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A New Generalization of Weighted Remkan Distribution Properties and Its Applications

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ABSTRACT

In this paper, a new three-parameter distribution, known as the Weighted Remkan Distribution. It extensively explores the probability density function (PDF), the cumulative distribution function (CDF), and its statistical properties, including the moment-generating function (MGF), various entropy measures, order statistics, Bonferroni and Lorenz curves, harmonic mean, survival and hazard functions, reverse hazard rate, and moments. The maximum likelihood estimation (MLE) method is utilized for parameter estimation. The application of the distribution is demonstrated using serum creatinine data from diabetic patients with chronic kidney disease data sets, and the goodness of fit for the weighted Remkan distribution has been compared with other existing distributions.

Keywords: Weighted Remkan Distribution, Moments, Bonferroni and Lorenz Curves, Serum Creatinine, Diabetic patients

1. Introduction

The concept of weighted distribution was given by Fisher (1934) later it was modified by Roa (1965) in a Unified manner, whereby weighted distribution many situations can be solved. As a result, weighted models were formulated in such situations to record the observations according to some weighted function. The weighted distribution reduces to length biased distribution as the weight function considers only the length of the units. The concept of length biased sampling, was first introduced by Cox (1969) and Zelen (1974). Van Deusen in (1986) arrived at Size-biased distribution theory. The statistical distributions are crucial in engineering, economics, biomedicine, medical science and other fields of study. Uwaeme et.al. and Akpan et. al. in 2024 proposed as Remkan distribution with its statistical properties and its applications. This investigates the goodness of fit of the weighted Remkan distribution using a real life data. The Remkan distribution have been better fit to complex real life data set. some recently developed new parameter distribution includes the Hamza, Samade, Alzoubi (Benrabia and Alzoubia), Darna, and Copoun distribution. Remkan distribution with three parameters (η, ϕ, c). This distribution is a three-component density of an exponential (η), gamma (3, ηx) distribution and gamma (4, ηx) distribution. A new generalization of the Remkan distribution, with applications to real-life scenarios, has been studied. Its statistical properties, including shapes for varying values of parameters, Survival function, hazard function, reverse hazard function, moments, and moment-generating function, have been examined. Finally, we discuss, the practical relevance of the proposed model is demonstrated through an application to a real medical dataset consisting of serum creatinine measurements from diabetic patients with chronic kidney disease (CKD). Serum creatinine is a critical biomarker for assessing kidney function, and its accurate modeling can aid in disease monitoring and prognosis. A comparative goodness-of-fit analysis shows that the

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Weighted Remkan distribution provides a better fit to this dataset than both the Area-biased Remkan distribution and the Remkan distribution.

The Probability density function of the Remkan distribution is given by

$$f(x;\eta,\phi)=\frac{\eta^2}{(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x};x>0,\ \eta>0,\ \phi>0 \tag{1}$$

The Cumulative Distribution Function of Remkan Distribution is given by

$$F(x;\eta,\phi)=1-\left[1+\frac{\eta^2x^3+(3+\phi)\eta^2x^3+(6+\phi)\eta x}{\eta+2\phi+6}\right]e^{-\eta} \tag{2}$$

2. Weighted Remkan Distribution

Suppose X is a non- negative random variable with probability density function f (x). Let w(x) be the non- negative weight function, then the probability density function of the weighted random variable X_w is given by

$$f_w(x)=\frac{w(x)f(x)}{E(w(x))}$$

where, the Weight function $w(x)$, that is non-negative and $E(w(x))=\int w(x)f(x)dx<\infty$.

In this paper, we consider the weight function as $w(x)=x^c$ to obtain the weighted Aradhana distribution. The probability density function of weighted Remkan distribution is given as:

$$f_w(x)=\frac{x^c}{E(x^c)} \tag{3}$$

where,

$$E(x^c)=\int_0^\infty x^c\,f(x;\eta,\phi)dx$$

$$E(x^c)=\int_0^\infty x^c\,\frac{\eta^2}{(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}dx$$

$$=\frac{\eta^2}{(\eta+2\phi+6)}\int_0^\infty x^c\,[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}dx$$

$$=\frac{\eta^2}{(\eta+2\phi+6)}\int_0^\infty x^c\,e^{-\eta x}dx+\phi\eta\int_0^\infty x^{c+2}\,e^{-\eta x}dx+\eta^2\int_0^\infty x^{c+3}\,e^{-\eta x}dx$$

$$=\frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{\Gamma c+1}{\eta^{c+1}}+\phi\eta\,\frac{\Gamma c+3}{\eta^{c+3}}+\eta^2\,\frac{\Gamma c+4}{\eta^{c+4}}\right]$$

$$=\frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{\Gamma c+1}{\eta^{c+1}}+\phi\,\frac{\Gamma c+3}{\eta^{c+2}}+\frac{\Gamma c+4}{\eta^{c+2}}\right]$$

$$=\frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{\Gamma c+1}{\eta^{c+1}}+\frac{\phi\Gamma c+3\Gamma c+4}{\eta^{c+2}}\right]$$

$$=\frac{\eta^2}{(\eta+2\phi+6)}\left[\frac{\eta\Gamma c+1+\phi\Gamma c+3\Gamma c}{\eta^{c+2}}\right]$$

On Simplification, we get

$$E(x^c) = \frac{\eta^{-c}(\eta c! + \phi(c+2)! + (c+3)!)}{\eta + 2\phi + 6} \tag{4}$$

Substitute (1) and (4) in equation (3), one gets the required probability density function of weighted Remkan distribution as

$$f_w(x;\eta,\phi,c) = \frac{x^c \frac{\eta^2}{(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2 x^3]e^{-\eta x}}{\frac{\eta^{-c}(\eta c!+\phi(c+2)!+(c+3)!)}{\eta+2\phi+6}}$$

$$f_w(x;\eta,\phi,c) = \frac{x^c \eta^2 (1+\phi\eta x^2+\eta^2 x^3)e^{-\eta x}}{\eta^{-c}[\eta c!+\phi(c+2)!+(c+3)!]}$$

$$f_w(x;\eta,\phi,c) = \frac{x^c \eta^{c+2} (1+\phi\eta x^2+\eta^2 x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!} \tag{5}$$

Now, the cumulative Distribution Function of the Weighted Remkan Distribution obtained as

$$F_w(x;\eta,\phi,c) = \int_0^\infty f_w(x;\eta,\phi,c)\,dx \tag{6}$$

$$= \int_0^\infty \frac{x^c \eta^{c+2} (1+\phi\eta x^2+\eta^2 x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!}\,dx$$

$$= \frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!} \int_0^\infty x^c\, e^{-\eta x} dx + \phi\eta \int_0^\infty x^{c+2}\, e^{-\eta x} dx + \eta^2 \int_0^\infty x^{c+3}\, e^{-\eta x} dx$$

where $\eta x = z,\ \eta dx = dz,\ x = \frac{z}{\eta}\ and\ \ dx = \frac{dz}{\eta}$

$$= \frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!} \int_0^{\eta x} \left(\frac{z}{\eta}\right)^c e^{-z}\frac{dz}{\eta} + \phi\eta \int_0^{\eta x} \left(\frac{z}{\eta}\right)^{c+2} e^{-z}\frac{dz}{\eta} + \eta^2 \int_0^{\eta x} \left(\frac{z}{\eta}\right)^{c+3} e^{-z}\frac{dz}{\eta}$$

$$= \frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!} \left[\left(\frac{1}{\eta^{c+1}}\right) \int_0^{\eta x} z^{(c+1)-1}\, e^{-z} dz + \phi\left(\frac{1}{\eta^{c+3}}\right) \int_0^{\eta x} \left(\frac{z}{\eta}\right)^{(c+3)-1} e^{-z} dz + \left(\frac{1}{\eta^{c+2}}\right) \int_0^{\eta x} \left(\frac{z}{\eta}\right)^{(c+4)-1} e^{-z} dz\right]$$

$$= \frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!} \left[\left(\frac{1}{\eta^{c+1}}\right)(\gamma(c+1),\eta x) + \phi\left(\frac{1}{\eta^{c+3}}\right)(\gamma(c+3),\eta x) + \left(\frac{1}{\eta^{c+2}}\right)(\gamma(c+4),\eta x)\right]$$

After simplification, one gets the cumulative distribution function of weighted Remkan distribution (WRD) as given below

$$F_w\,(x;\eta,\phi,c) = \frac{\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}{\eta c!+\phi(c+2)!+(c+3)!} \tag{7}$$

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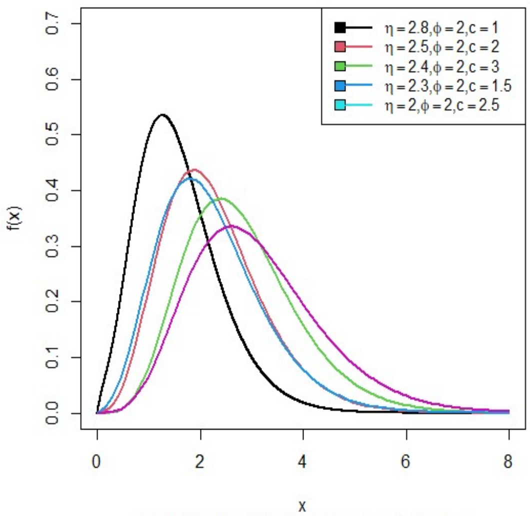

Fig.1:Pdf plot of Weighted Remkan distribution

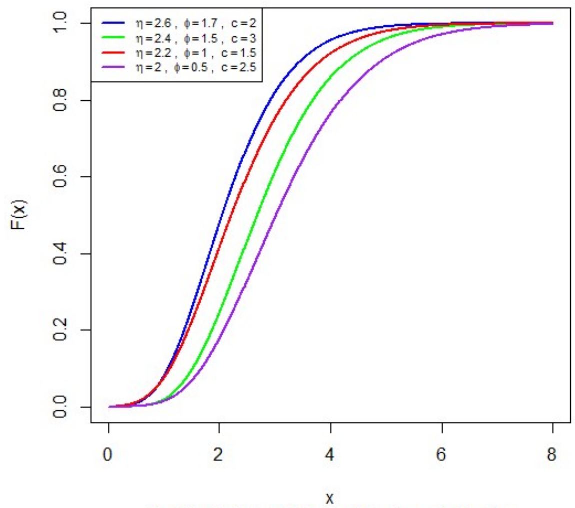

Fig.2 CDF plot of Weighted Remkan Distribution

3. Reliability Analysis

In this section, we discuss about the survival function, Hazard function, reverse hazard rate, cumulative hazard rate function, Odds rates function and Mills ratio of the weighted Remkan distribution (WRD)

Survival Function:

The survival function or the reliability function of the Weighted Remkan Distribution is provided by

$$S(x;\eta,\phi,c) = 1 - F_w(x;\eta,\phi,c)$$

$$= 1 - \frac{\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}{\eta c!+\phi(c+2)!+(c+3)!} \quad (8)$$

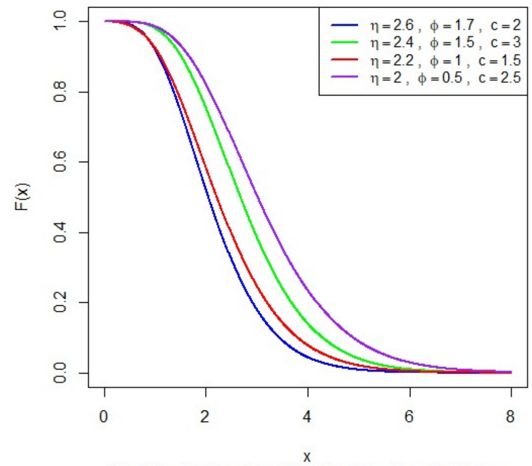

Fig.3 Survival Function of Weighted Remkan Distribution

Hazard Function (or) The Failure Rate

The Hazard function is also known as the hazard rate, instantaneous failure rate or force of mortality and is given by

$$H(x;\eta,\phi,c)=\frac{f_W(x;\eta,\phi,c)}{S(x;\eta,\phi,c)}$$

$$=\frac{\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!}}{\frac{\eta c!+\phi(c+2)!+(c+3)!-\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}{\eta c!+\phi(c+2)!+(c+3)!}}$$

$$=\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!-\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)} \quad (9)$$

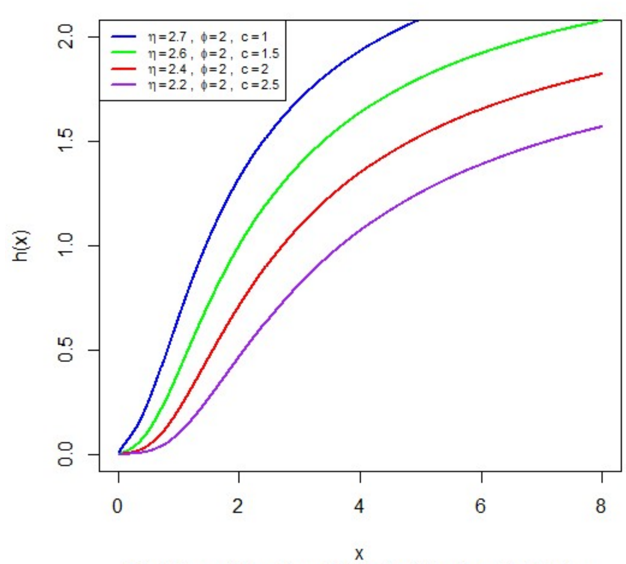

Fig.4 Hazard Function of Weighted Remkan Distribution

Reverse Hazard Function

The Reverse hazard rate function is given by

$$h_r(x;\eta,\phi,c)=\frac{f_w(x;\eta,\phi,c)}{F_w(x;\eta,\phi,c)}$$

$$=\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta}}{\eta\gamma(c+1,\eta x)+\phi\ \ (c+3,\eta x)+\gamma(c+4,\eta x)} \tag{10}$$

Odds Rates Function

The Odds Rates Function is given by

$$O(x;\eta,\phi,c)=\frac{F_w(x;\eta,\phi,c)}{1-F_w(x;\eta,\phi,c)}$$

$$=\frac{\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}{\eta c!+\phi(c+2)!+(c+3)!-\eta\ \ (c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)} \tag{11}$$

Cumulative Hazard Rate Function

The Cumulative Hazard Rate Function is given by

$$H_r(x;\eta,\phi,c)=-\ln\left(1-F_w(x;\eta,\phi,c)\right)$$

$$=-\ln\left(1-\frac{\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}{\eta c!+\phi(c+2)!+(c+3)!}\right) \tag{12}$$

Mills Ratio

The Mills Ratio of the weighted Remkan distribution is

$$M_r(x;\eta,\phi,c)=\frac{1}{h_r(x;\eta,\phi,c)}$$

$$=\frac{1}{\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}}{\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}}$$

$$=\frac{\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}} \tag{13}$$

Statistical Properties

In this section we discuss some statistical properties of weighted Remkan Distribution

Moments

Let X denotes the random variable of weighted Remkan distribution with parameters η,ϕ *and* c, then the r[th] order moment $E(X^r)$ of weigthed Remkan distribution is obtained as

$$E(X^r)=\mu_r'=\int_0^\infty x^r\,f_w(x;\eta,\phi,c)dx$$

$$\mu_r'=\int_0^\infty x^r\,\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!}dx$$

$$=\frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\int_0^\infty x^{r+c}\,e^{-\eta x}+\phi\eta\int_0^\infty x^{r+c+2}\,e^{-\eta x}+\eta^2\int_0^\infty x^{r+c+3}\,e^{-\eta x}dx$$

$$=\frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\left[\frac{\Gamma r+c+1}{\eta^{(r+c+1)}}+\phi\eta\frac{\Gamma r+c+3}{\eta^{(r+c+3)}}+\eta^2\frac{\Gamma r+c+4}{\eta^{(r+c+4)}}\right]$$

$$=\frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\left[\frac{\Gamma r+c+1}{\eta^{(r+c+1)}}+\phi\frac{\Gamma r+c+3}{\eta^{(r+c+2)}}+\frac{\Gamma r+c+4}{\eta^{(r+c+2)}}\right]$$

$$=\frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\left[\frac{\Gamma r+c+1}{\eta^{(r+c+1)}}+\frac{\phi\Gamma r+c+3+\Gamma r+c+}{\eta^{(r+c+2)}}\right]$$

$$=\frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\left[\frac{\eta\Gamma r+c+1+\phi\Gamma r+c+3+\Gamma r+c+4}{\eta^{(r+c+2)}}\right]$$

$$\mu_r'=\frac{\eta\Gamma r+c+1+\phi\Gamma r+c+3+\Gamma r}{\eta^r(\eta c!+\phi(c+2)!+(c+3)!)} \tag{14}$$

By Substituting values for r = 1,2, 3… in equation (14) one may get the distribution's moments

$$\mu_1' = \frac{\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c+5}{\eta(\eta c!+\phi(c+2)!+(c+3)!)}$$
$$\mu_2' = \frac{\eta\Gamma c+3+\phi\Gamma c+5+\Gamma c+}{\eta^2(\eta c!+\phi(c+2)!+(c+3)!)}$$
$$\mu_3' = \frac{\eta\Gamma c+4+\phi\Gamma c+6+\Gamma c+7}{\eta^2(\eta c!+\phi(c+2)!+(c+3)!)}$$

Therefore, Variance

$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$\mu_2 = \frac{\eta\Gamma c+3+\phi\Gamma c+5+\Gamma c+6}{\eta^2(\eta c!+\phi(c+2)!+(c+3)!)} - \left(\frac{\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c+5}{\eta(\eta c!+\phi(c+2)!+(c+3)!)}\right)^2 \quad (15)$$
$$\text{S. D}(\sigma) = \sqrt{variance}$$
$$\text{S. D}(\sigma) = \sqrt{\frac{\eta\Gamma c+3+\phi\Gamma c+5+\Gamma c+6}{\eta^2(\eta c!+\phi(c+2)!+(c+3)!)} - \left(\frac{\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c+5}{\eta(\eta c!+\phi(c+2)!+(c+3)!)}\right)} \quad (16)$$

Harmonic Mean

The Harmonic mean of the proposed model can be obtained as

$$E\left(\frac{1}{x}\right) = \int_0^\infty \frac{1}{x} f_w(x;\eta,\phi,c)dx$$
$$= \int_0^\infty \frac{1}{x}\left(\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!}\right)dx$$
$$= \frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\int_0^\infty x^{c-1}\, e^{-\eta x} + \phi\eta\int_0^\infty x^{c+1}\, e^{-\eta x} + \eta^2\int_0^\infty x^{c+2}\, e^{-\eta x}dx$$
$$= \frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\int_0^\infty x^{(c-1)+1}\, e^{-\eta x} + \phi\eta\int_0^\infty x^{(c+2)-1}\, e^{-\eta x} + \eta^2\int_0^\infty x^{(c+3)-1}\, e^{-\eta x}dx$$
$$= \frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\left[\frac{\Gamma c}{\eta^c} + \frac{\phi\eta\Gamma c+2}{\eta^{c+2}} + \frac{\eta^2\Gamma c+3}{\eta^{c+3}}\right]$$
$$= \frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\left[\frac{\Gamma c}{\eta^c} + \frac{\phi\Gamma c+}{\eta^{c+1}} + \frac{\Gamma c+3}{\eta^{c+1}}\right]$$
$$= \frac{\eta^2\Gamma c++\phi\Gamma c+5}{\eta c!+\phi(c+2)!+(c+3)!} \quad (17)$$

Moment Generating Function and Characteristic Function of weighted Remkan distribution

Let X have a weighted Remkan distribution, then the MGF of X is obtained as

$$M_X(t) = E(e^{tx})$$
$$= \int_0^\infty e^{tx}\ f(x;\eta,\phi)dx$$

Using Taylor's Series Expansion

$$M_{X_i}(t) = \int_0^\infty \left[1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right]$$
$$M_X(t) = \sum_{r=0}^\infty \left(\frac{t^r}{r!}\right)\int_0^\infty\ f(x;\eta,\phi)dx$$
$$M_X(t) = \sum_{r=0}^\infty \left(\frac{t^r}{r!}\right)\mu_r'$$
$$M_X(t) = \sum_{r=0}^\infty \left(\frac{t^r}{r!}\right)\left(\frac{\eta\Gamma r+c+1+\phi\Gamma r+c+3+\Gamma r}{\eta^r(\eta c!+\phi(c+2)!+(c+3)!)}\right) \quad (18)$$

Similarly, Characteristic Function of weighted Remkan distribution can be obtained as

$$\phi_X(it) = \left(e^{itx}\right)$$
$$\phi_X(it) = \int_0^\infty e^{itx} f(x;\eta,\phi)dx$$
$$\phi_X(it) = \sum_0^\infty \frac{(it)^r}{r!}\left(\frac{\eta\Gamma r+c+1+\phi\Gamma r+c+3+\Gamma r}{\eta^r(\eta c!+\phi(c+2)!+(c+3)!)}\right) \quad (19)$$

4. Order Statistics

Let $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ be the order statistics of a random sample drawn from the continuous population with probability density function $f_X(x)$and cumulative density function with $F_X(x)$, then the probability density function of r^th^ order statistics $X_{(r)}$ is given by

$$f_{X(r)}(x) = \frac{n!}{(r-1)(n-r)!} f_X(x)[F_X(x)]^{r-1}[1 - F_X(x)]^{n-r}$$

Using the equations (5) and (6) in equation (8), the probability density function of r^th^ order statistics $X_{(r)}$ of weighted Remkan distribution is given by

$$= \frac{n!}{(r-1)(1-1)!}\left(\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!}\right)\left(\frac{\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}{\eta c!+\phi(c+2)!+(c+3)!}\right)^{r-1}$$
$$\left(1 - \frac{\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}{\eta c!+\phi(c+2)!+(c+3)!}\right)^{n-r} \quad (20)$$

Therefore, the probability density function of higher order statistics $X_{(n)}$ can be obtained as

$$f_{X(n)}(x) = \frac{n!}{(r-1)(n-r)!}\left(\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!}\right)\left(\frac{\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}{\eta c!+\phi(c+2)!+(c+3)!}\right)^{n-1}$$ and

the probability density function of first order statistics $X_{(1)}$ can be obtained as

$$f_{X(1)}(x) = \frac{n!}{(1-1)(n-1)!}\left(\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!}\right)\left(1 - \frac{\eta\gamma(c+1,\eta x)+\phi\gamma(c+3,\eta x)+\gamma(c+4,\eta x)}{\eta c!+\phi(c+2)!+(c+3)!}\right)^{n-1}$$

Likelihood Ratio Test

Let $X_1, X_2, \dots, X_n$ be a random sample from the weighted Remkan Distribution. To test the hypothesis

$$H_0: f(x) = f(x;\eta,\phi,c) \qquad \text{against} \qquad H_1: f(x) = f_w(x;\eta,\phi,c)$$

In order to test whether the random sample of size n comes from the Remkan distribution or weighted Remkan distribution, the following test statistic is used

$$\Delta = \frac{L_1}{L_0} = \sum_{i=1}^{n} \frac{f_w(x;\eta,\phi,c)}{f(x;\eta,\phi)}$$
$$= \sum_{i=1}^{n} \frac{\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!}}{\frac{\eta^2}{(\eta+2\phi+6)}[1+\phi\eta x^2+\eta^2x^3]e^{-\eta x}}$$
$$= \sum_{i=1}^{n}\left(\frac{x^c\eta^{c+2}(\eta+2\phi+6)}{\eta c!+\phi(c+2)!+(c+3)!}\right)x_i^c$$

Thus, we reject the null hypothesis if

$$\Lambda = \left(\frac{x^c\eta^{c+2}(\eta+2\phi+6)}{\eta c!+\phi(c+2)!+(c+3)!}\right)^n \prod_{i=1}^{n} x_i^c > k$$

Or $\Lambda^* = \prod_{i=1}^{n} x_i^c > k\left(\frac{x^c\eta^{c+2}(\eta+2\phi+6)}{\eta c!+\phi(c+2)!+(c+3)!}\right)^n$

$$\Lambda^* = \prod_{i=1}^{n} x_i^c > k^*, where k^* = k\left(\frac{x^c\eta^{c+2}(\eta+2\phi+6)}{\eta c!+\phi(c+2)!+(c+3)!}\right)^n$$

Thus, for large sample size n, 2 log Δ is distributed as chi-square distribution with one degree of freedom and also p-value is obtained from the chi-square distribution. Thus, we reject the null hypothesis, when the probability value is $p(\Lambda^* > \beta^*), where\ \beta^* = \prod_{i=1}^{n} x_i^c$ is less than the specified level of significance and $\prod_{i=1}^{n} x_i^c$ is the observed value of the statistics Λ^*

5. Entropies

The concept of entropy is important in different areas such as probability and statistics, physics, communication theory and economics. Entropies quantify the diversity, uncertainty, or randomness of a system. Entropy of a random variable X is a measure of variation of the uncertainty

Renyi Entropy

The Renyi entropy is an important measure in ecology and statistics as index of diversity. The Renyi entropy is also important in quantum information, where it can be used as a measure of entanglement. For a given probability distribution, Renyi entropy is given by

$$R_\lambda = \frac{1}{(1-\lambda)} \log \int_0^\infty \left(f_w(x;\eta,\phi,c)\right)^\lambda dx$$

$$R_\lambda = \frac{1}{(1-\lambda)} \log \int_0^\infty \left(\frac{x^c \eta^{c+2}(1+\phi\eta x^2+\eta^2 x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!}\right)^\lambda dx$$

$$R_\lambda = \frac{1}{(1-\lambda)} \log \left(\frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\right)^\lambda \int_0^\infty x^{c\lambda}(1+\phi\eta x^2+\eta^2 x^3)^\lambda \, e^{-\lambda\eta x} dx$$

Using Binomial Expansion

$$(1+\phi\eta x^2+\eta^2 x^3)^\lambda = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} d^r$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} (b+c)^r$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} \sum_{s=0}^{n} \binom{n}{r} b^s c^{r-s}$$

$$= \sum_{i=0}^{n} \binom{\lambda}{i} \sum_{j=0}^{n} \binom{i}{j} (\phi\eta x^2)^j \, (\eta^2 x^3)^{i-j}$$

$$= \sum_{i=0}^{n} \binom{\lambda}{i} \binom{i}{j} \left(\phi^j \eta^j x^{2j}\right)\left(\eta^{2(i-J)} x^{3(i-j)}\right)$$

$$= \sum_{i=0}^{n} \binom{\lambda}{i} \binom{i}{j} \phi^j \eta^{i-2j} x^{3i-j}$$

$$= \sum_{i=0}^{n} \binom{\lambda}{i} \binom{i}{j} \int_0^\infty x^{c\lambda} \phi^j \eta^{i-2j} x^{3i-j} \, e^{-\lambda\eta x} dx$$

$$= \sum_{i=0}^{n} \binom{\lambda}{i} \binom{i}{j} \phi^j \eta^{i-2j} \int_0^\infty x^{c\lambda+3i-} \quad e^{-\lambda\eta x} dx$$

$$= \sum_{i=0}^{n} \binom{\lambda}{i} \binom{i}{j} \phi^j \eta^{i-2j} \left(\frac{\Gamma c\lambda+3i-j+}{\lambda\eta^{(c\lambda+3i-j+1)}}\right)$$

$$= \sum_{i=0}^{n} \binom{\lambda}{i} \binom{i}{j} \phi^j \eta^{i-2j} \left(\frac{1}{\lambda\eta}\right)^{c\lambda+3i-j+1} \quad \Gamma c\lambda+3i-j+1$$

$$= \frac{1}{(1-\lambda)} \log \left(\frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\right)^\lambda \sum_{i=0}^{n} \binom{\lambda}{i} \binom{i}{j} \phi^j \eta^{i-2j} \left(\frac{1}{\lambda\eta}\right)^{c\lambda+3i-j+1} \quad \Gamma c\lambda+3i-j+1 \tag{21}$$

Tsallis Entropy:

A generalization of Boltzmann-Gibbs (B-G) statistical measures initiated by Tsallis has focused a great deal to attention. This generalization of B-G statistics was proposed firstly by introducing the mathematical expression of Tsallis entropy (Tsallis, 1988) for a continuous random variable is defined as follows

$$T_\delta(x) = \frac{1}{\delta-1}\left(1-\int_0^\infty \left(f_w(x;\eta,\phi,c)\right)^\delta dx\right)$$

$$= \frac{1}{\delta-1}\left(1 - \int_0^\infty \left(\frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta}}{\eta c!+\phi(c+2)!+(c+3)!}\right)^\delta dx\right)$$
$$= \frac{1}{\delta-1}\left(1 - \left(\frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\right)^\delta \int_0^\infty x^{c\delta}\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}dx\right)$$
$$= \frac{1}{\delta-1}\left(1 - \left(\frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\right)^\delta \sum_{i=0}^{n}\binom{\delta}{i}\sum_{j=0}^{n}\binom{i}{j}\phi^j\eta^{i-2j}\left(\frac{1}{\lambda\eta}\right)^{c\delta+3i-j+1} \atop \Gamma c\delta+3i-j+1\right) \tag{22}$$

Bonferroni Curve and Lorenz curve

The Bonferroni and the Lorenz curves are not only used in economics in order to study the income and poverty, but it is also being used in other fields like reliability, medicine, insurance and demography. The Bonferroni and Lorenz curves are given as follows

$$B(P) = \frac{1}{\delta-1}\int_0^\infty x f_w(x;\eta,\phi,c)dx$$

where, $q = F^{-1}(P);\ \mu = E(x)$

Hence the Bonferroni Curve of Remkan distribution is given by

$$\mu = \frac{\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c+5}{\eta(\eta c!+\phi(c+2)!+(c+3)!)}$$
$$B(P) = \frac{\eta(\eta c!+\phi(c+2)!+(c+3)!)}{P\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c+5}\int_0^\infty \frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta}}{\eta c!+\phi(c+2)!+(c+3)!}dx$$
$$= \frac{\eta^{c+3}}{P\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c}\int_0^\infty x^c(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}dx$$
$$= \frac{\eta^{c+3}}{P\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c}\int_0^\infty x^c e^{-\eta x}dx + \phi\eta\int_0^\infty x^{c+2}e^{-\eta x}dx + \eta^2\int_0^\infty x^{c+3}e^{-\eta x}dx$$

Where $x = \frac{t}{\eta}, \eta x = t, dx = \frac{1}{\eta}dt$

$$= \frac{\eta^{c+3}}{P\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c}\int_0^{\eta x}\left(\frac{t}{\eta}\right)^c e^{-t}\frac{dt}{\eta} + \phi\eta\int_0^{\eta x}\left(\frac{t}{\eta}\right)^{c+2} e^{-t}\frac{dt}{\eta} + \eta^2\int_0^{\eta x}\left(\frac{t}{\eta}\right)^{c+3} e^{-t}\frac{dt}{\eta}$$
$$= \frac{\eta^{c+3}}{P\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c} * \frac{1}{\eta^{c+1}}\int_0^{\eta x}(t)^c e^{-t}dt + \phi\frac{1}{\eta^{c+2}}\int_0^{\eta x}(t)^{c+2}e^{-t}dt + \frac{1}{\eta^{c+2}}\int_0^{\eta x}(t)^{c+3}e^{-t}dt$$
$$= \frac{\eta^{c+3}}{P\eta\Gamma c+2+\phi\Gamma c+4} * \frac{1}{\eta^{c+1}}\gamma(c+1,\eta x) + \frac{1}{\eta^{c+2}}\phi\,\gamma(c+3,\eta x) + \frac{1}{\eta^{c+2}}\gamma(c+4,\eta x)$$
$$= \frac{1}{P\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c+} * \frac{\eta^{c+3}}{\eta^{c+1}}\gamma(c+1,\eta x) + \frac{\eta^{c+3}}{\eta^{c+2}}(\phi\,\gamma(c+3,\eta x) + \gamma(c+4,\eta x))$$
$$= \frac{1}{P\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c}\eta^2\gamma(c+1,\eta x) + \eta(\phi\,\gamma(c+3,\eta x) + \gamma(c+4,\eta x))$$
$$= \frac{\eta^2\gamma(c+1,\eta x)+\eta(\phi\,\gamma(c+3,\eta x)+\gamma(c+4,\eta x))}{P\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c+} \tag{23}$$

Lorenz curve

$$L(P) = \frac{1}{\mu}\int_0^\infty x f_w(x;\eta,\phi,c)dx$$
$$L(P) = \frac{\eta(\eta c!+\phi(c+2)!+(c+3)!)}{\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c+}\int_0^\infty \frac{x^c\eta^{c+2}(1+\phi\eta x^2+\eta^2x^3)e^{-\eta x}}{\eta c!+\phi(c+2)!+(c+3)!}dx$$
$$L(P) = \frac{\eta^2\gamma(c+1,\eta x)+\eta(\phi\,\gamma(c+3,\eta x)+\gamma(c+4,\eta x))}{\eta\Gamma c+2+\phi\Gamma c+4+\Gamma c+} \tag{24}$$

Maximum Likelihood Estimator

This is one of the most useful methods for estimating the different parameters of the distribution. Let $X_1, X_2, \dots, X_n$ be the random sample of size n drawn from the weighted Remkan distribution, then the likelihood function of weighted Remkan distribution is given as:

$$L(x) = \prod_{i=1}^{n} f_w\,(x_i; \eta, \phi, c)$$

$$L(x) = \prod_{i=1}^{n} \frac{x_i^c \eta^{c+2}(1+\phi\eta x^2+\eta^2 x^3)e^{-\eta\ \ i}}{\eta c!+\phi(c+2)!+(c+3)!}$$

$$= \frac{\eta^{c+2}}{\eta c!+\phi(c+2)!+(c+3)!}\prod_{i=1}^{n} x_i^c(1+\phi\eta x^2+\eta^2 x^3)e^{-\eta x_i}$$

The log likelihood function is

$$logL = n(c+2)log\eta - nlog\eta c! + \phi(c+2)! + (c+3)! + \sum_{i=1}^{n} logx_i^c - \eta\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} log\,(1+\phi\eta x^2+\eta^2 x^3) \tag{25}$$

The maximum likelihood estimates of $\eta, \phi\ and\ c$ can be obtained by partially differentiating equation (25) with respect to $\eta, \phi\ and\ c,$ and then equating the derivatives to zero, These estimates must satisfy the normal equations.

$$\frac{\partial logL}{\partial\eta} = \frac{n(c+2)}{\eta} - \frac{nc!}{\eta c!+\phi(c+2)!+(c+3)!} - \sum x_i + \sum_{i=1}^{n}\frac{\phi x_i^2+2\eta x_i^3}{1+\phi\eta x_i^2+\eta^2 x_i^3} = 0 \tag{26}$$

$$\frac{\partial logL}{\partial\phi} = \frac{n(c+2)!}{\eta c!+\phi(c+2)!+(c+3)!} + \sum_{i=1}^{n}\frac{\eta x_i^2}{1+\phi\eta x_i^2+\eta^2 x_i^3} = 0 \tag{27}$$

$$\frac{\partial logL}{\partial c} = nlog\eta - \frac{n.\eta c!\psi(c+1)+\phi(c+2)!\psi(c+3)+(c+3)!\psi(c+4)}{\eta c!+\phi(c+2)!+(c+3)!} + \sum_{i=1}^{n} logx_i = 0 \tag{28}$$

On solving equations (26), (27) and (28) we obtain the maximum likelihood estimator of parameters in weighted Remkan distribution. The equation, however, cannot be solved analytically, so they used R programming and a data set to solve it numerically.

6. Applications

In this section, we discuss the goodness of fit of the proposed Weighted Remkan distribution by applying it to a real medical dataset and comparison with the Area-biased Remkan distribution and the Remkan distribution. The dataset consists of serum creatinine measurements, relating to times (in months), from 32 diabetic patients with chronic kidney disease. These data were analyzed by Kaggle. Com website and the data set are shown below in table 1.

Table 1: Diabetic Serum Creatinine dataset

1.7, 1.9, 1.3, 2.5, 11.8, 4.4, 2.4, 13.4, 3.4, 3.3, 2.5, 1.3, 4.5, 6.8, 1.8, 2.8, 5.3, 2.1, 6.3, 1.5, 6.3, 1.3, 7.2, 1.5, 1.5, 10.8, 7.3, 5.2, 4.6, 16.9, 4.3, 4.3

The R software used to carry out the estimation of model comparison criterion values along with the estimation of unknown parameters. In order to compare the performance of weighted Remkan distribution with Remkan distributions, we consider the model selection values like Bayesian Information criterion (BIC), Akaike Information Criterion (AIC) and Akaike Information Criterion Corrected (AICC). The better distribution is which corresponds to lesser values of *AIC, BIC, AICC* and *-2logL*.

For calculating the criterion values *AIC*, *BIC*, *AICC* and *-2logL*, one can use the formulae as follows

$AIC = 2k - 2logL$ $\quad BIC = klogn - 2logL$ $\quad$ and $\quad AICC = AIC + \frac{2k(k+1)}{n-k-1}$

where k is the number of parameters, n is the sample size and $-2logL$ is the maximized value of log-likelihood function under the considered model and are shown in table 2.

Table 2: Parameter estimations, corresponding standard errors, Criterion values & goodness of fit test.

Distribution	MLE	S.E	-2logL	AIC	BIC	AICC
Weighted Remkan Distribution	$\hat{\eta} = 1.94521$ $\hat{\phi} = 0.47100$ $\hat{c} = 2.05044$	$\hat{\eta} = 0.29814$ $\hat{\phi} = 0.08821$ $\hat{c} = 0.34123$	**158.21454**	**163.21454**	**165.22410**	**163.02463**
Area-biased Remkan distribution	$\hat{\eta} = 1.99000$ $\hat{\phi} = 0.47840$	$\hat{\eta} = 0.30501$ $\hat{\phi} = 0.09051$	160.74112	164.74112	168.26200	165.26254
Remkan Distribution	$\hat{\eta} = 2.01534$ $\hat{\phi} = 0.48210$	$\hat{\eta} = 0.31142$ $\hat{\phi} = 0.09231$	162.38124	166.38124	169.90210	166.90234

From results given in table 2, it can be clearly stated that the Weighted Remkan distribution have the lesser *AIC*, *BIC*, *AICC* and *-2logL* values as compared to the Area-biased Remkan Distribution and Remkan distribution. Hence, it can be concluded that the Weighted Remkan distribution leads to a better fit than the Area-biased Remkan Distribution and Remkan distribution.

7. Conclusion

The present article proposes a new generalization of Remkan distribution known as weighted Remkan distribution has been studied and investigated. Its various statistical properties including its moments, order statistics, survival analysis, entropies, Bonferroni and Lorenz curves have been determined and discussed. For estimating its parameters, the method of maximum likelihood has been used. Finally, the significance of newly proposed distribution is illustrated using a real medical dataset comprising serum creatinine levels from diabetic patients diagnosed with chronic kidney disease (CKD). Serum creatinine is a clinical biomarker for kidney function assessment, and its accurate modeling is important for disease prognosis and treatment planning. and the result drawn from the data set proved that the weighted Remkan distribution fits better than the Area-biased Remkan Distribution and Remkan distribution in modeling this dataset, highlighting its potential applicability in medical research and clinical data analysis.

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