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Modelling the Reliability of Cement Sheath Data with the Poly-Exponential Weibull Distribution

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ABSTRACT

The objective of this study is to assess the applicability of the three-parameter Poly-Exponential Weibull distribution for modelling the reliability of cement sheath data, specifically based on Vickers hardness measurements (MPa). This research explores the theoretical properties of the Poly-Exponential Weibull distribution, including the derivation of its quantile function, incomplete moments, Rényi and q-entropies, mean deviations, and the Bonferroni and Lorenz curves. Parameter estimation is performed using the method of maximum likelihood. The findings suggest that the Poly-Exponential Weibull model offers a promising alternative to existing models in the literature, particularly for handling highly skewed reliability data.

Keywords: Weibull exponential; Poly-Weibull; moments; entropies; maximum likelihood estimation.

1. Introduction

There are many ways to infer properties and characteristics of life-testing experiments based on various probability distributions. Depending on the probability distribution, a theory about how well it estimates the parameters of the probability model is used to explain the life testing framework. However, this framework is lacking in various ways when the distribution has a large variance. To overcome this issue, various distributions have been studied in the statistics literature. The well-known family of probability distributions is the Weibull distribution for studying life-testing problems whose shapes are unimodal, skewed, and roughly symmetric. It is a versatile family of probability distributions that can have remarkable effects on the behavior of supplementary types of probability models based on the value of the shape parameter. The Weibull distribution is a widely used parametric family because it is a flexible parameter family structure that has been found in practice to be more flexible for fitting a wide range of real-world applications. The motivations for its extensive deployment for developing new probability distributions that are more flexible for modelling reliability/survival data, because the information can present a high degree of skewness and kurtosis. The Weibull distribution was established by Waloddi Weibull [1] and has been applied comprehensively to construct a new family of lifetime distributions. A random variable X has the classical Weibull distribution with two parameters α and β, $x > 0$, its cumulative distribution function (cdf) is given by

$$G(x;\alpha,\beta) = 1 - exp\left(-\alpha x^{\beta}\right), \tag{1}$$

the probability density function corresponding to (1) is given by

$$g(x;\alpha,\beta) = \alpha\beta\, x^{\beta-1} exp\left(-\alpha x^{\beta}\right), \tag{2}$$

where α and β are the scale and shape parameters of the Weibull distribution, respectively.

The Poly-Weibull Distribution was pioneered by Berger et al. [2] as a generalization of the standard Weibull distribution, who studied the Bayesian framework using informative priors. This research paper aims to provide a new bathtub-shaped failure rate distribution, namely the three-parameter Poly exponential Weibull distribution, and investigate the potential usefulness of this model which extends the recent work of Chesneau et al. [3]. This research examines the three-

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parameter Poly exponential Weibull distribution properties and parametric estimation using both graphical methods and maximum likelihood estimation. Khan et al. [4] studied the transmuted Weibull distribution with regression analysis and discussed the various structural properties with applications. Mudholkar and Srivastava [5] introduced the idea of exponentiated distribution and studied the exponentiated Weibull distribution as an extension of the Weibull distribution. Zaindin and Sarhan [6] introduced the new generalized Weibull distribution for modelling lifetime data. Xie and Lai [7] inspected the reliability of an additive Weibull model with a bathtub-shaped failure rate function. According to this approach, a random variable X is said to have a Poly exponential G-distribution if its cumulative distribution function (cdf) satisfies the relationship

$$F(x) = \frac{1}{1-(1-\theta)exp(\theta)}\{1-\left(1-\theta G(x)\right) exp\left(\theta G(x)\right)\}, \tag{3}$$

and

$$f(x) = \frac{1}{1-(1-\theta)exp(\theta)}\theta^2 G(x) e^{\theta G(x)} g(x), \tag{4}$$

where $G(x)$ is the cdf of the base distribution, $g(x)$ and $f(x)$ are the corresponding probability density function (pdf) associated with $G(x)$ and $F(x)$ respectively. The article is organized as follows: Section 2 presents the analytical shapes of the probability density and hazard function of the PEW distribution. Section 3 derives the incomplete moments, the moment generating function, and the quantile function. Maximum likelihood estimates (MLEs) of the unknown parameters and the asymptotic confidence intervals of the unknown parameters of the PEW models are discussed in Section 4. Rényi, q-entropies, mean deviation, Bonferroni, and Lorenz curves are derived in Section 5. The flexibility of the PEW family is illustrated using cement sheath data in Section 6. Concluding remarks are addressed in Section 7.

2. Poly Exponential Weibull Distribution

A random variable X is said to have Poly exponential Weibull distribution with parameters $\alpha > 0$ $\beta > 0$ and $\theta \in \mathbb{R}$, then X has the distribution function as

$$F(x;\alpha,\beta,\theta) = w\left[1-\left\{1-\theta\left(1-exp\left(-\alpha x^{\beta}\right)\right)\right\} exp\left(\theta-\theta exp\left(-\alpha x^{\beta}\right)\right)\right], \tag{5}$$

where $w = 1/\{1-(1-\theta)exp(\theta)\}$, the probability density function corresponding to (5) is given by

$$f(x;\alpha,\beta,\theta) = \alpha\beta\theta^2\, w\, x^{\beta-1} exp\left(-\alpha x^{\beta}-\theta exp\left(-\alpha x^{\beta}\right)\right)\left(1-exp\left(-\alpha x^{\beta}\right)\right). \tag{6}$$

The parameter β controls the shape of the distribution, whereas the parameter $\alpha\ and\ \theta$ control the scale of the distribution, respectively. If X has the Poly exponential Weibull distribution, then it can be denoted as $X{\sim}PEW(x;\alpha,\beta,\theta)$. Figure 1 shows the shape of the Poly exponential Weibull PDF with different choices of parameters. The x-axis represents the variable of interest, while the y-axis represents the probability. The PEW distribution curves have been overlaid on the graph, demonstrating how well the theoretical distribution fits the empirical data. The parameters of the PEW distribution provide insights into the shape and scale of the data. These parameters are crucial for understanding the distribution's behaviour and making predictions based on the PEW model.

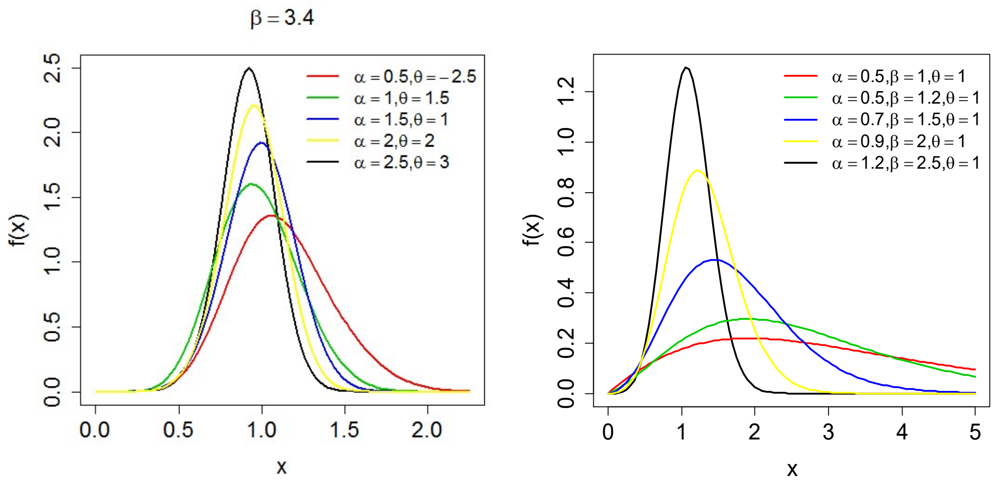

Figure 1: Plots of the Poly exponential Weibull PDF

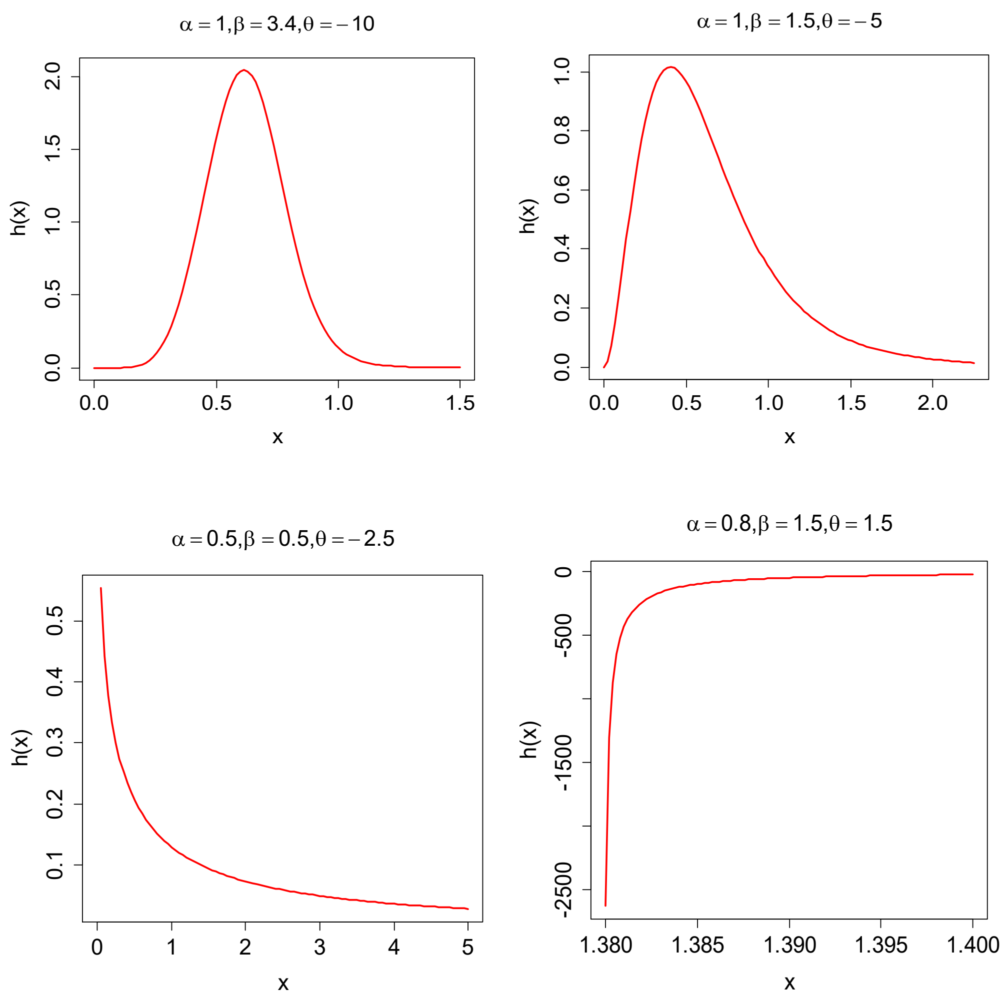

Figure 2: Plots of the Poly exponential Weibull Hazard Function

If X has the PEW distribution, then the reliability function and hazard functions are given by

$$R(x;\alpha,\beta,\theta)=1-w\left[1-\left\{1-\theta\left(1-exp\left(-\alpha x^{\beta}\right)\right)\right\}exp\left(\theta-\theta exp\left(-\alpha x^{\beta}\right)\right)\right], \quad (7)$$

and

$$h(x;\alpha,\beta,\theta)=\frac{\alpha\beta\theta^{2}\,w\,x^{\beta-1}exp\left(-\alpha x^{\beta}-\theta exp\left(-\alpha x^{\beta}\right)\right)\left(1-exp\left(-\alpha x^{\beta}\right)\right)}{1-w\left[1-\left\{1-\theta\left(1-exp\left(-\alpha x^{\beta}\right)\right)\right\}exp\left(\theta-\theta exp\left(-\alpha x^{\beta}\right)\right)\right]}. \quad (8)$$

Figure 2 displays the PEW hazard rates graph yields valuable insights into the failure characteristics of the system. The hazard graph illustrates the hazard function of the PEW model, providing insights into the failure pattern over time. The x-axis typically represents time, and the y-axis depicts the hazard rate, representing the probability of an event occurring at a specific point in time given survival up to that time. These features of the instantaneous failure rates illustrate that the PEW distribution has increasing, decreasing, and upside-down bathtub shapes with different choices of parameters.

3. Moments and Quantiles

This section presents the k^{th} incomplete moments, moment generating function, and derivation of the quantile model of the Poly exponential Weibull distribution.

Theorem 1: If X has the $PEW(x;\alpha,\beta,\theta)$, then the k^{th} incomplete moment of X say $\psi_{k(q)}$ is given as follows

$$\psi_{k(q)}=\left\{\sum_{i=0}^{\infty}T_{i,1,k,\beta,\theta}\,\gamma\left(\frac{k}{\beta}+1,\alpha(i+1)q^{\beta}\right)+\sum_{i=0}^{\infty}T_{i,2,k,\beta,\theta}\,\gamma\left(\frac{k}{\beta}+1,\alpha(i+2)q^{\beta}\right)\right\}.$$

Proof: The k^{th} incomplete moment of the PEW distribution as follows

$$\psi_{k(q)}=\int_{0}^{q}x^{k+\beta-1}\,\alpha\beta\theta^{2}\,w\,exp\left(-\alpha x^{\beta}-\theta exp\left(-\alpha x^{\beta}\right)\right)\left(1-exp\left(-\alpha x^{\beta}\right)\right)\,dx,$$

the above integral reduces to

$$\psi_{k(q)}=\frac{\alpha\beta\theta^{2}exp(\theta)}{\{1-(1-\theta)exp(\theta)\}}\int_{0}^{q}x^{k+\beta-1}\,exp\left(-\alpha x^{\beta}-\theta exp\left(-\alpha x^{\beta}\right)\right)\,dx+\frac{\alpha\beta\theta^{2}exp(\theta)}{\{1-(1-\theta)exp(\theta)\}}\int_{0}^{q}x^{k+\beta-1}\,exp\left(-2\alpha x^{\beta}-\theta exp\left(-\alpha x^{\beta}\right)\right)\,dx,$$

Using Taylor series expansions, the above integral reduces to

$$\psi_{k(q)}=\frac{\alpha\beta\theta^{2}exp(\theta)}{\{1-(1-\theta)exp(\theta)\}}\sum_{i=0}^{\infty}\frac{(-1)^{i}\theta^{i}}{i!}\int_{0}^{q}x^{k+\beta-1}\,exp\left(-\alpha x^{\beta}(i+1)\right)\,dx+\frac{\alpha\beta\theta^{2}exp(\theta)}{\{1-(1-\theta)exp(\theta)\}}\sum_{i=0}^{\infty}\frac{(-1)^{i}\theta^{i}}{i!}\int_{0}^{q}x^{k+\beta-1}\,exp\left(-\alpha x^{\beta}(i+2)\right)\,dx,$$

Finally, we obtain

$$\psi_{k(q)}=\left\{\sum_{i=0}^{\infty}T_{i,1,k,\beta,\theta}\,\gamma\left(\frac{k}{\beta}+1,\alpha(i+1)q^{\beta}\right)+\sum_{i=0}^{\infty}T_{i,2,k,\beta,\theta}\,\gamma\left(\frac{k}{\beta}+1,\alpha(i+2)q^{\beta}\right)\right\}. \quad (9)$$

where

$$T_{i,j,k,\beta,\theta}=\frac{\theta^{2}exp(\theta)\alpha^{-\frac{k}{\beta}}}{\{1-(1-\theta)exp(\theta)\}}\frac{(-1)^{i}\theta^{i}}{i!\,(i+j)^{\frac{k}{\beta}+1}}\Gamma\left(\frac{k}{\beta}+1\right),\qquad j=1,2.$$

Theorem 2: If X has the $PEW(x;\alpha,\beta,\theta)$, then the moment generating function of X, $M_X(t)$ is given as follows

$$M_X(t) = \frac{\theta^2 exp(\theta)}{\{1-(1-\theta)exp(\theta)\}} \sum_{m=0}^{\infty} \frac{t^m U_{m,\beta,\theta}}{m!\, \alpha^{\frac{m}{\beta}}} \Gamma\left(\frac{m}{\beta}+1\right),$$

where

$$U_{m,\beta,\theta} = \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{i!\,(i+1)^{\frac{m}{\beta}+1}} + \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{i!\,(i+2)^{\frac{m}{\beta}+1}}.$$

Proof: The moment generating function of the TW distribution as follows

$$M_X(t) = \int_0^{\infty} x^{\beta-1} \alpha\beta\theta^2\, w\, exp\left(tx - \alpha x^\beta - \theta exp(-\alpha x^\beta)\right)\left(1 - exp(-\alpha x^\beta)\right) dx,$$

the above expression can be written as

$$M_X(t) = \frac{\alpha\beta\theta^2 exp(\theta)}{\{1-(1-\theta)exp(\theta)\}} \int_0^{\infty} x^{\beta-1} exp\left(tx - \alpha x^\beta - \theta exp(-\alpha x^\beta)\right) dx + \frac{\alpha\beta\theta^2 exp(\theta)}{\{1-(1-\theta)exp(\theta)\}} \int_0^{\infty} x^{\beta-1} exp\left(tx - 2\alpha x^\beta - \theta exp(-\alpha x^\beta)\right) dx,$$

using the Taylor series expansions, the above integral reduces to

$$= \frac{\alpha\beta\theta^2 exp(\theta)}{\{1-(1-\theta)exp(\theta)\}} \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^{\infty} x^{m+\beta-1} exp\left(-\alpha x^\beta - \theta exp(-\alpha x^\beta)\right) dx + \frac{\alpha\beta\theta^2 exp(\theta)}{\{1-(1-\theta)exp(\theta)\}} \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^{\infty} x^{m+\beta-1} exp\left(-2\alpha x^\beta - \theta exp(-\alpha x^\beta)\right) dx,$$

The above integral for $M_X(t)$ can be finally obtained as

$$M_X(t) = \frac{\theta^2 exp(\theta)}{\{1-(1-\theta)exp(\theta)\}} \times \sum_{m=0}^{\infty} \frac{t^m}{m!\, \alpha^{\frac{m}{\beta}}} \Gamma\left(\frac{m}{\beta}+1\right) \left\{ \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{i!\,(i+1)^{\frac{m}{\beta}+1}} + \sum_{i=0}^{\infty} \frac{(-1)^i \theta^i}{i!\,(i+2)^{\frac{m}{\beta}+1}} \right\}. \quad (10)$$

which completes the proof.

Theorem 3: The qth quantile $F(x_q)$ of the PEW random variable is given by

$$F(x_q) = \left[-\frac{1}{\alpha} ln \left\{ 1 - \frac{1}{2\theta} \pm \frac{1}{2\theta} \sqrt{1 - 4ln\{1 - q(1-(1-\theta)exp(\theta))\}} \right\} \right]^{\frac{1}{\beta}}, \quad 0 < q < 1. \quad (11)$$

Proof: The qth quantile x_q of the PEW distribution is defined as

$$q = P(X \le x_q) = F(x_q), \qquad x_q \ge 0.$$

Using the distribution function of the PEW distribution we have

$$q = F(x_q) = w\left[1 - \{1 - \theta(1 - exp(-\alpha x^\beta))\} exp\left(\theta - \theta exp(-\alpha x^\beta)\right)\right],$$

that is

$$w\left[1 - \{1 - \theta(1 - exp(-\alpha x^\beta))\} exp\left(\theta - \theta exp(-\alpha x^\beta)\right)\right] + q = 0.$$

the above equation can be written as, by setting $D = exp(-\alpha x^\beta)$

$$\theta^2 D^2 - (2\theta^2 - \theta)D + \theta^2 - \theta + ln\{1 - q(1-(1-\theta)exp(\theta))\} = 0$$

Consider this as a quadratic equation as

$\Delta = 1 - 4ln\{1 - q(1-(1-\theta)exp(\theta))\}$.)

It has roots $\frac{(2\theta^2-\theta)\pm\theta\sqrt{\Delta}}{2\theta^2}$. These exist if Δ is positive, then

$$exp\left(-\alpha x^{\beta}\right) = \frac{(2\theta^2 - \theta) \pm \theta\sqrt{\Delta}}{2\theta^2}.$$

Finally, we obtain the qth quantile x_q of the PEW distribution as

$$x_q = \left[-\frac{1}{\alpha} ln\left\{1 - \frac{1}{2\theta} \pm \frac{1}{2\theta}\sqrt{\Delta}\right\}\right]^{\frac{1}{\beta}},$$

which completes the proof.

4. Parameter Estimation

Consider the random samples $x_1, x_2, \ldots, x_n$ consisting of n observations from the Poly exponential Weibull distribution, then the log-likelihood function $\mathcal{L} = \ln L$ of (6) is given by

$$\mathcal{L} = nln\alpha + nln\beta + 2n\, ln\theta + n\theta - n\, ln\left(1 - (1-\theta)exp(\theta)\right) + \\ (\beta - 1)\sum_{i=1}^{n} lnx_i - n\alpha\sum_{i=1}^{n} {x_i}^{\beta} + \sum_{i=1}^{n} ln\left(1 - exp\left(-\alpha {x_i}^{\beta}\right)\right) - n\theta\sum_{i=1}^{n} exp\left(-\alpha {x_i}^{\beta}\right) \quad . \quad (12)$$

Let $\Theta = (\alpha, \beta, \theta)^{\mathrm{T}}$ be the parameter vector. The associated score function is given by $U(\Theta) = (\partial\mathcal{L}/\partial\alpha, \partial\mathcal{L}/\partial\beta, \partial\mathcal{L}/\partial\theta)^{\mathrm{T}}$, where

$$\frac{\partial\mathcal{L}}{\partial\alpha} = \frac{n}{\alpha} - nln\beta - n\sum_{i=1}^{n} {x_i}^{\beta} + \sum_{i=1}^{n} \frac{{x_i}^{\beta}\, exp\left(-\alpha {x_i}^{\beta}\right)}{(1 - exp(-\alpha {x_i}^{\beta}))} + n\theta\sum_{i=1}^{n} {x_i}^{\beta}\, exp\left(-\alpha {x_i}^{\beta}\right),$$

$$\frac{\partial\mathcal{L}}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} lnx_i - n\alpha\sum_{i=1}^{n} {x_i}^{\beta} ln(x_i) \\ + \sum_{i=1}^{n} \frac{\alpha\, exp\left(-\alpha {x_i}^{\beta}\right) {x_i}^{\beta} ln(x_i)}{(1 - exp(-\alpha {x_i}^{\beta}))} + n\alpha\theta\sum_{i=1}^{n} exp\left(-\alpha {x_i}^{\beta}\right) {x_i}^{\beta} ln(x_i),$$

$$\frac{\partial\mathcal{L}}{\partial\theta} = \frac{2n}{\theta} + n - \frac{n\,\theta\, exp(\theta)}{1 - (1-\theta)exp(\theta)} - n\sum_{i=1}^{n} exp\left(-\alpha {x_i}^{\beta}\right).$$

The maximum likelihood estimates (MLEs) of the parameter vector $\Theta = (\alpha, \beta, \theta)^{\mathrm{T}}$ are obtained by solving the non-linear equations $\mathrm{U}(\Theta) = 0$. These systems of non-linear equations can be solved numerically by using softwares such as R, SAS, and MAPLE.

For interval estimation and hypothesis tests on the model parameters of the PEW distribution, we require the 3×3 unit observed information matrix is

$$I_n(\Theta) = -\begin{pmatrix} I_{\alpha,\alpha} & I_{\alpha,\beta} & I_{\alpha,\theta} \\ I_{\alpha,\beta} & I_{\beta,\beta} & I_{\beta,\theta} \\ I_{\alpha,\theta} & I_{\beta,\theta} & I_{\theta,\theta} \end{pmatrix},$$

The elements of the 3×3 information matrix $I_n(\Theta)$ are given by

$$\frac{\partial^2\mathcal{L}}{\partial\alpha^2} = -\frac{n}{\alpha^2} - \sum_{i=1}^{n} \frac{{x_i}^{2\beta}\, exp\left(-\alpha {x_i}^{\beta}\right)}{(1 - exp(-\alpha {x_i}^{\beta}))} - n\theta\sum_{i=1}^{n} exp\left(-\alpha {x_i}^{\beta}\right)\left({x_i}^{\beta}\right)^2$$

$$\frac{\partial^2\mathcal{L}}{\partial\beta^2} = -\frac{n}{\beta^2} - n\alpha\sum_{i=1}^{n} {x_i}^{\beta} ln(x_i)^2 + n\alpha\theta\sum_{i=1}^{n} exp\left(-\alpha {x_i}^{\beta}\right) {x_i}^{\beta} ln(x_i)^2 \left(1 - \alpha {x_i}^{\beta}\right) \\ + \alpha\sum_{i=1}^{n} \frac{exp\left(-\alpha {x_i}^{\beta}\right) {x_i}^{\beta} ln(x_i)^2 \left[1 - \alpha {x_i}^{\beta} - exp\left(-\alpha {x_i}^{\beta}\right)\left(1 - \alpha\, {x_i}^{\beta} + \alpha^2 {x_i}^{\beta} ln(x_i)\right)\right]}{(1 - exp(-\alpha {x_i}^{\beta}))^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial\theta^2} = -\frac{2n}{\theta^2} - \frac{n\ exp(\theta)\big(1+\theta-exp(\theta)\big)}{\big(1-(1-\theta)exp(\theta)\big)^2}$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\beta} = n\sum_{i=1}^{n} x_i^{\beta} ln(x_i) + 2\theta\sum_{i=1}^{n} exp\big(-\alpha x_i^{\beta}\big)\, x_i^{\beta} ln(x_i)\big(1-\alpha x_i^{\beta}\big) + \sum_{i=1}^{n}\frac{exp\big(-\alpha x_i^{\beta}\big)\, x_i^{\beta} ln(x_i)\left(1-\alpha x_i^{\beta}-exp\big(-\alpha x_i^{\beta}\big)\right)}{(1-exp(-\alpha x_i^{\beta}))^2},$$

$$\frac{\partial^2\mathcal{L}}{\partial\alpha\partial\theta} = -n\sum_{i=1}^{n} exp\big(-x_i^{\beta}\big),$$

and

$$\frac{\partial^2\mathcal{L}}{\partial\beta\partial\theta} = -n\sum_{i=1}^{n} exp\big(-\alpha x_i^{\beta}\big)\, x_i^{\beta} ln(x_i),$$

respectively.

The asymptotic multivariate normal $N_3(0, I_n(\Theta)^{-1})$ distribution can be used to construct the approximate confidence intervals and confidence region of individual parameters for the Poly exponential Weibull distribution. We can compute the maximum likelihood values to find likelihood ratio (LR) statistics for testing the PEW distribution against other lifetime distributions. For testing the hypothesis, we formulate the null hypothesis $H_0: \Theta = \Theta_0$ versus $H_A: \Theta \neq \Theta_0$ can be performed for LR statistics to compare the PEW distribution with other lifetime distributions, as $\Lambda = 2\{l(\hat{\alpha}, \hat{\beta}, \hat{\theta}) - l(\tilde{\alpha}, \tilde{\beta}, \tilde{\theta})\}$, where $\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$ are the MLEs under H_A and $\tilde{\alpha}, \tilde{\beta}, \tilde{\theta}$ are the estimates under H_0.

5. Entropies and Mean Deviation

The entropy of a random variable X with probability density $f(x)$ is a measure of the variation of the uncertainty. A large value of entropy indicates greater uncertainty in the data. The Rényi entropy [8] is defined as

$$I_R(\rho) = \frac{1}{1-\rho} log\{\textstyle\int_0^\infty f(x)^\rho dx\}, \quad (13)$$

where $\rho > 0$ and $\rho \neq 1$. The integral in $I_R(\rho)$ of the $PEW(x; \alpha, \beta, \theta)$ can be defined as

$$\int_0^\infty f(x)^\rho dx = \alpha^\rho\beta^\rho\theta^{2\rho}w^\rho \times \int_0^\infty x^{\rho(\beta-1)} exp\left(-\alpha\rho x^{\beta} - \theta\rho exp\big(-\alpha x^{\beta}\big)\right)\big(1-exp\big(-\alpha x^{\beta}\big)\big)^\rho\ dx,$$

The above integral reduces to

$$\int_0^\infty f(x)^\rho dx = \sum_{j,k=0}^{\infty} \mathcal{U}_{\theta,\beta,\rho,j}\ \frac{(-1)^{j,+k}(\theta\rho)^k}{k!}\int_0^\infty x^{\rho(\beta-1)} exp\left(-(\rho+j+k)\alpha x^{\beta}\right) dx,$$

where

$$\mathcal{U}_{\theta,\beta,\rho,j} = \binom{\rho}{j}\frac{\alpha^\rho\beta^\rho\theta^{2\rho}\ exp(\theta\rho)}{\big(1-(1-\theta)exp(\theta)\big)^\rho}.$$

$$\int_0^\infty f(x)^\rho dx = \sum_{j,k=0}^{\infty} \mathcal{U}_{\theta,\beta,\rho,j}\ \frac{(-1)^{j,+k}(\theta\rho)^k\{\alpha(\rho+j+k)\}^{-\rho+\frac{\rho}{\beta}+\beta-1}}{k!}\Gamma\left\{\rho-\frac{\rho}{\beta}-\beta+2\right\},$$

(14)

Finally, we obtain the Rényi entropy as

$$I_R(\rho)=\frac{\left(\frac{\rho}{\beta}+\beta-1\right)}{1-\rho}\log(\alpha)-\log(\beta)+\frac{2\rho}{1-\rho}\log(\theta)+\frac{1}{1-\rho}\log$$

$$\left\{\sum_{j,k=0}^{\infty}\binom{\rho}{j}\frac{(-1)^{j,+k}(\theta\rho)^k\,exp(\theta\rho)\,(\rho+j+k)^{-\rho+\frac{\rho}{\beta}+\beta-1}}{\left(1-(1-\theta)exp(\theta)\right)^{\rho}k!}\Gamma\left\{\rho-\frac{\rho}{\beta}-\beta+2\right\}\right\}. \tag{15}$$

The q-(or α entropy) was introduced by Havrda and Charvat [9] and Ullah [10] stated that q-(or α entropy) measures are the monotonic functions of the Rényi entropy and is defined as

$$I_H(q)=\frac{1}{q-1}\left\{1-\int_0^{\infty}f(x)^q\,dx\right\}, \tag{16}$$

where $q>0$ and $q\neq 1$. The integral in $I_H(q)$ of the PEW distribution can be defined as

$$\int_0^{\infty}f(x)^q dx=\alpha^q\beta^q\theta^{2\rho}w^q$$

$$\times\int_0^{\infty}x^{q(\beta-1)}exp\left(-\alpha qx^{\beta}-\theta q exp\left(-\alpha x^{\beta}\right)\right)\left(1-exp\left(-\alpha x^{\beta}\right)\right)^q\;dx,$$

Using (16), the above integral reduces to

$$\int_0^{\infty}f(x)^q\,dx=$$

$$\sum_{j,k=0}^{\infty}\binom{q}{j}\;\frac{(-1)^{j,+k}\alpha^q\beta^q\theta^{2q}\,exp(\theta q)\,(\theta q)^k\{\alpha(q+j+k)\}^{-q+\frac{q}{\beta}+\beta-1}}{\left(1-(1-\theta)exp(\theta)\right)^{q}k!}\Gamma\left\{q-\frac{q}{\beta}-\beta+2\right\}.$$

Finally, we can write $I_H(q)$ as

$$I_H(q)=\frac{1}{q-1}\left\{\begin{array}{c}1-\sum_{m=0}^{\infty}\binom{q}{j}\frac{\alpha^q\beta^q\theta^{2q}\,exp(\theta q)}{\left(1-(1-\theta)exp(\theta)\right)^q}\\ \frac{(-1)^{j,+k}(\theta q)^k\{\alpha(q+j+k)\}^{-q+\frac{q}{\beta}+\beta-1}}{k!}\Gamma\left\{q-\frac{q}{\beta}-\beta+2\right\}\end{array}\right\}. \tag{17}$$

The degree of scatter in a population is calculated by the totality of deviations from the mean and median. If X has the $PEW(x;\alpha,\beta,\theta)$, then we derive the mean deviation about the mean and about the median M can be obtained from the following equations

$$\delta_1=2[\mu\,F(\mu)-\psi(\mu)]\quad\text{and}\quad\delta_2=\mu-2\psi(\mathrm{M}) \tag{18}$$

where $\psi(q)$ can be obtained from (9) by substituting $k=1$

$$\psi(q)=\sum_{i=0}^{\infty}T_{i,1,1,\beta,\theta}\,\gamma\left(\frac{1}{\beta}+1,\alpha(i+1)q^{\beta}\right)+\sum_{i=0}^{\infty}T_{i,2,1,\beta,\theta}\,\gamma\left(\frac{1}{\beta}+1,\alpha(i+2)q^{\beta}\right). \tag{19}$$

The quantity $\psi(q)$ can also be used to determine the Bonferroni and Lorenz curves which have applications in insurance, econometrics and finance. For a given probability P, they are given by

$$B(P)=\frac{\psi(q)}{P\mu}\qquad\text{and}\qquad L(P)=\frac{\psi(q)}{\mu},$$

respectively, where $q=Q(P)$ follows from the quantile function.

Table 1 presents an analysis of the Rényi and q entropies for the Poly Exponential Weibull distribution. The distribution is characterized by three parameters α, β, and θ. The entropies are computed for selected parameter values, and the focus is on two different measures: Rényi entropy with parameter ρ (ρ=2 and ρ=3), and q entropy with parameter q (q=2 and q=3). Table 1 illustrates the effect of parameters on Rényi and q entropies for the Poly Exponential Weibull distribution. As observed from the results, varying the parameters α, β, and θ has a notable impact on both Rényi and q entropies. Different parameter combinations lead to different entropy values, indicating the sensitivity of entropy measures to distribution parameters. In terms of comparative analysis, the values of Rényi and q entropies provide insights into the distribution's uncertainty and variability. Comparative analysis between different parameter sets can help in understanding the distribution's behaviour under various conditions. These entropy measures can be valuable in applications such as risk assessment, information theory, and reliability analysis. The choice of ρ and q parameters allows for tailoring the analysis to specific requirements.

Table 1: Rényi and q entropies of selected parameter values for PEW distribution

α	β	θ	$I_R(\rho=2)$	$I_R(\rho=3)$	$I_H(q=2)$	$I_H(q=3)$
0.5	0.5	1	1.3988	1.2924	0.9601	0.4987
		1.5	1.4698	1.3748	0.9661	0.4991
		2	1.5302	1.4464	0.9705	0.4994
		3	1.6273	1.5568	0.9764	0.4996
1	1	1	0.5249	0.4882	0.7014	0.4472
		1.5	0.5419	0.5059	0.7129	0.4513
		2	0.5552	0.5194	0.7215	0.4543
		3	0.5734	0.5376	0.7329	0.4579
1	2	1	0.1784	0.1476	0.3369	0.2466
		1.5	0.1754	0.1441	0.3323	0.2425
		2	0.1710	0.1391	0.3254	0.2366
		3	0.1596	0.1269	0.3075	0.2213
2	1.5	1	0.1216	0.0901	0.2441	0.1698
		1.5	0.1248	0.0930	0.2498	0.1743
		2	0.1259	0.0938	0.2517	0.1754
		3	0.1238	0.0909	0.2481	0.1710

6. Application: Cement sheath data

This section provides the data analysis to assess the goodness-of-fit of the PEW distribution and compare this model with three different Weibull family of lifetime distributions for modelling the reliability of hardness measurements of cement sheath performance data. when various thermal cements were cured at 35 °C and then heated to 230 °C. The data set consists of 50 observations, which were originally reported by Jon. M and Karen. L [11]. In this study, I fitted the Poly-exponential Weibull (PEW), Exponentiated Weibull (EW), New Generalized Weibull (NGW), Additive Weibull (AW), Generalized Power Weibull (GPW), and Weibull (W) distributions by the method of maximum likelihood. The analysis of Maximum Likelihood estimates and information criteria for different distributions applied to cement sheath data provides insights into the goodness of fit and model complexity. The NGW and PEW distributions appear to be competitive models based on the presented information, with the NGW distribution demonstrating particularly favorable results. Further investigation and model refinement may be warranted to ensure the selected model accurately represents the underlying data distribution.

The required numerical evaluations are implemented using the R language [12]. The MLEs and the values of maximized log-likelihoods for Poly EW, EW, NGW, GPW, AW, and Weibull distributions are displayed in Table 2. Table 2 gives the MLEs of the unknown parameters (with their standard errors) and the AIC (Akaike Information Criterion), BIC (Bayesian information criterion), and the HQIC (Hannan-Quinn information criterion) goodness of fit tests. The NGW distribution exhibits lower AIC, BIC, and HQIC compared to the AW distribution, suggesting that the NGW model provides a better fit to the data. The GPW distribution has a relatively high AIC and BIC, indicating that the model may be less preferable in terms of goodness of fit and model complexity. The PEW distribution's parameter estimates are provided in Table 2, making it a reliable and better model fit for the hardness measurements of cement sheath data. For the PEW model, its AIC, BIC, and HQIC values are relatively low, suggesting a competitive fit.

Table 2: MLEs of the Parameters for the cement sheath data, the Corresponding SEs (in parenthesis) with the AIC, BIC and HQIC measures.

Distribution	Parameter Estimates				AIC	BIC	HQIC
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\eta}$			
NGW	10.5933 (8.6378)	0.0563 (0.0778)	0.0306 (0.0220)	1.5227 (0.1468)	321.08	328.72	323.99
AW	0.2886 (0.1751)	0.2216 (0.1258)	0.2708 (0.1521)	0.2008 (0.3612)	587.24	594.89	590.16
PEW	0.0156 (0.0077)	1.6515 (0.1325)	17.0136 (6.3305)	-	317.03	322.77	319.22
EW	12.3750 (3.9623)	1.7112 (0.1098)	0.0115 (0.0047)	-	318.34	324.07	320.52
GPW	3.6649 (11.0749)	0.0678 (0.2050)	-	-	509.58	513.41	511.04
W	0.6649 (0.0523)	0.0768 (0.0151)	-	-	390.23	394.05	391.68

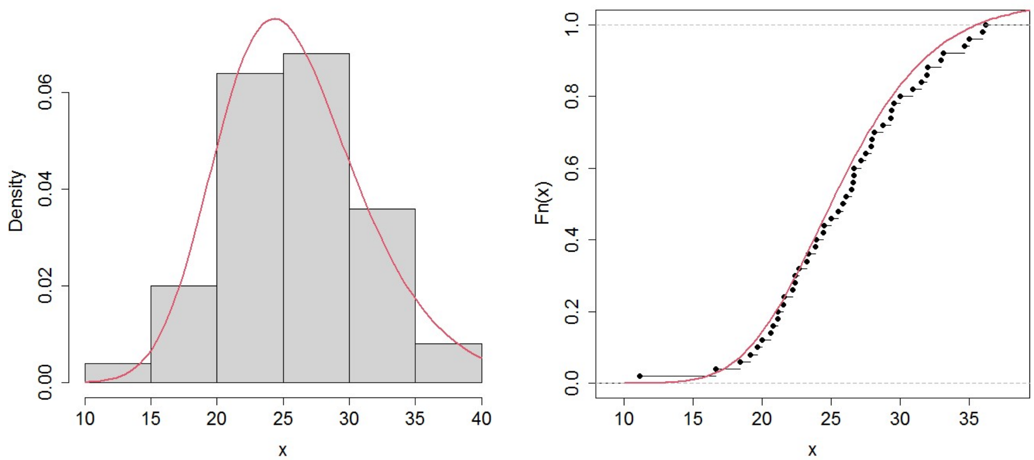

Figure 3: Plots of fitted Poly-exponential Weibull model for the reliability of hardness measurements of cement sheath data.

Comparing five distributions with the Poly-Exponential Weibull distribution using the AIC, BIC, and HQIC indicates that the PEW distribution provides a better fit for the reliability of hardness measurements of cement sheath performance data. The graphical goodness of fit displayed in Figure 3 indicates that the PEW distribution provides a better fit than the other five distributions. Using the maximum likelihood estimates of the unknown parameters for cement sheath data, we obtain the approximately 95% two-sided confidence interval for the parameters α, β and θ are [0.0001, 0.0311], [1.3852, 1.9178], and [4.2920, 29.7352], respectively.

7. Concluding Remarks

This article discusses the theoretical properties of the Poly exponential Weibull distribution and examines the potential usefulness of this model with application to the reliability of hardness measurements of cement sheath data. The PEW distribution has increasing, decreasing, and upside-down bathtub shapes with different choices of parameters. We derive the explicit expressions for the incomplete moments, moment generating function, quantile functions, Renyi, Shannon, and q-entropies, mean deviation, Bonferroni, and Lorenz curves. Based on the three goodness-of-fit measures, the PEW distribution provides a better fit than the other five lifetime distributions. The importance of the PEW model is illustrated by means of a real-life application. This research concludes that the poly-exponential transformation provides more flexibility in the PEW distribution for modelling the hardness measurements of cement sheath data.

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