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A Novel Flexible Exponentiated XLindley Distribution with applications to Modeling COVID-19 Mortality and Precipitation

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ABSTRACT

Probability models play an important role in modeling the real-life data, particularly, modeling complex nature data. The COVID-19 pandemic has resulted in a significant increase in mortality rates globally. In this study we propose a new probability model to address the complex nature of such data sets. The new model is termed as Exponentiated- Exponentiated XLindley (EEXL) distribution. We present some key statistical properties of this distribution including moments, generating functions, order statistics etc. Parameters of the model are estimated using Maximum Likelihood Estimation (MLE). The performance of the model is evaluated using Netherlands COVID-19 data covering monthly mortality rate for 30 days (31st march to April 30, 2020). Further, the model is also fitted to the precipitation data. We provide insights the distribution performance through simulation study. The proposed model provides efficient results as compared to well-known competitive distributions.

Keywords: Exponentiated distributions; COVID-19; Maximum likelihood Estimation; Order Statistics; Goodness of fit criteria

1. Introduction

Probability models have broad applications in many fields such as in the sports, medical, industries, engineering and other related sectors. Besides the traditional distributions, researchers have developed some useful models to adequately models some real-life data sets. For example, Alshenawy [1] used updated form of the inverse Weibull model for analyzing a breast cancer data set. Almetwally [2] used the odd Weibull inverse Topp-Leone model for analyzing COVID-19 data. Ahmad *et al*. [3] introduced novel statistical model for analyzing COVID-19 data in China. Penn *et al*. [4] applied a double Poisson model to predict football results. Alevizakos *et al*. [5] applied gamma distribution in engineering sector. Shafqat *et al*. [6] used the inverse Rayleigh model in the industrial sector.

The Lindley distribution derived from exponential distribution gained attention for its applicability in survival analysis and reliability engineering due to its ability to model time-to-event data effectively. Chouia and Zeghdoudi [7] introduced one of the most flexible and straightforward lifespan models called XLindley (XL) distribution which is combination of the Lindley distribution and exponential distribution. The cumulative density function (cdf) and probability density function (pdf) of XLindley distribution are expressed as

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$$F(z) = 1 - \left(1 + \frac{\delta z}{(1+\delta)^2}\right)\exp(-\delta z) \tag{1}$$

$$f(z) = \frac{\delta^2 (2 + \delta + z)}{(1+\delta)^2}\exp(-\delta z) \tag{2}$$

respectively.

Researchers have been attracted by introducing new families of distributions using different methods. Olanrewaju *et al*. [8] introduced Beta-G method. Tahir *et al*. [9] introduced Weibull-G method, Barco *et al*. [10] introduced inverse power transformation. Aljarrah and Felix [11] introduced T-X generator, Gomes-Silva *et.al* [12] introduced odd Lindley-G, Bantan *et al*. [13] introduced truncated Burr X-G, Eghwerido et al. [14] introduced Teissier-G.

The exponentiated family of distribution is widely used for developing flexible models by adding a new parameter to the family. Pal *et al*. [15] introduced exponentiated Weibull. Nadarajah [16] introduced exponentiated Gumbel distribution, Nadarajah *et al.* [17] presented exponentiated Gamma distribution. Rasool *et al.* [18] worked on exponentiated Power Inverse Lomax Distribution. Ashour *et al*. [19] produced exponentiated power Lindley distribution. Alomair *et al*. [20] introduced exponentiated XLindley distribution.

The main aim of this research is to introduce a novel, flexible, three-parameter distribution using exponentiated parameter indication technique. The new distribution is termed as Exponentiated -Exponentiated XLindley (EEXL) distribution.

The paper structured is as follows; in section 2 we proposed a new distribution, plots of the probability density function, cumulative density function. We derive survival function, hazard function and cumulative hazard functions and their plots. The statistical properties, estimation and ordered statistics from EEXL distribution are derived in section 3, 4, and 5 respectively. Simulation studies are also provided in section 6. For demonstrating the applicability of the proposed distribution, two datasets are considered in section 7.

2. Exponentiated- Exponentiated XLindley Distribution

A generalized form of XLindley distribution is proposed using power to cdf transformation $G(y) = \left\{\left[F(z)\right]^{\alpha}\right\}^{\beta}$, where Z is a random variable which follows XLindley distribution. The cdf of EEXL distribution is given as

$$G(y) = \left\{\left[1 - \left(1 + \frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^{\alpha}\right\}^{\beta} \tag{3}$$

where δ is the scale parameter and α, β are shape parameters of EEXL distribution.

The pdf of EEXL distribution is obtained as

$$g(y) = \frac{\alpha\beta\delta^2 (2 + y + \delta)}{-1 + \exp(-\delta y)(1+\delta)^2 - \delta(2 + y + \delta)}\left\{\left[1 - \left(1 + \frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^{\alpha}\right\}^{\beta} \tag{4}$$

The pdf is generally unimodal having a single peak curve that can shift depending on the values of the parameters α and β.

(a) (b)

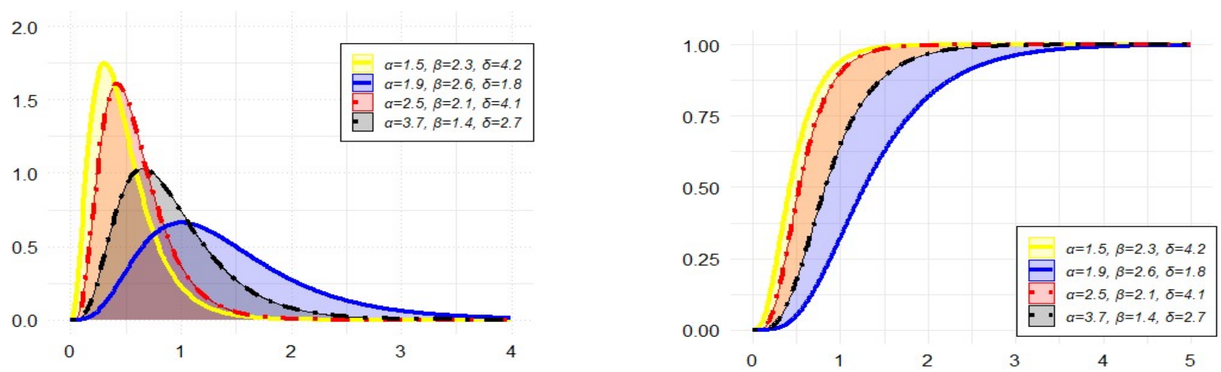


Figure 1. Plots for the (a) pdf and (b) cdf of EEXL distribution for the selected values of the parameters.

Plots of the EEXL pdf and cdf for the selected values of the parameters are given in the Figure 1. It is clear from the figure that in general, the distribution is positively skewed. However, various shapes of the model can be explored for different values of the shape parameters.

The survival function (sf) hazard function (hf) and cumulative hazard function (chf) of EEXL distribution are given by

$$S(y)=1-\left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(\delta y)\right]^{\alpha}\right\}^{\beta} \tag{5}$$

$$H(y)=\frac{\alpha\beta\delta^2(2+y+\delta)\left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^{\alpha}\right\}^{\beta}}{-1+\exp(\delta y)(1+\delta)^2-\delta(2+y+\delta)}\left\{1-\left[\left(1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right)^{\alpha}\right]^{\beta}\right\} \tag{6}$$

$$F(y)=-\log\left\{1-\left[\left(1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right)^{\alpha}\right]^{\beta}\right\} \tag{7}$$

(a) (b) (c)

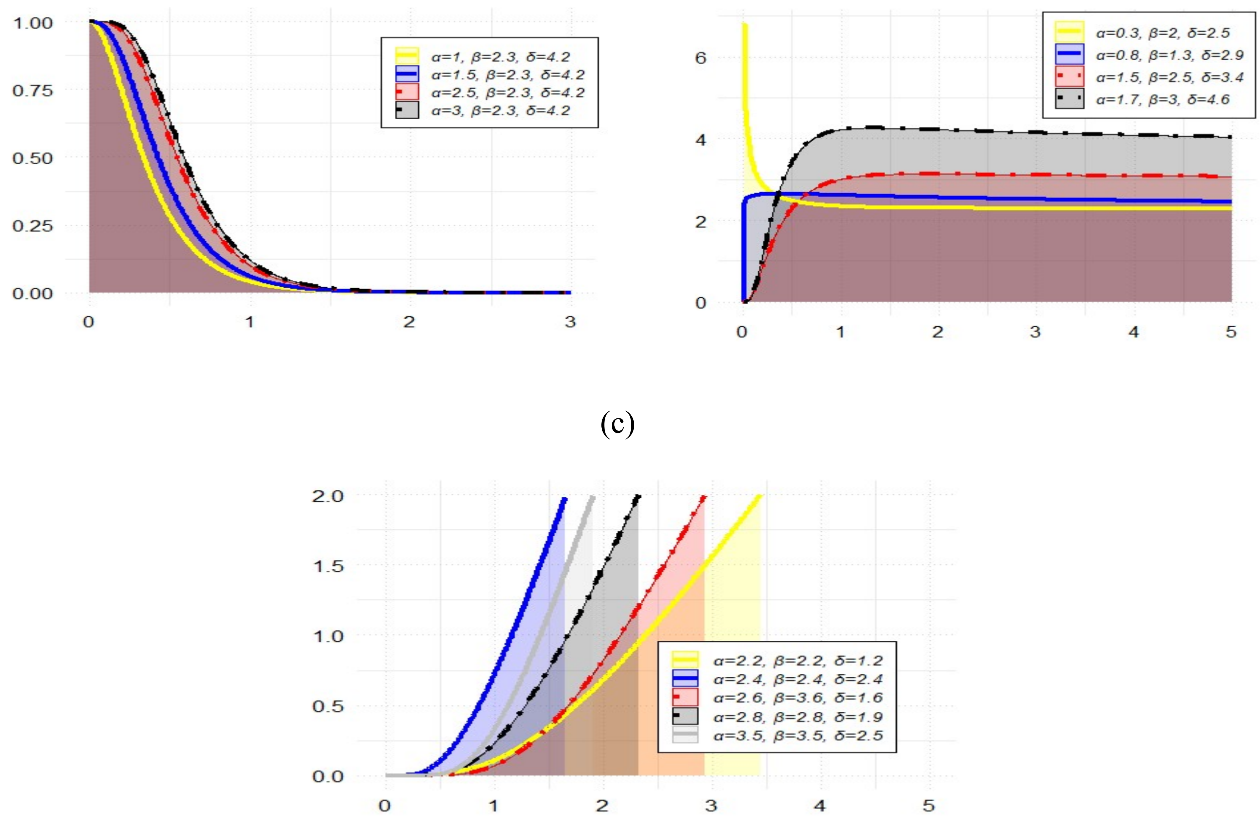


Figure 3. Plots for the (a) sf (b) hf (c) chf of EEXL distribution for the selected values of the parameters.

Plots of the EEXL survival function, hazard function and cumulative hazard function for the selected values of the parameters are given in Figure 3.

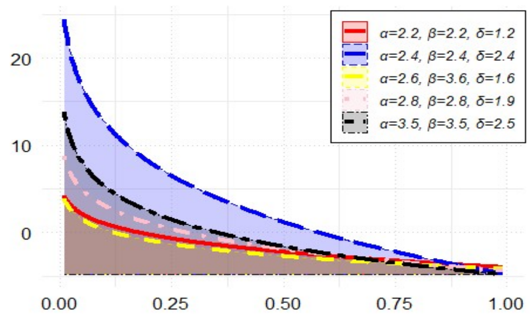


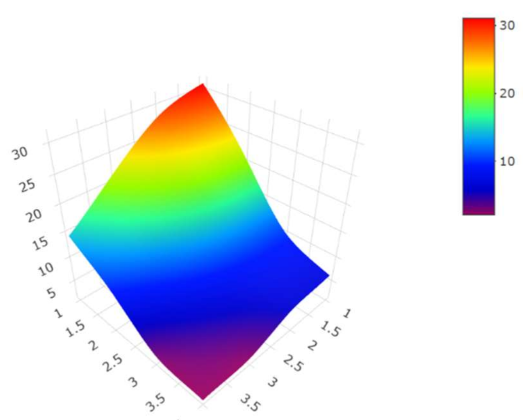
Figure.4. Plot of the EEXL for quantile function.

Plot of the EEXL for the quantile function for different values of the parameters are given in Figure 4.

Table 1. Some computational statistics of EEXL model for various parameter values.

δ	α	β	Mean	Variance	Skewness	Kurtosis
0.35	0.30	0.50	0.443783	1.678651	4.79705	31.0854
	0.40	1.00	0.234635	0.421264	4.57548	28.5386
	0.60	2.00	0.139551	0.119195	4.01450	22.0289
	1.00	5.00	0.072469	0.024434	3.28712	14.4808
0.50	0.30	0.50	0.628448	2.272722	3.98319	21.4339
	0.40	1.00	0.333009	0.567211	3.79588	19.6672
	0.60	2.00	0.198511	0.158217	3.31637	15.0547
	1.00	5.00	0.103356	0.031683	2.68181	9.57019
1.00	0.30	0.50	1.211889	3.767340	2.80644	10.3826
	0.40	1.00	0.649783	0.915997	2.68951	9.53439
	0.60	2.00	0.391102	0.238909	2.38371	6.99812
	1.00	5.00	0.205556	0.042191	1.99584	2.95884
1.50	0.30	0.50	1.722379	4.667202	2.37349	6.61094
	0.40	1.00	0.945618	1.079421	2.36797	5.85973
	0.60	2.00	0.577098	0.249165	2.40434	2.48172
	1.00	5.00	0.913097	0.308567	2.94950	2.21342

(a)



(b)

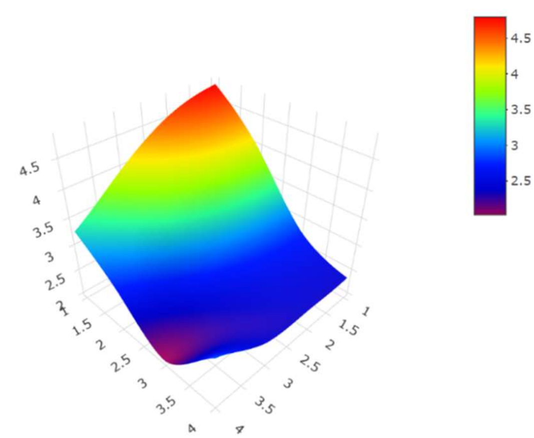


Figure.5. Plots of the EEXL for (a) Skewness (b) Kurtosis.

Some of the moments of the EEXL distribution for different values of the parameters are given in Table 1, and plots for the skewness and kurtosis are given in Figure 5.

3. Statistical Properties of EEXL distribution

This section offers different properties of EEXL distribution.

3.1. Quantile Function

The quantile function of EEXL distribution is

$$u = G(u)$$

$$u = \left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^{\alpha}\right\}^{\beta}$$

$$u^{\frac{1}{\alpha\beta}} = 1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)$$

$$\left(1-u^{\frac{1}{\alpha\beta}}\right)\exp(-\delta y) = 1+\frac{\delta y}{(1+\delta)^2}$$

$$y = \frac{(1+\delta)^2}{\delta}\left\{\left[1-u^{\frac{1}{\alpha\beta}}\right]\exp(-\delta u)-1\right\} \tag{8}$$

The median of the EEXL distribution can be obtained by putting $u = \frac{1}{2}$, that is

$$y = \frac{(1+\delta)^2}{\delta}\left\{\left[1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha\beta}}\right]\exp(-\delta u)-1\right\} \tag{9}$$

3.2. rth Moments

The r^{th} moment of EEXL is defined as

$$\mu_r' = \int_0^{\infty} y^r g(y)\,dy$$

$$\mu_r' = \int_0^{\infty} y^r \frac{\alpha\beta\delta^2(2+y+\delta)}{-1+\exp(-\delta y)(1+\delta)^2-\delta(2+y+\delta)}\left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^{\alpha}\right\}^{\beta} dy$$

Using binomial expansion, we get

$$\left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^{\alpha}\right\}^{\beta} = \sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^{j}(-1)^j y^j \exp(-\delta i y)$$

Thus the above expression modifies to

$$\mu_r' = \sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^{j}(-1)^j \int_0^{\infty} y^{r+j}\frac{\alpha\beta\delta^2(2+y+\delta)}{-1+\exp(y\delta)(1+\delta^2)-\delta(2+y+\delta)}\exp(-\delta i y)dy$$

$$\mu_r' = \sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^j (-1)^j \int_0^\infty y^{r+j} \frac{\alpha\beta\delta^2(2+y+\delta)}{-\delta y+(1+\delta)^2\{\exp(-\delta y)-1\}}\exp(-\delta iy)\,dy$$

$$\mu_r' = \sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^j (-1)^j \alpha\beta\delta^2 \int_0^\infty y^{r+j} \frac{(2+y+\delta)}{y\delta^2(2+\delta)}\exp(-\delta iy)\,dy$$

$$\mu_r' = \sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^j \frac{(-1)^j\alpha\beta}{(2+\delta)} \int_0^\infty y^{r+j-1}(2+y+\delta)\exp(-\delta iy)\,dy$$

$$\mu_r' = \sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^j \frac{(-1)^j\alpha\beta}{(2+\delta)}\times$$

$$\left\{2\int_0^\infty y^{r+j-1}\exp(-\delta iy)\,dy + \int_0^\infty y^{r+j+1-1}\exp(-\delta iy)\,dy + \delta\int_0^\infty y^{r+j-1}\exp(-\delta iy)\,dy\right\}$$

Using gamma function

$$\frac{\Gamma(\alpha)}{\lambda^\alpha} = \int_0^\infty x^{\alpha-1}\exp(-x\lambda)\,dx$$

$$\mu_r' = \sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^j \frac{(-1)^j\alpha\beta}{(2+\delta)}\left\{\frac{2\Gamma(r+j+1)}{(r+j)^{\delta i}} + \frac{\Gamma(r+j+1)}{(r+j+1)^{\delta i}} + \frac{\delta\Gamma(r+j)}{(r+j)^{\delta i}}\right\} \quad (10)$$

3.3. Moment Generating Function

The moment generating function is defined as

$$M_Y(t) = \int_0^\infty \exp(ty)\,g(y)\,dy$$

Using series expansion of $\exp(ty)$, we obtained

$$M_Y(t) = \sum_{m=0}^{\infty}\frac{t^m}{m!}\int_0^\infty y^m g(y)\,dy$$

$$M_Y(t) = \sum_{m=0}^{\infty}\frac{t^m}{m!}\mu_m'$$

$$M_Y(t) = \sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\sum_{m=0}^{\infty}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^j \frac{(-1)^i\alpha\beta t^m}{(2+\delta)m!}\left\{\frac{2\Gamma(r+j)}{(r+j)^{\delta i}} + \frac{\Gamma(r+j+1)}{(r+j+1)^{\delta i}} + \frac{\delta\Gamma(r+j)}{(r+j)^{\delta i}}\right\} \quad (11)$$

3.4. Characteristics Function

The characteristics function of EEXL distribution is

$$\phi_Y(t) = \int_0^\infty \exp(ity)\,g(y)\,dy$$

Using series expansion of $\exp(ity)$, we obtained

$$\phi_Y(t)=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\int_0^{\infty}y^m g(y)\,dy$$

$$\phi_Y(t)=\sum_{m=0}^{\infty}\frac{(it)^m}{m!}\mu'_m$$

$$\phi_Y(t)=\sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\sum_{m=0}^{\infty}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^j\frac{(-1)^i\alpha\beta(it)^m}{(2+\delta)m!}\left\{\frac{2\Gamma(r+j)}{(r+j)^{\delta i}}+\frac{\Gamma(r+j+1)}{(r+j+1)^{\delta i}}+\frac{\delta\Gamma(r+j)}{(r+j)^{\delta i}}\right\} \quad (12)$$

3.5. Probability Generating Function

The probability generating function of EEXL distribution can be given as

$$G(\alpha)=E(\alpha^y)$$

$$G(\alpha)=\int_0^{\infty}\exp\{y\log(\alpha)\}g(y)\,dy$$

Using series expansion of $\exp(y\log(\alpha))$, we obtained

$$G(\alpha)=\sum_{m=0}^{\infty}\frac{\{\log(\alpha)\}^m}{m!}\int_0^{\infty}y^m g(y)\,dy$$

$$G(\alpha)=\sum_{m=0}^{\infty}\frac{\{\log(\alpha)\}^m}{m!}\mu'_m$$

$$G(\alpha)=\sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\sum_{m=0}^{\infty}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^j\frac{(-1)^i\alpha\beta\{\log(\alpha)^m\}}{(2+\delta)m!}\left\{\frac{2\Gamma(r+j)}{(r+j)^{\delta i}}+\frac{\Gamma(r+j+1)}{(r+j+1)^{\delta i}}+\frac{\delta\Gamma(r+j)}{(r+j)^{\delta i}}\right\} \quad (13)$$

3.6. Mode

The mode of EEXL distribution can be obtained as

$$f'(y)=0$$

$$f'(y)=\frac{d}{dy}\left(\frac{\alpha\beta\delta^2(2+y+\delta)}{-1+\exp(-\delta y)(1+\delta)^2-\delta(2+y+\delta)}\left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^{\alpha}\right\}^{\beta}\right)=0$$

$$\frac{\alpha\beta\delta^2\left[-1+\alpha\beta\delta^2(2+y+\delta)^2-\exp(-\delta y)(1+\delta^2)\{-1+\delta(2+y+\delta)\}\right]}{1-\exp(-\delta y)(1+\delta)^2+\delta(2+y+\delta)}\times$$
$$\left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^{\alpha}\right\}^{\beta}=0 \quad (14)$$

The equation (14) does not have a general explicit solution. However, one can get numerical solution by using some iterative procedures.

4. Renyi Entropy

Renyi entropy is one of the famous measures that is used to measure the unpredictability of a probability distribution, high entropy shows maximum uncertainty and low entropy show minimum uncertainty. Zero value of entropy indicates completely certain information.

Let $Y \sim EEXL(y;\alpha,\beta,\delta)$, then the entropy can be derived as

$$H(\theta)=\frac{1}{1-\theta}\log\int_0^\infty \left(g(y)\right)^\theta dy$$

$$H(\theta)=\frac{1}{1-\theta}\log\int_0^\infty\left[\frac{\alpha\beta\delta^2(2+y+\delta)}{-1+\exp(-\delta y)(1+\delta)^2-\delta(2+y+\delta)}\left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^\alpha\right\}^\beta\right]^\theta$$

$$\left[\left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^\alpha\right\}^\beta\right]^\theta=\sum_{i=0}^{\alpha\beta\theta}\sum_{j=0}^{i}\binom{\alpha\beta\theta}{i}\binom{i}{j}\left(\frac{\delta}{(1+\delta)^2}\right)^j(-1)^j y^j\exp(-\delta iy)$$

$$H(\theta)=\frac{1}{1-\theta}\log\sum_{i=0}^{\alpha\beta\theta}\sum_{j=0}^{i}\binom{\alpha\beta\theta}{i}\binom{i}{j}\left(\frac{\delta}{(1+\delta)^2}\right)^j(-1)^j\int_0^\infty\frac{\alpha^\theta\beta^\theta\delta^{2\theta}(2+y+\delta)^\theta y^j\exp(-\delta iy)}{-1+\exp(-\delta\theta y)(1+\delta)^{2\theta}-\delta^\theta(2+y+\delta)^\theta}dy$$

$$H(\theta)=\frac{1}{1-\theta}\log\sum_{i=0}^{\alpha\beta\theta}\sum_{j=0}^{i}\binom{\alpha\beta\theta}{i}\binom{i}{j}\left(\frac{\delta}{(1+\delta)^2}\right)^j(-1)^j\left(\alpha\beta\delta^2\right)^\theta\int_0^\infty\frac{y^j(2+y+\delta)^\theta\exp(-\delta iy)}{y\delta^{2\theta}(2+\delta)^\theta}dy$$

$$H(\theta)=\frac{1}{1-\theta}\log\sum_{i=0}^{\alpha\beta\theta}\sum_{j=0}^{i}\binom{\alpha\beta\theta}{i}\binom{i}{j}\left(\frac{\delta}{(1+\delta)^2}\right)^j\frac{(-1)^j(\alpha\beta)^\theta}{(2+\delta)}\int_0^\infty y^{j-1}(2+y+\delta)^\theta\exp(-\delta iy)\,dy$$

Using Binomial expansion

$$\left[(2+\delta)+y\right]^\theta=\sum_{k=0}^{\theta}\binom{\theta}{k}y^k(2+\delta)^{\theta-k}$$

$$H(\theta)=\frac{1}{1-\theta}\log\sum_{i=0}^{\alpha\beta\theta}\sum_{j=0}^{i}\sum_{k=0}^{\theta}\binom{\alpha\beta\delta}{i}\binom{i}{j}\binom{\theta}{k}\left(\frac{\delta}{(1+\delta)^2}\right)^j\frac{(-1)^j(\alpha\beta)^\theta}{(2+\delta)^{\theta-k+1}}\frac{\Gamma(j+k)}{(\delta i)^{j+k}} \tag{15}$$

5. Mean Residual Life

The extra life time that is expected for the survival of an object of interest is termed mean residual life.

$$\mu(t)=\frac{1}{S(t)}\int_t^\infty yg(y)-t$$

Considering

$$\int_t^\infty yg(y)\,dy=\int_t^\infty y\frac{\alpha\beta\delta^2(2+y+\delta)}{-1+\exp(-\delta y)(1+\delta)^2-\delta(2+y+\delta)}\left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^\alpha\right\}^\beta dy$$

Using binomial expansion

$$\left\{\left[1-\left(1+\frac{\delta y}{(1+\delta)^2}\right)\exp(-\delta y)\right]^{\alpha}\right\}^{\beta}=\sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^{j}(-1)^{j}y^{j}\exp(-\delta iy)$$

$$=\sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^{j}(-1)^{j}\alpha\beta\delta^{2}\int_{t}^{\infty}y^{r+j}\frac{(2+y+\delta)}{y\delta^{2}(2+\delta)}\exp(-\delta iy)\,dy$$

$$=\sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^{j}\frac{(-1)^{j}\alpha\beta}{(2+\delta)}\int_{t}^{\infty}y^{j}(2+y+\delta)\exp(-\delta iy)\,dy$$

$$=\sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^{j}\frac{(-1)^{j}\alpha\beta}{(2+\delta)}\left\{2\int_{t}^{\infty}y^{j}\exp(-\delta iy)\,dy+\int_{t}^{\infty}y^{j+1}\exp(-\delta iy)\,dy+\delta\int_{t}^{\infty}y^{j}\exp(-\delta iy)\,dy\right\}$$

Using

$$\int_{a}^{\infty}x^{k}\exp(-x)\,dx=\Gamma(k+1,a)$$

Solve each integral separately. Take the first integral

$$2\int_{t}^{\infty}y^{j}\exp(-\delta iy)\,dy$$

By substitution

$$u=\delta iy \quad \frac{du}{\delta i}=dy$$

$$2\int_{t}^{\infty}y^{j}\exp(-\delta iy)\,dy=\frac{2}{(\delta i)^{j+1}}\int_{\delta it}^{\infty}u^{j}\exp(-u)\,du$$

$$2\int_{t}^{\infty}y^{j}\exp(-\delta iy)\,dy=\frac{2}{(\delta i)^{j+1}}\Gamma(j+1,\delta it)$$

The second integral

$$\int_{t}^{\infty}y^{j+1}\exp(-\delta iy)=\frac{1}{(\delta i)^{j+2}}\Gamma(j+2,\delta it)$$

The third integral

$$\delta\int_{t}^{\infty}y^{j}\exp(-\delta iy)\,dy=\frac{\delta}{(\delta i)^{j+1}}\Gamma(j+1,\delta it)$$

$$\int_{t}^{\infty}yg(y)\,dy=$$

$$\sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^2}\right\}^{j}\frac{(-1)^{j}\alpha\beta}{(2+\delta)}\left\{\frac{2}{(\delta i)^{j+1}}\Gamma(j+1,\delta it)+\frac{1}{(\delta i)^{j+2}}\Gamma(j+2,\delta it)+\frac{\delta}{(\delta i)^{j+1}}\Gamma(j+1,\delta it)\right\}$$

$$\mu(t)=$$

$$\frac{1}{S(t)}\left[\sum_{i=0}^{\alpha\beta}\sum_{j=0}^{i}\binom{\alpha\beta}{i}\binom{i}{j}\left\{\frac{\delta}{(1+\delta)^{2}}\right\}^{j}\frac{(-1)^{j}\alpha\beta}{(2+\delta)}\left\{\frac{2}{(\delta i)^{j+1}}\Gamma(j+1,\delta it)+\frac{1}{(\delta i)^{j+2}}\Gamma(j+2,\delta it)+\frac{\delta}{(\delta i)^{j+1}}\Gamma(j+1,\delta it)\right\}\right]-t \tag{16}$$

6. Estimation

To apply maximum likelihood estimation method for estimation the parameters of EEXL distribution, let $y_1, y_2, y_3, \ldots, y_n$ be n observations of random sample form EEXL distribution. The likelihood function is given by

$$L=\prod_{i=1}^{n} g(y_i;\alpha,\beta,\delta) \tag{17}$$

$$L=\prod_{i=1}^{n}\left\{\frac{\alpha\beta\delta^{2}(2+y_i+\delta)}{-1+\exp(-y_i\delta)(1+\delta)^{2}-\delta(2+y_i+\delta)}\left[\left(1-\left(1+\frac{\delta y_i}{(1+\delta)^{2}}\right)\exp(-\delta y_i)\right)^{\alpha}\right]^{\beta}\right\}$$

The Log-likelihood function can be written as

$$\log L=n\log(\alpha)+n\log(\beta)+n\log(\delta^{2})+\sum_{i=1}^{n}\log(2+\delta+y_i)-\sum_{i=1}^{n}\log\left\{-1+\exp(-\delta y_i)(1+\delta)^{2}-\delta(2+\delta+y_i)\right\}$$

$$+\alpha\beta\sum_{i=1}^{n}\log\left(1-\exp(-\delta y_i)\left(1+\frac{\delta y_i}{(1+\delta)^{2}}\right)\right) \tag{18}$$

Now differentiating with respect to α to have

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\alpha}+\beta\sum_{i=1}^{n}\log\left\{1-\exp(-\delta y_i)\left(1+\frac{\delta y_i}{(1+\delta)^{2}}\right)\right\}$$

Now $\frac{\partial\log L}{\partial\alpha}=0$ and solve for α we get

$$\hat{\alpha}=-\frac{n}{\beta\sum_{i=0}^{n}\log\left\{1-\exp(-\delta y_i)\left(1+\frac{\delta y_i}{(1+\delta)^{2}}\right)\right\}} \tag{19}$$

Differentiating with respect to β we get

$$\frac{\partial\log L}{\partial\beta}=\frac{n}{\beta}+\alpha\sum_{i=1}^{n}\log\left\{1-\exp(-\delta y_i)\left(1+\frac{\delta y_i}{(1+\delta)^{2}}\right)\right\}$$

Now $\frac{\partial\log L}{\partial\beta}=0$ and solve for β we get

$$\hat{\beta}=-\frac{n}{\alpha\sum_{i=1}^{n}\log\left\{1-\exp(-\delta y_i)\left(1+\frac{\delta y_i}{(1+\delta)^{2}}\right)\right\}} \tag{20}$$

Differentiating with respect to δ .

$$\frac{\partial \log L}{\partial \delta} = \frac{2n}{\delta} + \sum_{i=1}^{n} \frac{1}{(2+\delta+y_i)} - \sum_{i=1}^{n} \frac{-2-2\delta+\exp(\delta y_i)(2+2\delta)-y_i+\exp(\delta y_i)(1+\delta)^2 y_i}{-1+\exp(\delta y_i)(1+\delta)^2-\delta(2+\delta+y_i)} +$$
$$\alpha\beta\sum_{i=1}^{n} \frac{-\exp(\delta y_i)\left(-\frac{2\delta y_i}{(1+\delta)^3}+\frac{y_i}{(1+\delta)^2}\right)+\exp(-\delta y_i)\, y_i\left(1+\frac{\delta y_i}{(1+\delta)^2}\right)}{1-\exp(-\delta y_i)\left(1+\frac{\delta y_i}{(1+\delta)^2}\right)} \tag{21}$$

There is no such close form of the equation (21). Therefore, some numerical methods may be used to get the numerical solution.

6.1. Fisher Information Matrix

The Fisher Information Matrix provides an approximation of the variance-covariance matrix of the maximum likelihood estimators (MLE) for the parameters α, β and δ . The inverse of the Fisher Information Matrix gives the variance-covariance matrix of the estimators.

$$I_0^{-1} = \begin{bmatrix} \frac{\partial^2 \log L}{\partial \alpha^2} & \frac{\partial^2 \log L}{\partial \alpha \partial \beta} & \frac{\partial^2 \log L}{\partial \alpha \partial \delta} \\ \frac{\partial^2 \log L}{\partial \beta \partial \alpha} & \frac{\partial^2 \log L}{\partial \beta^2} & \frac{\partial^2 \log L}{\partial \beta \partial \delta} \\ \frac{\partial^2 \log L}{\partial \delta \partial \alpha} & \frac{\partial^2 \log L}{\partial \delta \partial \beta} & \frac{\partial^2 \log L}{\partial \delta^2} \end{bmatrix}^{-1} \tag{22}$$

$$I_0^{-1} = \begin{bmatrix} \text{var}(\hat{\alpha}) & \text{cov}(\hat{\alpha},\hat{\beta}) & \text{cov}(\hat{\alpha},\hat{\delta}) \\ \text{cov}(\hat{\beta},\hat{\alpha}) & \text{var}(\hat{\beta}) & \text{cov}(\hat{\beta},\hat{\delta}) \\ \text{cov}(\hat{\delta},\hat{\alpha}) & \text{cov}(\hat{\delta},\hat{\beta}) & \text{var}(\hat{\delta}) \end{bmatrix}^{-1} \tag{23}$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = -\frac{n}{\alpha^2}$$

$$\frac{\partial^2 \log L}{\partial \beta^2} = -\frac{n}{\beta^2}$$

$$\frac{\partial^2 \log L}{\partial \delta^2} =$$
$$\frac{\begin{gathered} y\alpha\beta\delta^2\{-4-y-6\delta+y(-2+\delta)\delta+2\delta^3 \\ -exp(\delta y)\,(1+\delta)^2\{2(-2+\delta)+y(1+\delta)[-1+\delta(5+y+(3+y)\delta+\delta^2)]\}\} \\ +n(1+\delta)^2\{-2-4(2+y)\delta-\{10+y(4+y)\}\delta^2-2(2+y)\delta^3-2\exp(2y\delta)(1+\delta)^2(1+2\delta)\}+ \\ \exp(y\delta)\left\{4+\delta\{y^3\delta^2(1+\delta)^2+2y\{2+\delta(2+\delta)\}+4\{4+\delta(5+2\delta)\}+y^2\delta(1+\delta)\left[-1+\delta\left(5+\delta(3+\delta)\right)\right]\}\right\} \end{gathered}}{\{\delta^2(1+\delta)^2[1-\exp(y\delta)(1+\delta)^2+\delta(2+y+\delta)]^2\}}$$
$$-\sum_{i=1}^{n}\frac{1}{(2+\delta+y_i)^2}$$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\beta} = \log\left\{1-\exp\left(-\delta y\right)\left(1+\frac{y\delta}{\left(1+\delta\right)^2}\right)\right\}$$

$$\frac{\partial^2 \log L}{\partial\alpha\partial\delta} = -\frac{y\beta\delta\left\{4+y+\left(3+y\right)\delta+\delta^2\right\}}{\left(1+\delta\right)\left\{1-\exp\left(-\delta y\right)\left(1+\delta\right)^2+\delta\left(2+y+\delta\right)\right\}}$$

$$\frac{\partial^2 \log L}{\partial\beta\partial\delta} = -\frac{y\alpha\delta\left\{4+y+\left(3+y\right)\delta+\delta^2\right\}}{\left(1+\delta\right)\left\{1-\exp\left(-\delta y\right)\left(1+\delta\right)^2+\delta\left(2+y+\delta\right)\right\}}$$

7. Order Statistics

Let $Y_1, Y_2, Y_3, \ldots Y_n$ denote the order statistics obtained from a random sample $y_1, y_2, y_3, \ldots y_n$ taken from continuous population with cdf $G(y,\phi)$ and pdf $g(y,\phi)$ as follows:

$$g(y_i) = \frac{n!}{(i-1)!(n-i)!} g\left(y_i\right)\left\{G(y_i)\right\}^{i-1}\left\{1-G(y_i)\right\}^{n-i}$$

7.1. PDF of Minimum, Median and Maximum Order Statistics

The pdf of minimum order statistics is as follows

$$g(y_1) = n\left\{1-\left[\left(1-\left(1+\frac{\delta y_1}{(1+\delta)^2}\right)\exp\left(\delta y_1\right)\right)^{\alpha}\right]^{\beta}\right\}^{n-1} \times\left\{\frac{\alpha\beta\delta^2\left(2+y_1+\delta\right)}{-1+\exp\left(\delta y_1\right)\left(1+\delta\right)^2-\delta\left(2+y_1+\delta\right)}\right\}\times\left\{\left[1-\left(1+\frac{\delta y_1}{(1+\delta)^2}\right)\exp\left(\delta y_1\right)\right]^{\alpha}\right\}^{\beta} \tag{24}$$

The pdf of median order statistics is as follow

$$g_{m+1}(y_i) = \frac{(2m+1)!}{m!m!}\left\{\frac{\alpha\beta\delta^2(2+y+\delta)}{-1+\exp(\delta y)(1+\delta)^2-\delta(2+y+\delta)}\right\}\times \left\{\left[\left(1-\left(1+\frac{\delta y_1}{(1+\delta)^2}\right)\exp\left(\delta y_1\right)\right)^{\alpha}\right]^{\beta}\right\}^{m}\times\left\{1-\left[\left(1-\left(1+\frac{\delta y_1}{(1+\delta)^2}\right)\exp\left(\delta y_1\right)\right)^{\alpha}\right]^{\beta}\right\}^{m} \tag{25}$$

The pdf of maximum order statistics is as follow

$$g(y_n) = n\left[\left(1-\left(1+\frac{\delta y_n}{(1+\delta)^2}\right)\exp\left(\delta y_n\right)\right)^{\alpha}\right]^{\beta}\times \left\{\frac{\alpha\beta\delta^2\left(2+y_n+\delta\right)}{-1+\exp\left(\delta y_n\right)\left(1+\delta\right)^2-\delta\left(2+y_n+\delta\right)}\right\}\times\left[\left(1-\left(1+\frac{\delta y_n}{(1+\delta)^2}\right)\exp\left(\delta y_n\right)\right)^{\alpha}\right]^{\beta} \tag{26}$$

7.2. Joint PDF of Minimum and Maximum Order Statistics

The joint pdf of the minimum and maximum order statistics when a random sample of size n is drawn from the EEXL distribution. These pdfs can obtain by solving the following equations.

$$g_{i,j}\left(y_i, y_j\right) = k\left\{G(y_i)\right\}^{i-1}\left\{G(y_j) - G(y_i)\right\}^{j-i-1}\left\{1 - G(y_i)\right\}^{n-j} g\left(y_i\right) g\left(y_j\right)$$

where

$$k = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

The joint pdf of the minimum and maximum, that is, i^{th} and j^{th} order statistics from EEXL distribution, is

$$\begin{aligned} g_{i,j}(y_i, y_j) = k &\left\{\left[\left(1-(1+\frac{\delta y_i}{(1+\delta)^2})\exp(\delta y_i)\right)^{\alpha}\right]^{\beta}\right\}^{i-1} \times \\ &\left\{\left[\left(1-(1+\frac{\delta y_j}{(1+\delta)^2})\exp(\delta y_j)\right)^{\alpha}\right]^{\beta} - \left[\left(1-(1+\frac{\delta y_i}{(1+\delta)^2})\exp(\delta y_i)\right)^{\alpha}\right]^{\beta}\right\}^{j-i-1} \times \\ &\left\{1-\left[\left(1-(1+\frac{\delta y_j}{(1+\delta)^2})\exp(\delta y_j)\right)^{\alpha}\right]^{\beta}\right\}^{n-j} \times \\ &\left\{\left(\frac{\alpha\beta\delta^2(2+y_i+\delta)}{-1+\exp(\delta y_i)(1+\delta)^2-\delta(2+y_i+\delta)}\right)\left[\left(1-(1+\frac{\delta y_i}{(1+\delta)^2})\exp(\delta y_i)\right)^{\alpha}\right]^{\beta}\right\} \times \\ &\left\{\left(\frac{\alpha\beta\delta^2(2+y_j+\delta)}{-1+\exp(\delta y_j)(1+\delta)^2-\delta(2+y_j+\delta)}\right)\left[\left(1-(1+\frac{\delta y_j}{(1+\delta)^2})\exp(\delta y_j)\right)^{\alpha}\right]^{\beta}\right\} \end{aligned} \quad (27)$$

For the joint pdf of minimum and maximum order statistics we put $i = 1$ and $j = n$ we get

$$\begin{aligned} g_{1,n}(y_1, y_n) = n(n-1) &\left\{\left[\left(1-(1+\frac{\delta y_n}{(1+\delta)^2})\exp(\delta y_n)\right)^{\alpha}\right]^{\beta} - \left[\left(1-(1+\frac{\delta y_1}{(1+\delta)^2})\exp(\delta y_1)\right)^{\alpha}\right]^{\beta}\right\}^{n-2} \times \\ &\left\{\left(\frac{\alpha\beta\delta^2(2+y_1+\delta)}{-1+\exp(\delta y_1)(1+\delta)^2-\delta(2+y_1+\delta)}\right)\left[\left(1-(1+\frac{\delta y_1}{(1+\delta)^2})\exp(\delta y_1)\right)^{\alpha}\right]^{\beta}\right\} \times \\ &\left\{\left(\frac{\alpha\beta\delta^2(2+y_n+\delta)}{-1+\exp(\delta y_n)(1+\delta)^2-\delta(2+y_n+\delta)}\right)\left[\left(1-(1+\frac{\delta y_n}{(1+\delta)^2})\exp(\delta y_n)\right)^{\alpha}\right]^{\beta}\right\} \end{aligned} \quad (28)$$

8. Simulations

In this section simulation study is conducted to evaluate the performance of MLEs of parameters of EEXL distribution. Random data were generated from EEXL distribution with different parameter values. As we increase the sample size n, it is clearly that the bias tends to zero and MSE tends to decreases.

The results of the simulation study of the EEXL distribution are presented in **Table 2** and **3**. The results of the simulation study of the EEXL distribution are also illustrated graphically in **Figure 6** and **7**. From the numerical results (i.e., Table 2 and 3) and graphical illustration (i.e., Figures 6 and 7) of the simulation studies, we can observe that as we increase the sample size, the

- MLEs tends to stable
- Biases tends to zero
- MSEs decreases

Table 2. The numerical results of the SS of the EEXL model for $\alpha = 1.4$, $\beta = 1.0$ and $\delta = 0.4$.

n	**Parameters**	**MLEs**	**Biases**	**MSEs**
30	α	1.407242	0.007241	0.009215
	β	1.014428	0.014428	0.011655
	δ	0.403323	0.003323	0.000988
60	α	1.408397	0.008397	0.003346
	β	1.018312	0.018312	0.006182
	δ	0.405851	0.005851	0.001796
90	α	1.401692	0.001691	0.010526
	β	0.992475	-0.007524	0.005049
	δ	0.396850	-0.003149	0.000702
150	α	1.405399	0.005399	0.002070
	β	1.005694	0.005694	0.001372
	δ	0.402002	0.002002	0.000335
240	α	1.401711	0.001711	0.000596
	β	1.006511	0.006510	0.000777
	δ	0.403691	0.003691	0.000316
330	α	1.400772	0.000771	0.000590
	β	1.007899	0.007899	0.001436
	δ	0.401974	0.001974	0.000155
420	α	1.400425	0.000424	0.000305
	β	1.003620	0.003620	0.000433
	δ	0.402814	0.002813	0.000252
480	α	1.404177	0.004177	0.000492
	β	1.002947	0.002947	0.000412
	δ	0.401054	0.001054	0.000075
510	α	1.404208	0.004208	0.000508
	β	1.002608	0.002608	0.000210
	δ	0.399595	-0.00040	0.000061
570	α	1.400934	0.000933	0.000130

	β	1.002442	0.002442	0.000225
	δ	0.400789	0.000789	0.000036
600	α	1.401691	0.001691	0.000164
	β	1.002645	0.002645	0.000225
	δ	0.400884	0.000884	0.000051
1000	α	1.400230	-0.000184	0.000287
	β	1.001337	0.001337	0.000263
	δ	0.400205	0.000205	0.000046

(a) (b)

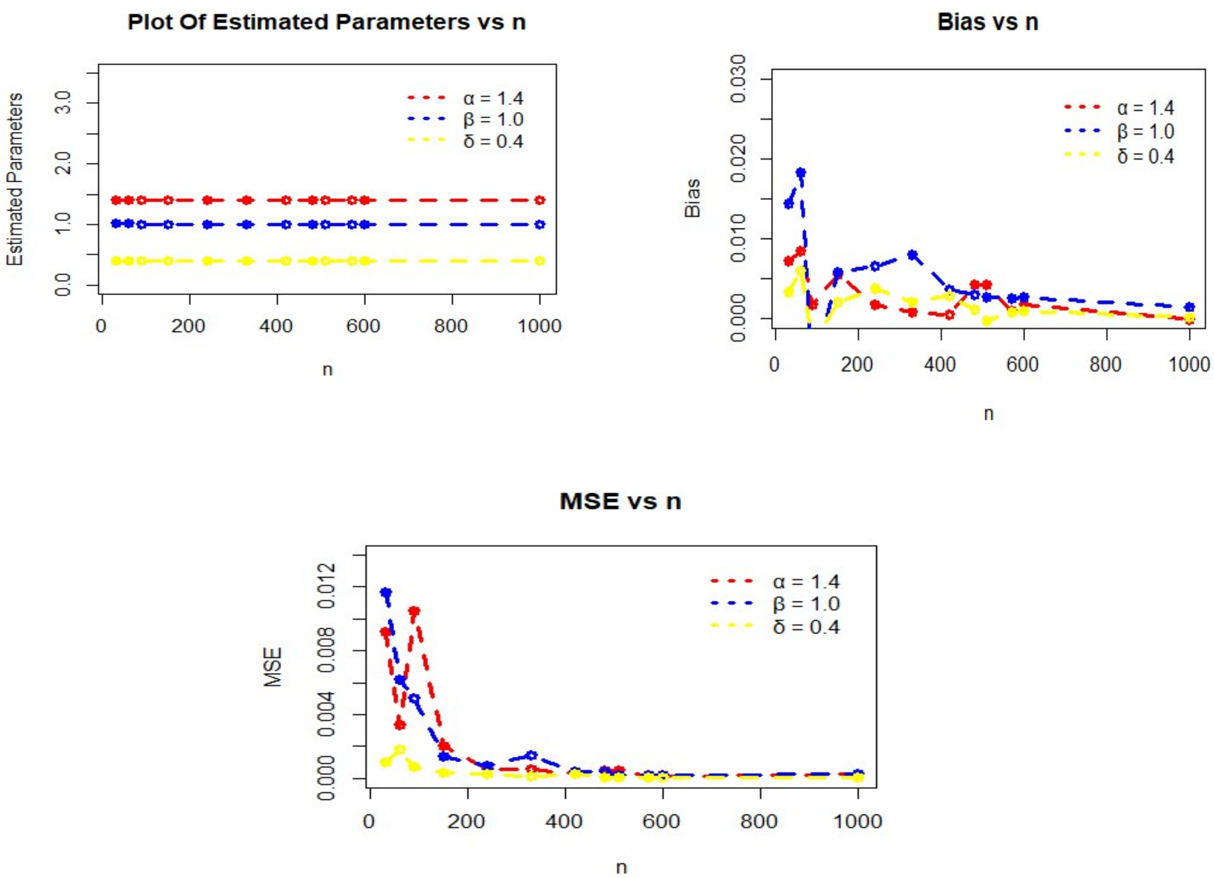
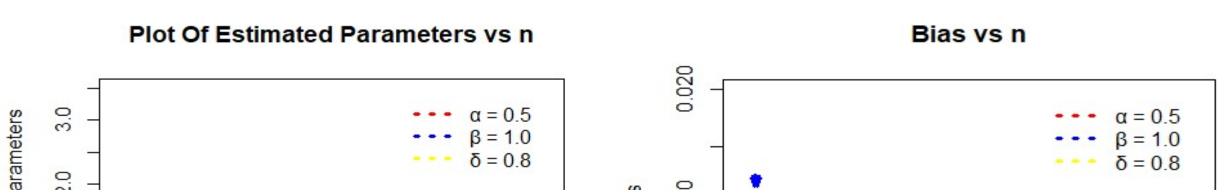


Figure.6. Visual display of the numerical results of the Simulation study of the EEXL distribution for $\alpha = 1.4$, $\beta = 1.0$ and $\delta = 0.4$.

Table 3. The numerical results of the SS of the EEXL model for $\alpha = 0.5$, $\beta = 1.0$ and $\delta = 0.8$.

n	Parameters	MLEs	Biases	MSEs
30	α	0.505935	0.005935	0.005212
	β	1.006594	0.006594	0.014721
	δ	0.785644	-0.014355	0.003012
60	α	0.507822	0.007822	0.000893
	β	1.012197	0.012197	0.003026
	δ	0.800162	0.000162	0.000528
90	α	0.506516	0.006516	0.001049
	β	1.006332	0.006332	0.002612
	δ	0.798578	-0.001422	0.000698
150	α	0.501563	0.0015628	0.000313
	β	1.000639	0.000638	0.0004577
	δ	0.798688	-0.001312	0.000348
240	α	0.502360	0.002360	0.000229
	β	1.002891	0.002891	0.000739
	δ	0.799030	-0.000960	0.000193
330	α	0.501726	0.001726	0.000152
	β	1.002681	0.002680	0.000235
	δ	0.799861	-0.000139	0.000065
420	α	0.504302	0.004302	0.000249
	β	1.006227	0.006227	0.000508
	δ	0.800127	0.000126	0.000117
480	α	0.501384	0.001384	0.000205
	β	1.001815	0.001815	0.000525
	δ	0.799529	-0.000470	0.000103
510	α	0.502839	0.002839	0.000302
	β	1.002704	0.002703	0.000713
	δ	0.801430	0.001430	0.000356
570	α	0.501419	0.001418	0.000113
	β	0.999752	-0.000248	0.000179
	δ	0.798441	-0.001558	0.000070
600	α	0.504734	0.004734	0.000278
	β	1.004582	0.004582	0.000503
	δ	0.800730	0.000730	0.000086
1000	α	0.502360	0.002360	0.000170
	β	1.002608	0.002607	0.000219
	δ	0.800760	0.000768	0.000042

(a) (b)



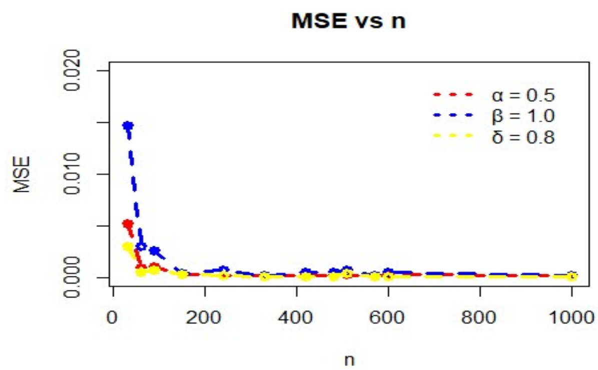


Figure.7. Visual display of the numerical results of the Simulation study of the EEXL distribution for $\alpha = 0.5$, $\beta = 1.0$ and $\delta = 0.8$.

9. Applications

In this section two real data sets have been used to compare the performance of the EEXL distribution with other distributions such as exponential distribution (Exp), Lindley distribution (L), XLindley distribution (XL), generalized Lindley distribution (GL), Weibull distribution, Power Lindley distribution (PL), Nadaraj-Haghighi distribution (NH) and Exponentiated XLindley distribution (EXL).

To evaluate the goodness of fit for the fitted distributions we considered the following measures: Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC) and Kolmogorov Smirnov (KS) respectively.

Data 1: The first data set consists of 30 observations, representing the COVID-19 mortality rates in Netherlands for 30 days (31 March to April 30, 2020). This data discussed by Almongy *et al*. [21]. The data set consist of the following observations; 14.918, 10.656, 12.274, 10.289, 10.832, 7.968, 7.584, 5.555, 6.027, 7.099, 5.928, 13.211, 4.097, 3.611, 4.960, 7.498, 6.940, 5.307, 5.048, 2.857, 2.254, 5.431, 4.462, 3.883, 3.461, 3.647, 1.974, 1.273, 1.416 and 4.235. Further, the visual display of the COVID-19 mortality rates dataset provided in Figure 8. We also present a visual comparison using fitted pdf, cdf, survival function (sf) and Q-Q plot given in Figure 9. The MLEs, Standard errors and goodness of fit for this data are given in Table 4.

(a) (b)

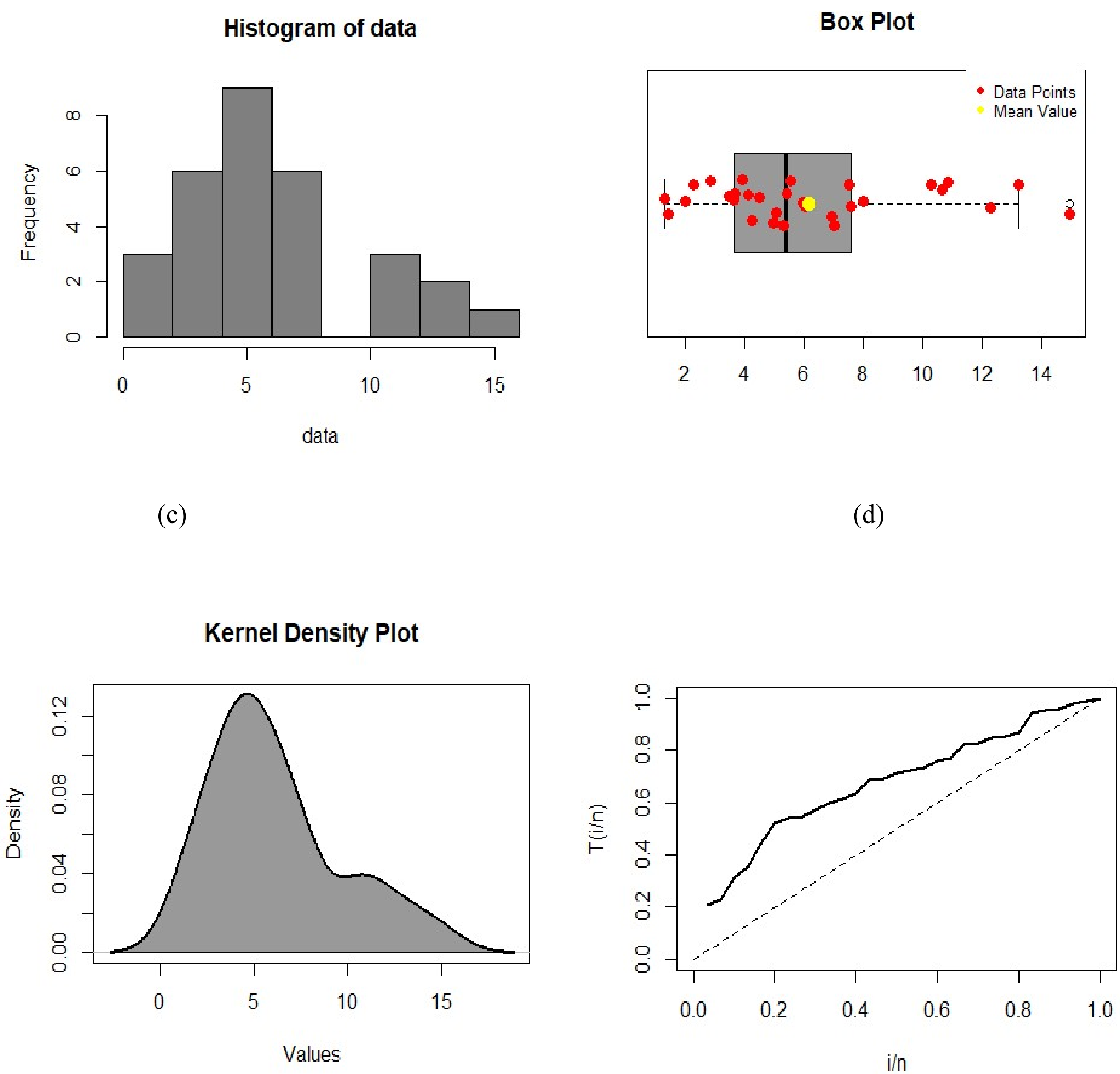


Figure.8. (a) histogram, (b) Kernel density, (c) boxplot and (d) TTT plot. A visual display of the behavior of the first dataset.

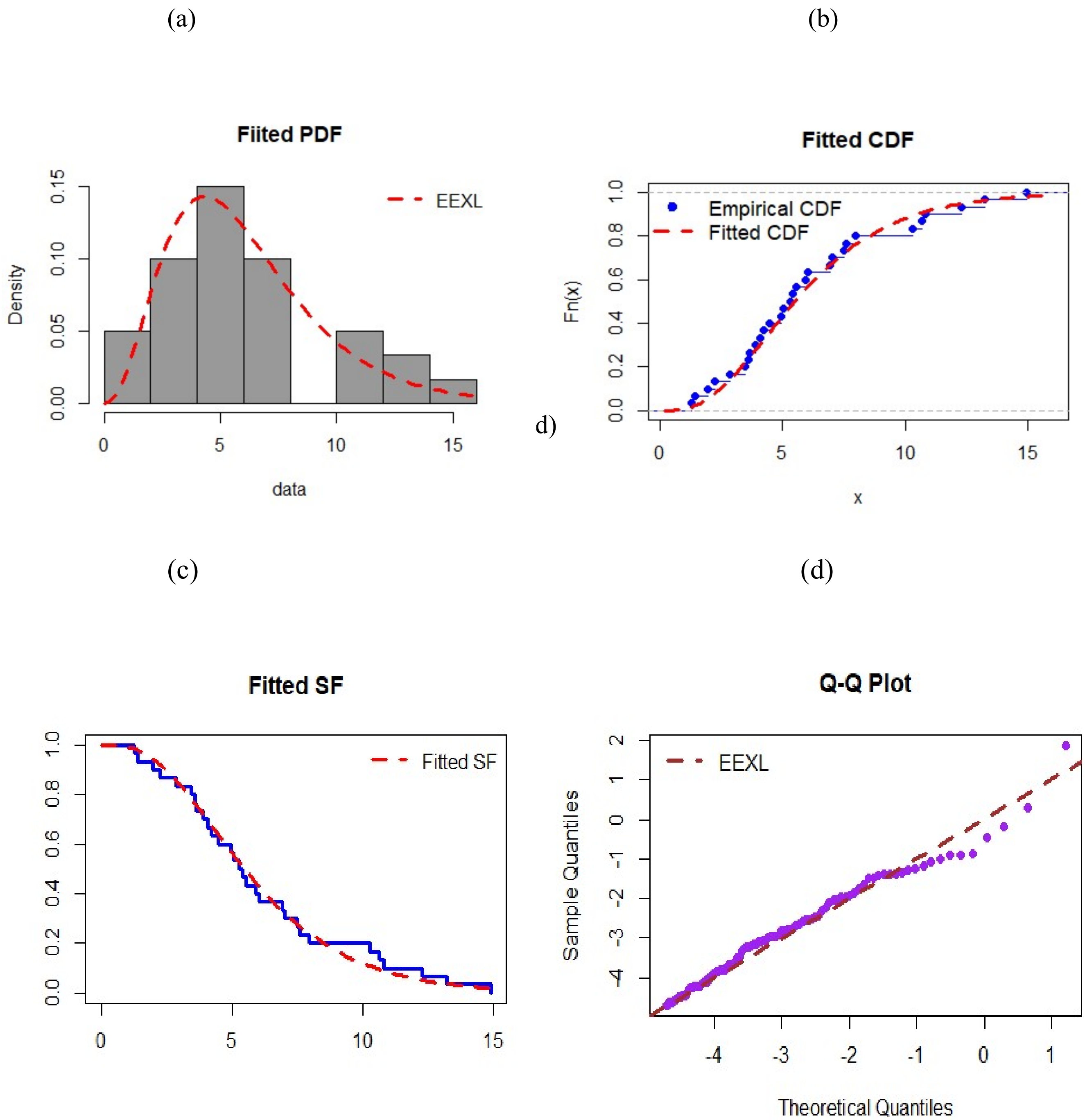


Figure.9. Illustration of the fitted (a) pdf, (b) cdf, (c) sf, (d) Q-Q plot of the EEXL model for the first data.

Table 4. Maximum Likelihood Estimates and Goodness of Fit measures for the first data.

Distribution	Parameter Estimates	S. E	Log-Lik.	AIC	BIC	KS Statistic	sig
EXP	0.1625	0.0297	-84.525	171.050	172.452	0.263	0.025
LD	0.2885	0.0377	-79.964	161.928	163.329	0.179	0.257
XLD	0.2631	0.0347	-81.261	164.523	165.924	0.201	0.152
GLD	0.4180 2.2660	0.0678 0.7058	-76.732	157.464	160.267	0.080	0.981
Weibull	0.0260 1.8801	0.0153 0.2590	-77.034	158.069	160.871	0.100	0.8976
PL	1.3774 0.1421	0.1648 0.0494	-77.009	158.019	160.821	0.092	0.942
NH	30.353 0.0035	20.443 0.0023	-79.913	163.826	166.628	0.193	0.190
EXLD	2.7254 0.4103	0.8215 0.0649	-76.713	157.426	160.229	0.080	0.982
EEXLD	1.3358 2.3897 0.4377	0.2155 0.7955 0.0680	-75.2901	156.5803	160.7839	0.091	0.941

Data 2: The second data set given below is precipitation in inches. The data was use and discuss by Hinkley [22]. The data set consists of the following observations; 0.77, 1.74, 0.8, 1.20, 3.37, 2.2, 3.0, 3.1, 1.51, 2.1, 0.52, 1.95, 1.20, 0.47, 1.43, 1.62, 1.31, 0.32, 0.59, 0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9 and 2.05. Further, the visual display of the precipitation in inches dataset provided in Figure 10. We also present a visual comparison using fitted pdf, cdf, survival function (sf) and Q-Q plot given in Figure 11. The MLEs, Standard errors and goodness of fit for this data are given in Table 5.

(a) (b)

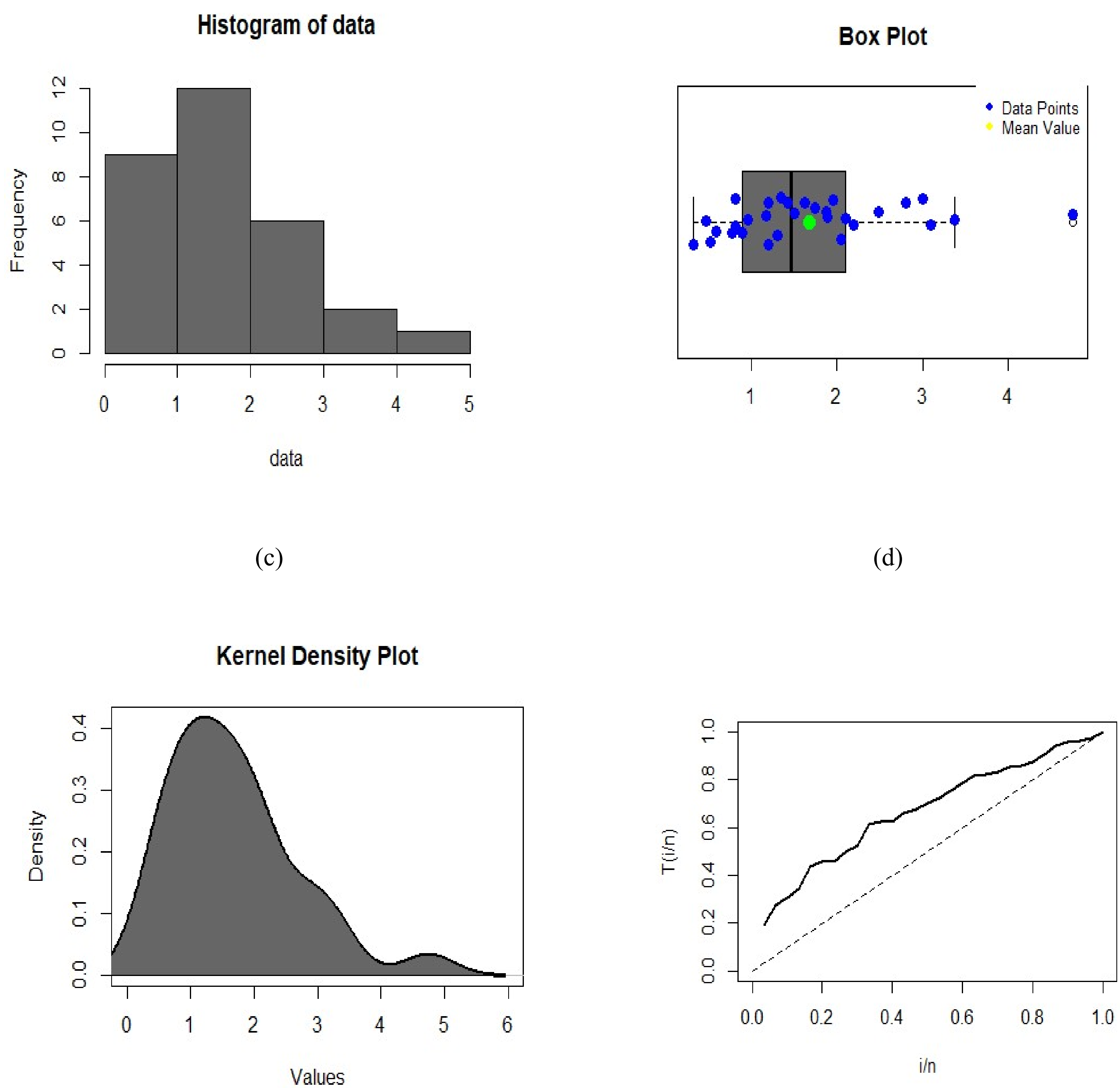


Figure.10. (a) histogram, (b) Kernel density, (c) boxplot and (d) TTT plot. A visual display of the behavior of the second dataset.

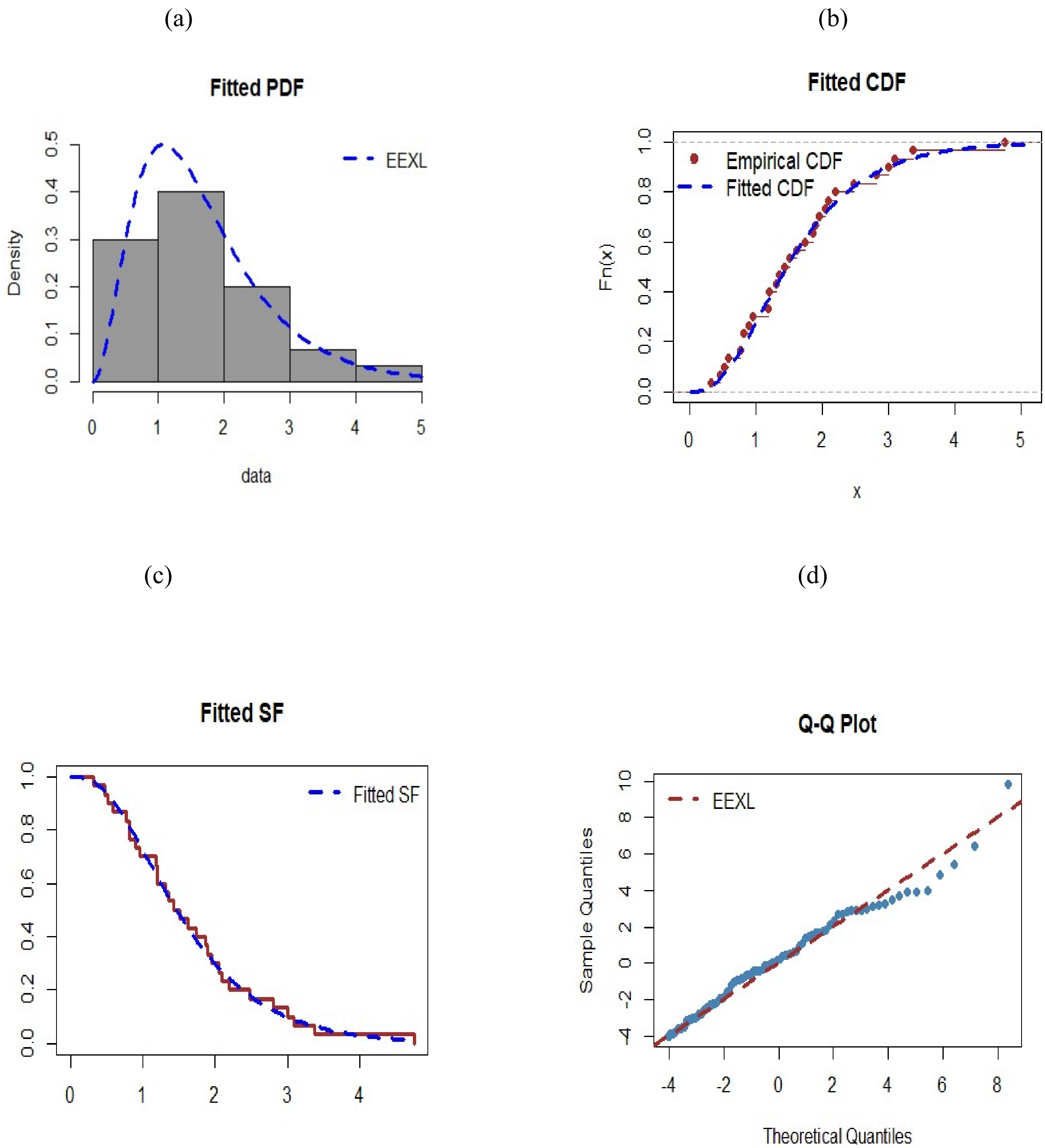


Figure.11. Illustration of the fitted (a) pdf, (b) cdf, (c) sf, (d) Q-Q plot of the EEXL model for the second data.

Table 5. Maximum Likelihood Estimates and Goodness of Fit measures for the second data.

Distribution	Parameter Estimates	S. E	Log-Lik.	AIC	BIC	KS Statistic	sig
EXP	0.5970	0.1090	-45.474	92.949	94.350	0.2352	0.072
LD	0.9097	0.1274	-43.144	88.287	89.689	0.1883	0.238
XLD	0.7799	0.1108	-44.548	91.906	92.497	0.2142	0.128
GLD	1.4528 2.8188	0.2327 0.8830	-38.120	80.240	83.043	0.0872	0.999
Weibull	0.3154 1.8090	0.0906 0.2491	-38.643	81.287	84.089	0.0689	0.999
PL	1.5263 0.6460	0.1924 0.1243	-38.872	81.745	84.548	0.0682	0.999
NH	22.071 0.0175	29.338 0.0239	-41.428	86.856	89.659	0.1579	0.442
EXLD	3.2795 1.3370	1.0301 0.2144	-38.088	80.175	82.978	0.0626	1.000
EEXLD	4.3793 3.3075 2.0000	0.8690 1.0767 0.2629	-38.087	78.17564	79.57684	0.102	0.91

10.Conclusion:

In this study, the Exponentiated-Exponentiated XLindley (EEXL) distribution was proposed by incorporating an additional shape parameter to enhance the flexibility of EEXL distribution. Mathematical expressions for various properties have been derived. These include density expansion, quantile function, survival function, hazard function, cumulative hazard function, moments, generating function, Mode, Renyi Entropy, Mean-Residual life. The model parameters were estimated using ML estimation. Simulation study was used to assess the consistency in biasness and mean square error (MSE). The results revealed that as we increase the sample size, the estimators become closer and closer to the actual values of the parameters.

The applications of the EEXL distribution was demonstrated using two datasets from distinct fields, including COVID-19 mortality rates and precipitation. The proposed EEXL distribution was compared with eight some existing distributions. The analysis reveals that the EEXL distribution performs better as compared to other distributions using well known goodness of fit criteria.

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