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A New One-Parameter Distribution with Two Turning Points and Bathtub Shaped Failure Rate Function

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ABSTRACT

The frequent development of new lifetime distributions is borne of the desire to obtain adequate fits to complex data sets across various fields of study. A new one-parameter mixture distribution called Emrem distribution is proposed in this study. Mathematical properties of the distribution such as the moments and other related measures, the moment generating function, mean absolute deviations, entropy and reliability indices are discussed extensively. The pdf of the distribution can have two turning points while the hazard rate function has a bathtub shape. The maximum likelihood approach is used to estimate the parameter of the Emrem distribution. We have demonstrated the consistency property of the maximum likelihood estimate through simulations. Results obtained by comparing the fit of the Emrem distribution and fits of the competing distributions to a real data set indicate that the proposed model is capable of outperforming well-known and widely used continuous one-parameter distributions in several data analysis cases.

Keywords: Algorithm, bathtub shape; maximum likelihood method, one-parameter mixture distribution; turning points.

1. Introduction

One-parameter continuous distributions are arguably useful in solving a variety of problems that emanate from diverse fields. For example, the famous exponential distribution (Epstein, 1958) is of immense application in queuing theory (Nair, Sreelatha and Ushakumari, 2021; Soorya and Sreelatha, 2021; Suleiman, Burodo and Ahmed, 2022) and reliability engineering (Abboudi, Al–Mashhadani and Salman, 2020; Duan *et al.*, 2021; Zagurskiy *et al.*, 2023) (Abboudi, Al–Mashhadani and Salman, 2020; Duan *et al.*, 2021; Zagurskiy *et al.*, 2023). However, it is not an ideal distribution for data that do not have a constant hazard rate function (hrf).

In lifetime data modelling, distributions possessing the bathtub shaped, upside down bathtub shaped, increasing and decreasing hazard rate functions are occasionally needed (Nassar and Dobbah, 2020; Sharma, Singh and Shekhawat, 2022; Khan, Bhattacharyya and Mitra, 2023; Uwaeme, Akpan and Orumie, 2023). The Lindley distribution (Lindley, 1958), which was initially formulated to handle problems in Bayesian statistics, has been adopted as an alternative to the exponential distribution. Properties of the distribution have been studied in an extensive manner (Ghitany, Atieh and Nadarajah, 2008). Specifically, the distribution is right-skewed and leptokurtic. Another interesting thing about it is that it is a one-parameter continuous mixture distribution obtained by mixing exponential and gamma distributions. Other one-parameter continuous distributions which are also mixtures of exponential and gamma distributions are in statistical science literature. They include Akash distribution (Shanker, 2015), Shambhu distribution (Shanker, 2016), Ishita distribution (Shanker and Shukla, 2017), Akshaya distribution (Shanker, 2017a), Suja distribution (Shanker, 2017b), Pranav distribution (Shukla, 2018), Odoma

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distribution (Odom and Ijomah, 2019), Nwikpe distribution (Nwikpe, Isaac and Emeka, 2021), Iwueze distribution (Elechi *et al.*, 2022), Juchez distribution (Echebiri and Mbegbu, 2022), Chris-Jerry distribution (Onyekwere and Obulezi, 2022). The common properties of these mixture distributions are they all positively skewed and leptokurtic.

A strong motivation for introducing a mixture distribution is to proffer adequate fits to data that arise from heterogeneous populations (Szulczewski and Jakubowski, 2018; Bagui, Liu and Zhang, 2020). As far as we are aware, none of the existing one-parameter continuous mixture distributions has a probability density function (pdf) with two turning points and hazard rate function with the bathtub shape. The crux of this paper is to propose a new one-parameter lifetime distribution called the Emrem distribution whose probability density function (pdf) and hazard rate function, respectively, have two turning points and a bathtub shape, among other notable properties. The rest of the paper is arranged as follows. Section 2 is predicated on the basic definition of the distribution via its pdf and the associated mathematical properties. Two procedures of estimating the Emrem model parameter are explicated in Section 3. Numerical results corresponding to the Emrem distribution are presented in Section 4, with emphasis on simulation results and real life application of the proposed model. Section 5 contains the conclusion.

2. Mathematical Properties of the Emrem Distribution

We shall begin this section by defining the proposed distribution. Let X be a positive random variable. Then X is said to follow the Emrem distribution with parameter θ if its pdf is given by

$$f(x)=\left(\frac{\theta^2}{\theta+1}\right)\left(1+\frac{\theta^2x^3}{6}\right)e^{-\theta x}, x>0, \theta>0 \tag{1.1}$$

For brevity, whenever X has the distribution, we write $X \sim \text{Emrem}\ (\theta)$.

The Emrem distribution is essentially a mixture of the Exp (θ) and Gamma $(4,\theta)$ distributions with mixing proportions $\frac{\theta}{\theta+1}$ and $\frac{1}{\theta+1}$ respectively. The pdf of the Emrem distribution is graphed in Figure 1 using different values of the parameter. It is obvious that the pdf of the distribution can be a decreasing function or non-monotonic function.

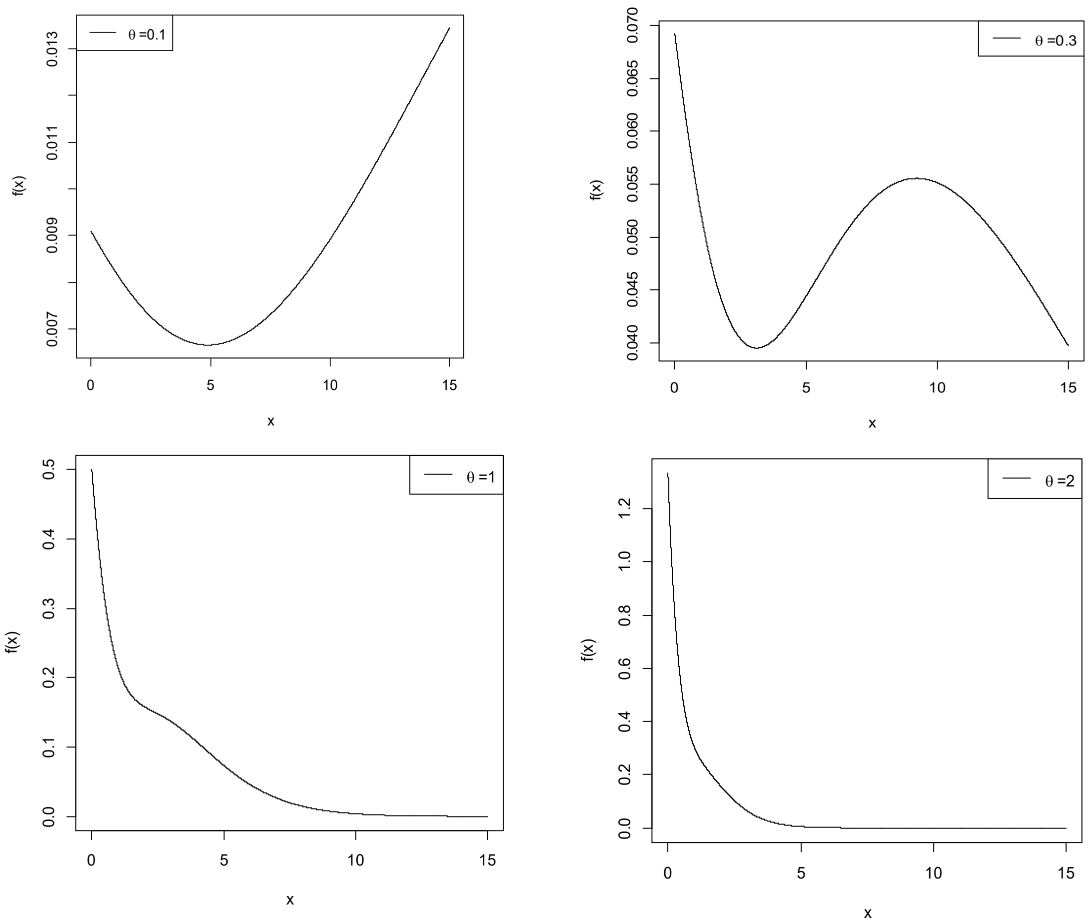

Figure 1: Plots of the pdf of the Emrem Distribution

The cdf of the Emrem distribution is

$$F(x)=\int_0^x f(u)\,du=1-\left(1+\frac{\theta^3x^3+3\theta^2x^2+6\theta}{6(\theta+1)}\right)e^{-\theta x} \tag{1.2}$$

The survival function $\left(\overline{F}(x)\right)$ of the distribution is obtained by subtracting $F(x)$ from 1.

Hence

$$\overline{F}(x)=\left(1+\frac{\theta^3x^3+3\theta^2x^2+6\theta}{6(\theta+1)}\right)e^{-\theta x} \tag{1.3}$$

The hazard rate function of the Emrem distribution is defined by

$$h(x)=\frac{f(x)}{\overline{F}(x)}=\frac{6\theta^2+\theta^4x^3}{\theta^3x^3+3\theta^2x^2+6\theta x+6(\theta+1)} \tag{1.4}$$

We present Figure 2 to substantiate the possession of the bathtub shaped hazard rate function by the Emrem distribution.

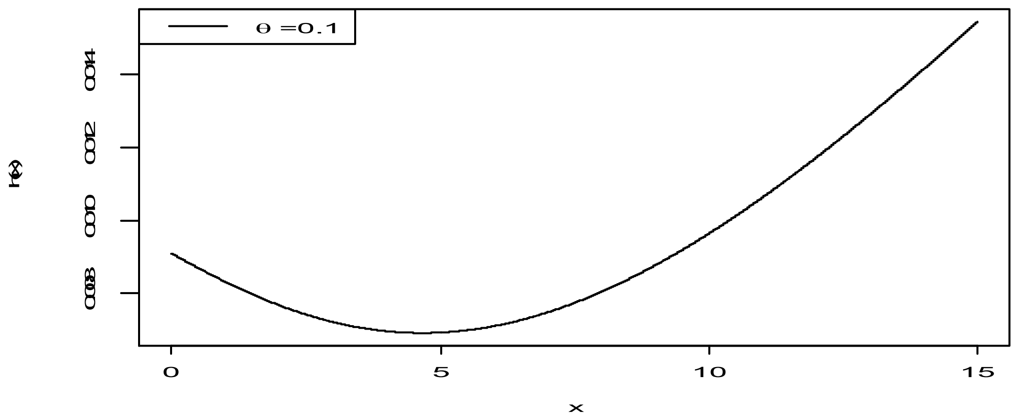

Figure 2: Plots of the hazard rate function of the Emrem Distribution

In what follows, we derive other important properties of the distribution under consideration.

2.1 Raw Moments and the related measures

If $X \sim Emrem(\theta)$, the rth raw moment of X is

$$\begin{aligned}\mu_r' = E\left(X^r\right) &= \int_0^\infty x^r \left(\frac{\theta^2}{\theta+1}\right)\left(1+\frac{\theta^2 x^3}{6}\right)e^{-\theta x} \\ &= \frac{r!}{6(\theta+1)\theta^r}\left(6\theta+(r+3)(r+2)(r+1)\right)\end{aligned} \tag{1.5}$$

For the central moments, we obtain

$$\mu_r' = E(X-\mu)^r = \sum_{i=0}^{r}(-1)^i \binom{r}{i}\mu^r \mu_{r-i}' \tag{1.6}$$

The mean and variance of X, respectively, are obtained from (15) as

$$\mu = E(X) = \frac{\theta+4}{\theta(\theta+1)}$$

and

$$V(X) = \mu_2 = \frac{\theta^2+14\theta+4}{\theta^2(\theta+1)^2}$$

For the coefficient of variation for the Emrem distribution, we have

$$CV = \frac{\sqrt{V(X)}}{E(X)} = \frac{\sqrt{\theta^2 + 14\theta + 4}}{\theta + 4}.$$

It is easy to establish that CV=1 if $\theta = 2$.

In order to describe the dispersion feature of the Emrem distribution, we consider its index of dispersion. Symbolically, the index of dispersion is

$$I = \frac{V(X)}{E(X)} = \frac{\theta^2 + 14\theta + 4}{\theta(\theta+1)(\theta+4)}.$$

The distribution is said to be equidispersed if its variance and mean are equal. Equating the mean of the distribution to its variance and solving for θ gives the condition for equidispersion of the distribution as $\theta = 2$. Consequently, the Emrem distribution is underdispersed if $\theta > 2$ and overdispersed when $\theta < 2$.

In the case of the Emrem distribution, the skewness coefficient is given by

$$S = \frac{\mu_3}{\left(\sqrt{\mu_2}\right)^3} = \frac{2\left(\theta^3 + 33\theta^2 + 9\theta + 4\right)}{\left(\sqrt{\theta^2 + 14\theta + 4}\right)^3} \tag{1.7}$$

Hence, the Emrem distribution is right-skewed.

We find the coefficient of kurtosis for the Emrem distribution to be

$$K = \frac{\mu_4}{\mu_2^{\ 2}} = \frac{9\left(\theta^4 + 52\theta^3 + 64\theta^2 + 48\theta + 8\right)}{\left(\theta^2 + 14\theta + 4\right)^2} \tag{1.8}$$

It can be easily deduced from (1.8) that the Emrem distribution is leptokurtic.

2.2 Moment generating function and Incomplete Moments

The moment generating function of the Emrem distribution is

$$M_X(t) = E\left(e^{tX}\right) = \int_0^\infty x^r \left(\frac{\theta^2}{\theta+1}\right)\left(1 + \frac{\theta^2 x^3}{6}\right) e^{-(\theta-t)x} dx$$

$$= \left(\frac{\theta^2}{\theta+1}\right)\left(\frac{\theta^2}{(\theta-t)^4} - \frac{1}{\theta - t}\right), t < \theta \tag{1.9}$$

Incomplete moments are of great importance in statistical theory and applications. For example, the first incomplete moment can be used to determine mean deviations and measures of inequality like Bonferroni and Lorenz curves.

The rth incomplete moment of the Emrem distribution is defined as

$$J_r(y)=\int_0^y x^r f(x)\,dx$$

$$=\left(\frac{\theta^2}{\theta+1}\right)\int_0^y x^r\left(1+\frac{\theta^2x^3}{6}\right)e^{-\theta x}dx$$

$$=\left(\frac{\theta^2}{\theta+1}\right)\left(\frac{\gamma(r+1,\theta y)}{\theta^{r+1}}+\frac{\theta^2\gamma(r+5,\theta y)}{6\theta^{r+4}}\right),$$

where $\gamma(s,x)$ is the lower incomplete gamma function defined by

$$\gamma(s,x)=\int_0^x y^{s-1}e^{-y}dy\ . \tag{1.10}$$

2.3 Mean deviations

Mean deviation about the mean (δ_1) and the mean deviation about the median (δ_2) are two measures of dispersion. Let the mean and median of an Emrem distributed random variable X be μ and $\tilde{X}$ respectively. Then

$$\delta_1=E(|X-\mu|)=\int_0^\infty |X-\mu| f(x)\,dx$$

$$=2\mu F(\mu)-2J_1(\mu)$$

$$\delta_2=E(|X-\tilde{X}|)=\int_0^\infty |X-\tilde{X}| f(x)\,dx \tag{1.11}$$

$$=\mu-2J_1(\tilde{X})$$

2.4 Bonferroni and Lorenze curves

Bonferroni and Lorenz curves are used to study and visualize income inequality (Huang and Oluyede, 2014).

Let p be a probability value. If $q=F^{-1}(p)$, then the Bonferroni curve for the Emrem distribution is

$$B(p)=\frac{\int_0^q tf(t)\,dt}{p\mu}=\frac{J_1(q)}{p\mu}. \tag{1.12}$$

The related Lorenz curve is

$$L(p)=\frac{\int_0^q tf(t)\,dt}{\mu}=\frac{J_1(q)}{\mu}. \tag{1.13}$$

2.5 Mean residual life function

The mean residual life function refers to the expected additional lifetime given that a component has survived until time t. In accordance with the Emrem distribution, this function is defined as

$$m(x) = \frac{1}{F(x)} \int_x^\infty \overline{F}(t)\,dt \tag{1.14}$$

$$= \left(\left[1 + \frac{\theta\text{^}3x\text{^}3 + 3\theta\text{^}2x\text{^}2 + 6\theta x}{6(\theta+1)}\right]\right)^{-1} \mathrm{e}^{\theta x} \int_x^\infty \left[1 + \frac{\theta\text{^}3t\text{^}3 + 3\theta\text{^}2t\text{^}2 + 6\theta t}{6(\theta+1)}\right] \mathrm{e}^{-\theta t}\,dt$$

$$= \frac{\left[6(\theta+1) + \theta\text{^}3x\text{^}3 + 6\theta\text{^}2x\text{^}2 + 18\theta x + 18\right]}{\theta\left[6(\theta+1) + \theta\text{^}3x\text{^}3 + 3\theta\text{^}2x\text{^}2 + 6\theta x\right]}.$$

It is easy to deduce that $m(0) = \mu$ and $m(\infty) = \frac{1}{\theta}$.

3.6 Rényi Entropy

An entropy measure quantifies the uncertainty inherent in a random variable. Here, we obtain the Rényi entropy for the Emrem distribution as

$$R_E(\beta) = \frac{1}{1-\beta} \ln \int_0^\infty f^\beta(x)\,dx, \quad \beta \neq 1, \beta > 0 \tag{1.15}$$

$$= \frac{1}{1-\beta} \ln\left(\left(\frac{\theta^2}{\theta+1}\right)^\beta \int_0^\infty \left(1 + \frac{\theta^2 x^3}{6}\right)^\beta e^{-\beta\theta x}\,dx\right)$$

$$= \frac{1}{1-\beta} \ln\left(2\beta In\theta - \beta In(\theta+1)\right) + S\frac{1}{1-\beta} \ln\left(\sum_{i=0}^{\beta} \binom{\beta}{i} \frac{\Gamma(3i+1)}{6^i \beta^{3i+1} \theta^{i+1}}\right). \ \beta \neq 1, \beta > 0 .$$

3. Estimation

In this section, two point estimation procedures, namely the method of moments and maximum likelihood method are given due consideration.

3.1 Method of moments

Method of moments estimator $(\hat{\theta})$ of the parameter of the Emrem distribution is obtained by equating the first theoretical moment to the corresponding sample moment, leading to the equation

$$\frac{\theta+4}{\theta(\theta+1)} = \overline{X} .$$

Solving the equation for θ, we obtain the estimator

$$\hat{\theta}=\frac{-\left(\bar{X}-1\right)+\sqrt{\left(\bar{X}-1\right)^{2}+16\bar{X}}}{2\bar{X}}$$

3.2 Maximum likelihood method

Given a random sample $X_1, X_2, ..., X_n$ from the Emrem distribution, the log-likelihood function is

$$l=In\left(\prod_{i=1}^{n}\left(\left(\frac{\theta^{2}}{\theta+1}\right)\left(1+\frac{\theta^{2}x_i^{3}}{6}\right)e^{-\theta x_i}\right)\right) \quad (1.16)$$

The maximum likelihood estimator $\hat{\theta}_{MLE}$ is obtained by solving the following equation for θ

$$\frac{2n}{\theta}-\frac{n}{\theta+1}+2\theta\sum\frac{x_i^{3}}{6+\theta_i^{2}x_i^{3}}-\sum x_i=0 \quad (1.17)$$

It is worthy of note that only the numerical solution of (1.17) can be obtained.

4. Numerical Illustrations

4.1 Simulation

In this section, we use the algorithm below to simulate a random sample $\left(X_i, i=1,2,...,n\right)$ of size n from the Emrem distribution:

(i) Generate $U_i \sim U\left(0,1\right), i=1,2,...,n$

(ii) Generate $V_i \sim E\left(\theta\right), i=1,2,...,n$

(iii) Generate $W_i \sim Gamma\left(4,\theta\right), i=1,2,...,n$

(iv) If $U_i \le \frac{\theta}{\theta+1}$, set $X_i = V_i$. Otherwise set $X_i = W_i$

We compute the average estimate (AE), average bias (AB) and mean squared error (MSE) that are associated with each parameter value, sample size and 5000 samples. Let $\hat{\theta}_j$ denote the maximum likelihood estimate of θ corresponding to the jth sample $j=1,2,...,N$. Then

(i) $AE=\frac{\sum_{j=1}^{N}\hat{\theta}_j}{N}$;

(ii) $AB=\frac{\sum_{j=1}^{N}\left(\hat{\theta}_j-\theta\right)}{N}$;

(iii) $MSE=\frac{\sum_{j=1}^{N}\left(\hat{\theta}_j-\theta\right)^{2}}{N}$.

Simulation results corresponding to the sample sizes $n = 20, 50, 100$ and $\theta = 1.6$ is present in the Table 1 so as to illustrate consistency of the maximum likelihood estimate of the parameter of Emrem distribution. Obviously, the MSE associated with the parameter estimate decreases as the sample size increases, implying the consistency of the estimate.

Table 1: simulation result

N	AE	AB	MS
20	3.1287	0.1287	0.3656
50	3.0459	0.0459	0.1222
100	3.0190	0.0190	0.0575
200	3.0108	0.0108	0.0271

4.2 Real Life Application

Having obtained some theoretical results on the Emrem distribution, we proceed to consider its application using a real data set obtained from (Bekker, Roux and Mosteit, 2000). The data, which comprise the survival times (in years) of a group of patients given chemotherapy treatment alone are given below:

0.047, 0.115, 0.121, 0.132, 0.164, 0.197, 0.203, 0.260, 0.282, 0.296, 0.334, 0.395, 0.458, 0.466, 0.501, 0.507, 0.529, 0.534, 0.540, 0.641, 0.644, 0.696, 0.841, 0.863,1.099, 1.219, 1.271, 1.326, 1.447, 1.485, 1.553, 1.581, 1.589, 2.178, 2.343, 2.416, 2.444, 2.825, 2.830, 3.578, 3.658, 3.743, 3.978, 4.003, 4.033.

The fit of the distribution to the data is compared to the fit of each exponential, logarithmic Lindley, xgamma and weighted xgamma distributions. The popular AIC, BIC, Komogorov-Smirnov (KS) statistic, Anderson-Darling statistic (A*) and Cramér-von Mises (W*) statistic are used to ensure the effective comparison of the concerned distributions. The distribution that correspond to the minimum value of the statistics becomes the most suitable distribution for the given data.

Table 2 contains the maximum likelihood estimate of the parameter of each of the distributions and estimated values of the corresponding goodness of fit statistics.

Table 2: Maximum likelihood estimates and the corresponding results for distributions fitted to the data

Distribution	Est	-l	KS	A*	W*	p-value
E	0.75	-58.23	0.09	0.06	0.44	0.82
LL	0.97	-58.16	0.10	0.07	0.50	0.70
XG	1.36	-58.01	0.11	0.08	0.55	0.66
WXG	2.27	-66.14	0.25	0.72	5.49	0.01
EMREM	**1.59**	**-57.51**	**0.08**	**0.05**	**0.46**	**0.91**

The parameter estimates and the goodness of fit statistics for the different models based on the data set are presented in Table 2. From the results, the Emrem distribution performed better than the competing distributions since it its fit leads to the minimum values of the used goodness of fit statistics.

5. Conclusion

In this study, we have introduced and derived properties of a one-parameter distribution using a mixture of the exponential and gamma distribution. The pdf of the proposed distribution can be a decreasing function or non-monotonic function with two turning points while the related hazard rate function can be bathtub shaped. The distribution is positively skewed and leptokurtic. It can be underdispersed, equidersed or overdispersed, depending on whether the value of its parameter is greater than 2, equal to 2 or less than 2 respectively. We empirically showed that the new model can serve as a better distribution for modelling positively skewed data than several well-known and widely used one-parameter distributions, including the exponential, logarithmic Lindley, xgamma and weighted xgamma distributions.

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