F(x_{(i)j})]^{m-i}\, f(x_{(i)j}, y_{[i]j})$$

Thus, the likelihood function is given as

$$L(\Theta) = \prod_{j=1}^{r}\prod_{i=1}^{m} f_{i:m}(x_{(i)j}, y_{[i]j}),$$

$$L(\Theta) \propto \prod_{j=1}^{r}\prod_{i=1}^{m}[F((x_{(i)j})]^{i-1}[1-F((x_{(i)j})]^{m-i}\, f(x_{(i)j}, y_{[i]j}).$$

Now, we introduce the likelihood function in the case of the Marshall-Olkin class of bivariate distributions for the first time by taking into consideration the three scenarios of the experiment variables as follows:

$$L(\Theta) \propto \prod_{j=1}^{r}\prod_{i=1}^{m}[f_1(x_{(i)j}, y_{[i]j})]^{\delta_{1i}}\,[f_2(x_{(i)j}, y_{[i]j})]^{\delta_{2i}}\,[f_3(x_{(i)j})]^{\delta_{3i}}$$
$$.\ [F((x_{(i)j})]^{i-1}[\bar{F}(x_{(i)j})]^{m-i}$$

Where $\delta_{1i}, \delta_{2i}$ *and* $\delta_{3i}$ are event indicators for the $j^{th}$ cycle such that

$\delta_{1i} = \begin{cases} 1, & x_{(i)j} < y_{[i]j} \\ 0, & otherwise \end{cases}$, $\delta_{2i} = \begin{cases} 1, & x_{(i)j} > y_{[i]j} \\ 0, & otherwise \end{cases}$ and

$\delta_{3i} = \begin{cases} 1, & x_{(i)j} = y_{[i]j} \\ 0, & otherwise \end{cases}$.

So, we can have

$m_1 = \sum_{i=1}^{m}\delta_{1i}$ , $m_2 = \sum_{i=1}^{m}\delta_{2i}$ and $m_3 = \sum_{i=1}^{m}\delta_{3i}$ such that $m = m_1 + m_2 + m_3$.

## 3. Maximum Likelihood Estimation for the BGB model

This section deals with parametric estimation for the BGB distribution based on SRS and RSS. Two algorithms for generating bivariate random samples are provided.

**3.1. MLE based on Simple Random Sample**

Suppose that there are n independent pairs of components $(X_i, Y_i)$, $i = 1 \dots n$ under experiment, and each of them has $BGB(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$ distribution. The following is an algorithm that describes how to get an SRS from a BGB distribution

| **Algorithm 1: Generate SRS from BGB distribution** |
|---|
| **Step 1.** Generate $U_{1i}$ , $U_{2i}$ and $U_{3i}$ , $i = 1..n$ from U(0,1). <br> **Step 2.** Compute $Z_{1i} = \left[\delta({U_{1i}}^{1/\vartheta_1} - 1)\right]^{\frac{1}{\alpha}}$, $Z_{2i} = \left[\delta({U_{2i}}^{1/\vartheta_2} - 1)\right]^{\frac{1}{\alpha}}$ and $Z_{3i} = \left[\delta({U_{3i}}^{1/\vartheta_3} - 1)\right]^{\frac{1}{\alpha}}$, <br> **Step3.** Obtain $X_i = min\,(Z_{1i}, Z_{3i})$ and $Y_i = min\,(Z_{2i}, Z_{3i})$ , $i = 1..n$. <br> **Step4**. Define the indicator functions as $\delta_{1i} = \begin{cases} 1; & x_i < y_i \\ 0; & otherwise \end{cases}$, $\delta_{2i} = \begin{cases} 1; & x_i > y_i \\ 0; & otherwise \end{cases}$ $and$ $\delta_{3i} = \begin{cases} 1; & x_i = y_i \\ 0; & otherwise \end{cases}$. <br> **Step5**. For the SRS $(X_i, Y_i), i = 1,2, \dots, n$, the corresponding sample size n must satisfy $n = n_1 + n_2 + n_3$ Such that $n_1 = \sum_{i=1}^{n} \delta_{1i}$, $n_2 = \sum_{i=1}^{n} \delta_{2i}$ $and$ $n_3 = \sum_{i=1}^{n} \delta_{3i}$. |

The likelihood function of the SRS of size $n$ [ $(X_i, Y_i), i = 1,2, \dots, n$] is given by

$$L(\theta) = \prod_{i=1}^{n} [f_1(x_{1i}, x_{2i})]^{\delta_{1i}}\, [f_2(x_{1i}, x_{2i})]^{\delta_{2i}}\, [f_3(x_{1i}, x_{2i})]^{\delta_{3i}}.$$

Where $n_1 = \sum_{i=1}^{n} \delta_{1i}$ , $n_2 = \sum_{i=1}^{n} \delta_{2i}$ and $n_3 = \sum_{i=1}^{n} \delta_{3i}$ such that $n = n_1 + n_2 + n_3$.

The log-likelihood function of the SRS of size $n$ from the BGB distribution is given by

$$\begin{aligned} l(\Theta) = {} & (2n_1 + 2n_2 + n_3)(\log\alpha - \log\delta) + n_1 log\vartheta_1 + n_1 log\vartheta_{23} + n_2 log\vartheta_2 \\ & + n_2 log\vartheta_{13} + n_3 log\vartheta_3 + (\alpha - 1)\{\textstyle\sum_{i=1}^{n}(\delta_{1i} + \delta_{2i})\, logx_i y_i + \delta_{3i}\ logx_i\} \\ & -(\vartheta_1 + 1)\textstyle\sum_{i=1}^{n} \delta_{1i}\ \log\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right) - (\vartheta_{23} + 1)\sum_{i=1}^{n} \delta_{1i}\ \log\left(1 + \frac{{y_i}^{\alpha}}{\delta}\right) \\ & -(\vartheta_{13} + 1)\textstyle\sum_{i=1}^{n} \delta_{2i}\ \log\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right) - (\vartheta_2 + 1)\sum_{i=1}^{n} \delta_{2i}\ \log\left(1 + \frac{{y_i}^{\alpha}}{\delta}\right) \\ & -(\vartheta_{123} + 1)\textstyle\sum_{i=1}^{n} \delta_{3i}\ \log\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right). \end{aligned} \tag{1}$$

The first derivatives of the log- likelihood function with respect to $\vartheta_1, \vartheta_2, \vartheta_3$, $\alpha$ and $\delta$ are as follows:

$$\frac{\partial l}{\partial \vartheta_1} = \frac{n_1}{\vartheta_1} + \frac{n_2}{\vartheta_{13}} - \sum_{i=1}^{n} \delta_{1i} \ \log\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right) - \sum_{i=1}^{n} \delta_{2i} \ \log\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right) - \sum_{i=1}^{n} \delta_{3i} \ \log\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right),$$

$$\frac{\partial l}{\partial \vartheta_2} = \frac{n_2}{\vartheta_2} + \frac{n_1}{\vartheta_{23}} - \sum_{i=1}^{n} \delta_{1i} \ \log\left(1 + \frac{{y_i}^{\alpha}}{\delta}\right) - \sum_{i=1}^{n} \delta_{2i} \ \log\left(1 + \frac{{y_i}^{\alpha}}{\delta}\right) - \sum_{i=1}^{n} \delta_{3i} \ \log\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right),$$

$$\frac{\partial l}{\partial \vartheta_3} = \frac{n_3}{\vartheta_1} + \frac{n_1}{\vartheta_{23}} + \frac{n_2}{\vartheta_{13}} - \sum_{i=1}^{n} \delta_{1i} \ \log\left(1 + \frac{{y_i}^{\alpha}}{\delta}\right) - \sum_{i=1}^{n} \delta_{2i} \ \log\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right)$$

$$- \sum_{i=1}^{n} \delta_{3i} \ \log\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right),$$

$$\frac{\partial l}{\partial \delta} = -\frac{2n_1 + 2n_2 + n_3}{\delta} + (\vartheta_1 + 1) \sum_{i=1}^{n} \delta_{1i} \ \frac{{x_i}^{\alpha}/\delta^2}{\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right)} + (\vartheta_{23} + 1) \sum_{i=1}^{n} \delta_{1i} \ \frac{{y_i}^{\alpha}/\delta^2}{\left(1 + \frac{{y_i}^{\alpha}}{\delta}\right)}$$

$$+ (\vartheta_{13} + 1) \sum_{i=1}^{n} \delta_{2i} \ \frac{{x_i}^{\alpha}/\delta^2}{\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right)} + (\vartheta_2 + 1) \sum_{i=1}^{n} \delta_{2i} \ \frac{{y_i}^{\alpha}/\delta^2}{\left(1 + \frac{{y_i}^{\alpha}}{\delta}\right)} + (\vartheta_{123} + 1) \sum_{i=1}^{n} \delta_{2i} \ \frac{{x_i}^{\alpha}/\delta^2}{\left(1 + \frac{{x_i}^{\alpha}}{\delta}\right)},$$

$$\frac{\partial l}{\partial \alpha} = \frac{2n_1 + 2n_2 + n_3}{\alpha} + \sum_{i=1}^{n} (\delta_{1i} + \delta_{2i}) \ log x_i y_i + \delta_{3i} \ log x_i \text{-} (\vartheta_1 + 1) \sum_{i=1}^{n} \delta_{1i} \ \frac{{x_i}^{\alpha} \log x_i}{\delta + {x_i}^{\alpha}}$$

$$\text{-}(\vartheta_{23} + 1) \sum_{i=1}^{n} \delta_{1i} \ \frac{{y_i}^{\alpha} \log y_i}{\delta + {y_i}^{\alpha}} - (\vartheta_{13} + 1) \sum_{i=1}^{n} \delta_{2i} \ \frac{{x_i}^{\alpha} \log \ _i}{\delta + {x_i}^{\alpha}}$$

$$\text{-}(\vartheta_2 + 1) \sum_{i=1}^{n} \delta_{2i} \ \frac{{y_i}^{\alpha} \mathrm{lo} \ \ _i}{\delta + {y_i}^{\alpha}} \text{-} (\vartheta_{123} + 1) \sum_{i=1}^{n} \delta_{3i} \ \frac{{x_i}^{\alpha} \log x_i}{\delta + {x_i}^{\alpha}}.$$

### 3.2. MLE based on Ranked Set Sampling

In this sub-section, we discussed the MLE for BGB distribution parameters based on RSS as explained in Algorithm 2. Assume $[(x_{(i)j}, y_{[i]j}), i = 1,2, \dots, m \ , j = 1,2, \dots, r]$ denote RSS from $BGB(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$ distribution, for simplicity, assume $x_{ij} = x_{(i)j}$ and $y_{ij} = y_{[i]j}$ .

| **Algorithm 2: Generate RSS from BGB distribution** |
|---|
| **Step 1.** Generate a bivariate random sample $(X_i, Y_i), i = 1,2, \dots, m^2$ using Algorithm 1 for $j^{th}$ cycle.<br>**Step 2.** Divide the units in the sample randomly into *m* sets of size *m* each<br>**Step3.** Rank the units in each set from the smallest to the largest according to the variable *X*<br>**Step 4.** Select the order statistics $X_{(i)}$ and its concomitant $Y_{[i]}$ from the $i^{th}$ set<br>**Step 5.** Repeat Steps 1-4 *r* times if we need to obtain a sample of size $n = mr$.<br>$[(X_{(i)j}, Y_{[i]j}), i = 1,2, \dots, m \ , j = 1,2, \dots, r$.<br>Where $X_{(i)j}$ is the $i^{th}$ order statistic of *X* in the $j^{th}$ cycle and $Y_{[i]j}$ be its concomitant of $Y$. |

The likelihood function for the RSS $\{[(X_{(i)j}, Y_{[i]j}), i = 1,2, \dots, m \ , j = 1,2, \dots, r\}$ of size $n = rm$

is given as:

$$L(\Theta) \propto \prod_{j=1}^{r} \prod_{i=1}^{m} [f_1(x_{ij}, y_{ij})]^{\delta_{1i}} [f_2(x_{ij}, y_{ij})]^{\delta_{2i}} [f_3(x_{ij})]^{\delta_{3i}} [F((x_{ij})]^{i-1} [\bar{F}(x_{ij})]^{m-i} .$$

The log-likelihood function $l(\Theta) = log\, L(\Theta)$ of the RSS of size $n = r\, m$ from the BGB distribution is given by:

$$\begin{aligned} l(\Theta) &= (2rm_1 + 2rm_2 + rm_3) log\alpha - (2rm_1 + 2rm_2 + rm_3) log\delta + rm_1 log\vartheta_1 \\ &+rm_1 log\vartheta_{23} + rm_2 log\vartheta_2 + rm_2 log\vartheta_{13} + rm_3 log\vartheta_3 \\ &+(\alpha - 1)\{\textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} (\delta_{1i} + \delta_{2i} + \delta_{3i}) \, logx_{ij} + (\delta_{1i} + \delta_{2i}) \, logy_{ij}\} \\ &-(\vartheta_1 + 1) \textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{1i} \, \log(1 + \frac{x_{ij}{}^{\alpha}}{\delta}) - (\vartheta_{23} + 1) \sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{1i} \, \log(1 + \frac{y_{ij}{}^{\alpha}}{\delta}) \\ &-(\vartheta_{13} + 1) \textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{2i} \, \log(1 + \frac{x_{ij}{}^{\alpha}}{\delta}) - (\vartheta_2 + 1) \sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{2i} \, \log(1 + \frac{y_{ij}{}^{\alpha}}{\delta}) \\ &-(\vartheta_{123} + 1) \textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{3i} \, \log(1 + \frac{x_{ij}{}^{\alpha}}{\delta}) \\ &-(\vartheta_{13} + 1) \textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} (m - i) \log(1 + \frac{x_{ij}{}^{\alpha}}{\delta}) \\ &+\textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} (i - 1) log\, [1 - (1 + \frac{x_{ij}{}^{\alpha}}{\delta})^{-\vartheta_{13}}]. \end{aligned} \tag{2}$$

The first derivatives of the log- likelihood function with respect to $\vartheta_1, \vartheta_2, \vartheta_3$, $\alpha$ and $\delta$ are as follows

$$\begin{aligned} \frac{\partial l}{\partial \vartheta_1} &= \frac{r\, m_1}{\vartheta_1} + \frac{r\, m_2}{\vartheta_{13}} - \textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{1i} \, \log\left(1 + \frac{x_{ij}{}^{\alpha}}{\delta}\right) - \sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{2i} \, \log\left(1 + \frac{x_{ij}{}^{\alpha}}{\delta}\right) \\ &- \textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{3i} \, \log\left(1 + \frac{x_{ij}{}^{\alpha}}{\delta}\right) - \sum_{j=1}^{r} \sum_{i=1}^{m} (m - i) \log\left(1 + \frac{x_{ij}{}^{\alpha}}{\delta}\right) \\ &- \textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} (i - 1) \frac{(1+\frac{x_{ij}{}^{\alpha}}{\delta})^{-\vartheta_{13}} \log\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right)}{1-\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right)^{-\vartheta_{13}}}, \end{aligned}$$

$$\begin{aligned} \frac{\partial l}{\partial \vartheta_2} &= \frac{r\, m_1}{\vartheta_{23}} + \frac{r\, m_2}{\vartheta_2} - \textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{1i} \, \log\left(1 + \frac{y_{ij}{}^{\alpha}}{\delta}\right) - \sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{2i} \, \log\left(1 + \frac{y_{ij}{}^{\alpha}}{\delta}\right) \\ &- \textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{3i} \, \log\left(1 + \frac{x_{ij}{}^{\alpha}}{\delta}\right), \end{aligned}$$

$$\frac{\partial l}{\partial \vartheta_3} = \frac{r\, m_1}{\vartheta_{23}} + \frac{r\, m_2}{\vartheta_{13}} + \frac{r\, m_3}{\vartheta_3} - \textstyle\sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{1i} \, \log\left(1 + \frac{y_{ij}{}^{\alpha}}{\delta}\right) - \sum_{j=1}^{r} \sum_{i=1}^{m} \delta_{2i} \, \log\left(1 + \frac{x_{ij}{}^{\alpha}}{\delta}\right)$$

$$-\sum_{j=1}^{r}\sum_{i=1}^{m}\delta_{3i}\ \log\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right)-\sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\log\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right)$$

$$-\sum_{j=1}^{r}\sum_{i=1}^{m}(i-1)\frac{\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right)^{-\vartheta_{13}}\log\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right)}{1-\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right)^{-\vartheta_{13}}}.$$

## 4. Bayesian Estimation for the BGB model

This section deals with Bayesian estimation for the BGB distribution parameters based on RSS and SRS. Unlike those of MLEs, the Bayes estimators are obtained explicitly under the squared error loss function for both cases.

### 4.1. Bayesian Estimation based on Ranked Set Sampling

The Bayesian estimators for the BGB distribution are derived explicitly in the case of RSS in this sub-section as follows.

**The prior assumption:**

When the shape parameter $\alpha$ is known, we assume the same conjugate prior on $\vartheta_1, \vartheta_2$ and $\vartheta_3$

Assume $\vartheta_1, \vartheta_2$ and $\vartheta_3$ are independent, and distributed as gamma as following

$$\pi_i(\lambda_i)=\frac{b^{a_i}}{\Gamma(a_i)}\ \vartheta_i{}^{a_i-1}e^{-b\vartheta_i}\ , i=1,2,3\ , \vartheta_i>0\ .$$

The joint prior density of $\vartheta_1, \vartheta_2$ and $\vartheta_3$,

$$\pi_0(\vartheta_1,\vartheta_2,\vartheta_3)=\prod_{i=1}^{3}\frac{b^{a_i}}{\Gamma(a_i)}\ \vartheta_i{}^{a_i-1}e^{-b\,\vartheta_i}.$$

**Posterior Analysis and Bayesian Inference**

Assume we have a bivariate RSS from $BGB(\vartheta_1,\vartheta_2,\vartheta_3,\alpha,\delta)$ distribution, and it is denoted as follows

$D=[(x_{(i)j},y_{[i]j}), i=1,2,\dots,m\ ,\ j=1,2,\dots,r]$. Again, for simplicity, assume $x_{ij}=x_{(i)j}$ and $y_{ij}=y_{[i]j}$ .

Let $m=m_1+m_2+m_3$ , $\vartheta_{123}=\vartheta_1+\vartheta_2+\vartheta_3$ , $\vartheta_{13}=\vartheta_1+\vartheta_3$ and $\vartheta_{23}=\vartheta_2+\vartheta_3$.

Then the likelihood function given in Equation (2) can be rewritten as

$$L(D\backslash\Theta)=\ Exp(\log L(D\backslash\Theta))$$

$$L(D\backslash\Theta)\propto\sum_{k=1}^{rm_1}\sum_{s=1}^{rm_2}\binom{rm_1}{k}\binom{rm_2}{s}\prod_{j=1}^{r}\prod_{i=1}^{m}\sum_{l=1}^{i-1}(-1)^l\binom{m-i}{l}$$
$$.\vartheta_1{}^{rm_2+s}\ e^{-\vartheta_1T_1(\alpha,\delta)}\vartheta_2{}^{rm_2+k}e^{-\vartheta_2T_2(\alpha,\delta)}.\vartheta_3{}^{rm-k-}\ e^{-\vartheta_3T_3(\alpha,\delta)}.$$

Where

$$T_1(\alpha,\delta)=Z_1(\alpha,\delta)+Z_3(\alpha,\delta)+Z_4(\alpha,\delta)+Z_5(\alpha,\delta)\ ,$$
$$T_2(\alpha,\delta)=Z_2(\alpha,\delta)+Z_3(\alpha,\delta)+Z_4(\alpha,\delta)\ ,$$
$$T_3(\alpha,\delta)=Z_2(\alpha,\delta)+Z_3(\alpha,\delta)+Z_4(\alpha,\delta)+Z_5(\alpha,\delta)\ ,$$
$$Z_1(\alpha,\delta)=\sum_{j=1}^{r}\sum_{i=1}^{m}\delta_{1i}\log\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right), Z_2(\alpha,\delta)=\sum_{j=1}^{r}\sum_{i=1}^{m}\delta_{1i}\log\left(1+\frac{y_{ij}{}^{\alpha}}{\delta}\right),$$

$$Z_3(\alpha,\delta)=\sum_{j=1}^{r}\sum_{i=1}^{m}\delta_{3i}\log\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right),\ Z_4(\alpha,\delta)=\sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\log\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right),$$
$$Z_5(\alpha,\delta)=(i-l-1)\log\left(1+\frac{x_{ij}{}^{\alpha}}{\delta}\right).$$

Since Bayes theory states that: $f(D,\Theta)=\ L(D\backslash\Theta)\,\pi_0(\Theta)$

and since $\quad f(D)=\int f(D\backslash\Theta)d\,\Theta=\int \pi_0(\Theta)L(D\backslash\Theta)d\,\Theta.$

Hence, the joint posterior density function of $\Theta=(\vartheta_1,\vartheta_2,\vartheta_3,\alpha,\delta)$ given the data D, denoted by $\pi_1(\ \Theta\backslash D)$ can be written as

$$\pi_1(\ \Theta\backslash D)\ =\frac{f(D,\Theta)}{f(D)},$$

$$\pi_1(\ \Theta\backslash D)\propto A.\sum_{k=1}^{rm_1}\ \sum_{s=1}^{rm_2}\binom{rm_1}{k}\binom{rm_2}{s}\prod_{j=1}^{r}\prod_{i=1}^{m}\sum_{l=1}^{i-1}(-1)^l\binom{m-i}{l}$$
$$.\vartheta_1{}^{a_{1s}-1}\ e^{-\vartheta_1[T_1(\alpha,\delta)+b_1]}\vartheta_2{}^{a_{2k}-1}e^{-\vartheta_2[T_2(\alpha,\delta)+b_2]}.\vartheta_3{}^{a_{3s}-1}e^{-\vartheta_3[T_3(\alpha,\delta)+b_3]}. \qquad (3)$$

Where

$$A=\frac{1}{\sum_{k=1}^{rm_1}\sum_{s=1}^{rm_2}\prod_{j=1}^{r}\prod_{i=1}^{m}\sum_{l=1}^{i-1}A_{iksl}},$$

and $A_{iksl}\ =(-1)^l\binom{rm_1}{k}\binom{rm_2}{s}\binom{m-i}{l}.\frac{\Gamma(a_{1s})}{[b_1+T_1(\alpha,\delta)]^{a_{1s}}}.\frac{\Gamma a_{2k})}{[b_2+T_2(\alpha,\delta)]^{a_{2k}}}.\frac{\Gamma(a_{3s}\ )}{[b_3+T_3(\alpha,\delta)]^{a_{3s}}},$

$a_{1s}=a_1+rm_1+s,$
$a_{2k}\ =a_2+rm_2+k,$
and $a_{3sk}=a_3+rm-k-s.$

Therefore, under the assumption of independence of $\vartheta_1,\vartheta_2$and $\vartheta_3$ and $\alpha$ and $\delta$ are assumed to be known. It is possible to get the Bayes estimators of $\vartheta_1,\vartheta_2$and $\vartheta_3$ explicitly under the square error loss function using Equation (3), and they will be as follows:

$$\check{\vartheta}_1=\frac{A}{b_1+T_1}\sum_{k=1}^{rm_1}\sum_{s=1}^{rm_2}\prod_{j=1}^{r}\prod_{i=1}^{m}\sum_{l=1}^{i-1}A_{iksl}\ \ a_{1s},$$

$$\check{\vartheta}_2=\frac{A}{b_2+T_2}\sum_{k=1}^{rm_1}\sum_{s=1}^{rm_2}\prod_{j=1}^{r}\prod_{i=1}^{m}\sum_{l=1}^{i-1}A_{iksl}\ \ a_{2k},$$

and

$$\check{\vartheta}_3=\frac{A}{b_3+T_3}\sum_{k=1}^{rm_1}\sum_{s=1}^{rm_2}\prod_{j=1}^{r}\prod_{i=1}^{m}\sum_{l=1}^{i-1}A_{iksl}\ \ a_{3sk}.$$

### 4.2. Bayesian Estimation based on Simple Random Samples

In this sub-section, the Bayesian estimators are derived explicitly for the BGB distribution under the square error loss function in the case of SRS

We consider the same prior assumption as in the RSS case with known parameters $\alpha$ and $\delta$

So, the joint prior density of the independent parameters $\vartheta_1,\vartheta_2$ and $\vartheta_3$, is given as

$$\pi_0(\vartheta_1,\vartheta_2,\vartheta_3)=\prod_{i=1}^{3}\frac{b^{a_i}}{\Gamma(a_i)}\ \vartheta_i{}^{a_i-1}e^{-b\,\vartheta_i}.$$

Assume we have a bivariate SRS from $BGB(\vartheta_1, \vartheta_2, \vartheta_3, \alpha, \delta)$ distribution, and it is denoted as

$$D = [(x_i, y_i), i = 1,2, \dots, n].$$

Let $n = n_1 + n_2 + n_3$ , $\vartheta_{123} = \vartheta_1 + \vartheta_2 + \vartheta_3$ , $\vartheta_{13} = \vartheta_1 + \vartheta_3, \vartheta_{23} = \vartheta_2 + \vartheta_3$ and $\Theta = (\vartheta_1, \vartheta_2, \vartheta_3)$.

Then the likelihood function given in Equation (1) can be rewritten as

$$L(D\backslash\Theta) = Exp(\log L(D\backslash\Theta))$$

$$L(D\backslash\Theta) = \sum_{s=1}^{n_2} \sum_{k=1}^{n_1} \binom{n_1}{k} \binom{n_2}{s} . \vartheta_1{}^{n_1+s} e^{-\vartheta_1 T_1(\alpha,\delta)} . \vartheta_2{}^{n_2+k} e^{-\vartheta_2 T_2(\alpha,\delta)} . \vartheta_3{}^{n-k-s} e^{-\vartheta_3 T_3(\alpha,\delta)},$$

Where

$$T_1(\alpha, \delta) = Z_1(\alpha, \delta) + Z_3(\alpha, \delta) + Z_5(\alpha, \delta) ,$$
$$T_2(\alpha, \delta) = Z_2(\alpha, \delta) + Z_4(\alpha, \delta) + Z_5(\alpha, \delta) ,$$
$$T_3(\alpha, \delta) = Z_3(\alpha, \delta) + Z_4(\alpha, \delta) + Z_5(\alpha, \delta) ,$$
$$Z_1(\alpha, \delta) = \sum_{i=1}^{n} \delta_{1i} \log\left(1 + \frac{x_i^\alpha}{\delta}\right), Z_2(\alpha, \delta) = \sum_{i=1}^{n} \delta_{2i} \log\left(1 + \frac{y_i^\alpha}{\delta}\right),$$
$$Z_3(\alpha, \delta) = \sum_{i=1}^{n} \delta_{2i} \log\left(1 + \frac{x_i^\alpha}{\delta}\right), Z_4(\alpha, \delta) = \sum_{i=1}^{n} \delta_{1i} \log\left(1 + \frac{x_i^\alpha}{\delta}\right),$$
$$Z_5(\alpha, \delta) = \sum_{i=1}^{n} \delta_{3i} \log\left(1 + \frac{x_i^\alpha}{\delta}\right).$$

Furthermore, the joint posterior density function of $\Theta$ given the data D, denoted by $\pi_1(\Theta\backslash D)$ can be written by gamma densities as follows

$$\pi_1(\Theta\backslash D) \propto A \sum_{s=1}^{n_2} \sum_{k=1}^{n_1} A_{ks} \; Gamma[\vartheta_1; a_{1s}, \; b_1 + T_1(\alpha, \delta)]. \\ .Gamma\,[\vartheta_2;\, a_{2k}\,,\; b_2 + T_2(\alpha, \delta)].\, Gamma\,[\vartheta_3\,;\, a_{3k}\,,\; b_3 + T_3(\alpha, \delta)], \tag{4}$$

where $A = \frac{1}{\sum_{s=1}^{n_2} \sum_{k=1}^{n_1} A_{ks}}$,

$$A_{ks} = \binom{n_1}{k} \binom{n_2}{s} . \frac{\Gamma(a_{1k})}{[b_1 + T_1(\alpha, \delta)]^{a_{1s}}} . \frac{\Gamma a_{2s})}{[b_2 + T_2(\alpha, \delta)]^{a_{2k}}} . \frac{\Gamma(a_{3s}\;)}{[b_3 + T_3(\alpha, \delta)]^{a_{3sk}}},$$
$$a_{1s} = a_1 + n_1 + s,$$
$$a_{2k} = a_2 + n_2 + k,$$
and $a_{3sk} = a_3 + n - s - k$.

Hence, using Equation (4), the Bayes estimators of $\vartheta_1, \vartheta_2$and $\vartheta_3$ are obtained explicitly under the square error loss function based on SRS as follows:

$$\check{\vartheta}_1 = \frac{A}{b_1 + T_1} \sum_{s=1}^{n_2} \sum_{k=1}^{n_1} A_{ks} \; a_{1s},$$

$$\check{\vartheta}_2 = \frac{A}{b_2 + T_2} \sum_{s=1}^{n_2} \sum_{k=1}^{n_1} A_{ks} \; a_{2k},$$

and

$$\check{\vartheta}_3 = \frac{A}{b_3 + T_3} \sum_{s=1}^{n_2} \sum_{k=1}^{n_1} A_{ks} \; a_{3sk}.$$

## 5. Conclusions

Based on ranked set samples, this paper aims to deriving the likelihood function and to full Bayesian and non- Bayesian estimation procedures for the dependent variables described by using Marshal – Olkin bivariate models in general and considering Marshall–Olkin bivariate generalized Burr distribution especially. The Bayesian estimates are obtained in explicit forms. But non-Bayesian ones cannot be obtained in explicit form.

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