Journal of Probability
and Statistical Science
23(1), 1-15 Aug. 2025

The Bandages Problem

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ABSTRACT

A new probability problem, named the Bandages Problem, is described and solved. The problem involves repeatedly selecting and removing an item at random from a finite population that initially consists of a known configuration of single and paired items. For each selection, the probability that the chosen item is single is found. Generalizations are suggested.

Keywords: Probability problem; Probability puzzler; Mathematical induction; The bandages problem

1. Introduction

The idea for this probability puzzler arose when one of the authors bought a box of 60 bandages that contained 30 attached pairs. Suppose he uses one bandage per day. On day 1, there are only pairs, so he detaches one from a pair, putting the remaining single bandage back in the box. The next day, he reaches into the box, where there was now 29 pairs and 1 single. Each day, he continues the selection process until the box is empty.

There is a longstanding tradition of using small, ordinary-appearing, and everyday situations for touchstones in the subject of probability and in courses about it. The Monty Hall and Buffon's needle problems come to mind, but there are many others. For the history and generalizations of the Monty Hall problem, see [11]. The Buffon's needle problem is very old, dating from 1733. Often, it is used as the prototypical geometric probability example. For its history and some applications, see [7] and [10]. For contemporary uses and generalizations, see [6]. The website [12] lists 31 of these exemplar probability problems and provides links to websites about each of them. The classic book by Mosteller [3] contains many, as does [1], which is from a course in probability methods. The problem that is presented here might be placed within that historical convention.

Assume that for each daily selection, all remaining bandages are equally likely to be selected, whether they are still attached or not and that, if a bandage from a pair is selected, the non-selected bandage from the pair is still available for later sampling. This might be accomplished by randomly scattering the bandages on a table, so that none are overlapping, and selecting, for example, with eyes closed or some other technique to ensure randomness. See Figure 1. Although applications can be found for other types of objects, for simplicity, the term "bandage" is used throughout.

Generalize to having initially N bandages in any mixture of d pairs and s singles in the box. The goal is to find an expression for the daily probability of selecting a bandage that is a single. The probability $P(n; N, s)$ of choosing a single bandage on day n is a function of the

- Received December 30, 2024, in final form August 2025.
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initial N and s, where n ranges from 1 to $N = 2d + s$. In order to insure that d is an integer, N and s must be compatible in the sense that they have the same parity.

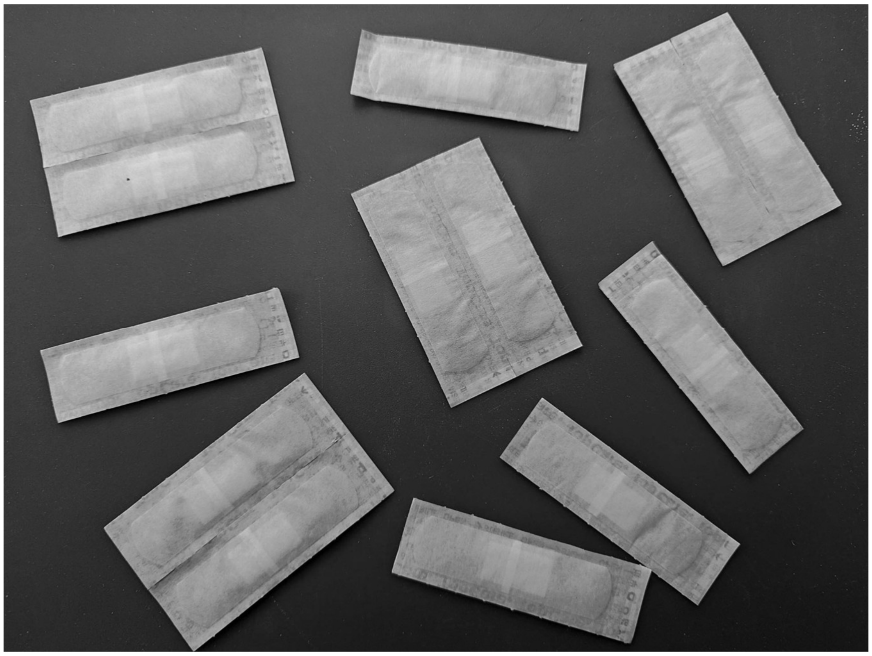

FIGURE 1: Four pairs and five single bandages remaining for the next selection

2. A formula for $P(n; N, s)$

The formula for $P(n; N, s)$ is confirmed by Theorem 1, below. Carefully working out some examples gives a feeling for the probability and yields the formula in the theorem. Numerical examples lead to the correct conjecture that $P(n; N, s)$ is a linear function of n. Multiplying the conditional probabilities that grow in number with n accompanied by binary splitting for the selection of a single or a paired bandage yields the probabilities [5, pp. 11–13]. For example, for $N = 7$ and $s = 3$, the probabilities $P(n; 7,3)$ are

$$^{9}/_{21}, {}^{11}/_{21}, {}^{13}/_{21}, {}^{15}/_{21}, {}^{17}/_{21}, {}^{19}/_{21}, \text{ and } {}^{21}/_{21} = 1,$$

respectively, for $n = 1, 2, \ldots, N = 7$. For instance,

$$P(\text{Single on day 2}) = P(\text{Single on day 2}|\text{Single on day 1})P(\text{Single on day 1}) + P(\text{Single on day 2}|\text{From pair on day 1})P(\text{From pair on day 1})$$

$$= \frac{2}{6}\,\frac{3}{7} + \frac{4}{6}\,\frac{4}{7} = \frac{11}{21}.$$

These probabilities form an arithmetic progression, which indicates that $P(n; N, s)$ is a linear function of n. Theorem 1 confirms that $P(n; N, s)$ is a linear function of n for any compatible pair (N, s). Because the line contains the points $(n, P(n; N, s))$ for $n = 1$ and for $n = N$, that is, $\left(1, P(1; N, s) = {}^{s}/_{N}\right)$ and $(N, P(N; N, s) = 1)$, the linear function is

$$P(n: N, s) = \frac{1 - \frac{s}{N}}{N - 1}(n - 1) + \frac{s}{N} = \frac{N - s}{N(N - 1)}n + \frac{s - 1}{N - 1},$$

which is validated by Theorem 1.

Theorem 1. If $N \geq 2$, then $P(n; N, s)$ increases linearly with n from s/N to 1 as n increases from 1 to N. In particular,

$$P(n; N, s) = \frac{N-s}{N(N-1)} n + \frac{s-1}{N-1} \tag{1}$$

for $n \in \{1, 2, \dots, N\}$.

Proof. Because

$$P(1; N, s) = \frac{s}{N}$$

from the physical setup, (1) is true for $n = 1$. For $n = 1, 2, \dots, N$, let S_n be the number of single bandages on day n, so that $S_1 = s$. For $n \geq 2$,

$$S_n = \begin{cases} S_{n-1} - 1 \text{ with probability } \frac{S_{n-1}}{N-n+2} \\ S_{n-1} + 1 \text{ with probability } 1 - \frac{S_{n-1}}{N-n+2} \end{cases}.$$

Then,

$$E(S_n | S_{n-1}) = (S_{n-1} - 1)\left(\frac{S_{n-1}}{N-n+2}\right) + (S_{n-1} + 1)\left(1 - \frac{S_{n-1}}{N-n+2}\right) = \frac{N-n}{N-n+2} E(S_{n-1}) + 1.$$

By the conditioning principle [5, pp. 100–101],

$$E(S_n) = E\big(E(S_n | S_{n-1})\big) = \frac{N-n}{N-n+2} E(S_{n-1}) + 1. \tag{2}$$

The unique solution of (2) is

$$E(S_n) = \frac{(N-n+1)(N-n)}{N(N-1)} s + \frac{(n-1)(N-n+1)}{N-1} \tag{3}$$

for $n \in \{1, 2, \dots, N\}$, which can be seen as follows.

Designate the right-hand side of (3) by c_n. Observe that

$$c_n = \frac{N-n}{N-n+2}\left(\frac{(N-n+2)(N-n+1)}{N(N-1)} s + \frac{(n-2)(N-n+2)}{N-1}\right) + 1 = \frac{N-n}{N-n+2} c_{n-1} + 1. \tag{4}$$

Because there is at most one ordered n-tuple of real numbers $(x_1, x_2, \dots, x_n)$ satisfying

$$x_1 = s \text{ and } x_n = \frac{N-n}{N-n+2} x_{n-1} + 1$$

for $n = 2, 3, \dots, N$, from (4) it follows that $x_n = c_n$ for $n = 2, 3, \dots, N$. Thus, (2) implies (3). Consequently, because the number of bandages present from which to select on day n is $N - n + 1$,

$$\begin{aligned} P(n; N, s) &= \frac{E(S_n)}{N - n + 1} \\ &= \frac{1}{N - n + 1}\left(\frac{(N - n + 1)(N - n)}{N(N - 1)} s + \frac{(n - 1)(N - n + 1)}{N - 1}\right) \\ &= \frac{N-n}{N(N-1)} s + \frac{n-1}{N-1} = \frac{N-s}{N(N-1)} n + \frac{s-1}{N-1}. \quad \blacksquare \end{aligned}$$

Appendix A contains an alternative proof by the method of mathematical induction.

Example 1 (Initially only paired bandages in the box). From (1), when $s = 0$, the probability of selecting a single bandage on the n^{th} day is

$$P(n; N, 0) = \frac{1}{N-1}n - \frac{1}{N-1} = \frac{n-1}{N-1}.$$

Thus, for $N = 60$ in the original purchase, the probability of choosing a single bandage on day n is $\frac{1}{59}n - \frac{1}{59}$.

3. Concluding comments

Possible generalizations to this bandages problem abound. One is to have more than one box of bandages available at each selection. These boxes might be said to be scattered about the house. Each box could start out with a different configuration of singles and pairs. A box, which can be partially depleted, is chosen at random, then a bandage is picked from the chosen box. This provides an example of length- or size-based sampling, i.e., each bandage is not equally likely to be selected, because those in boxes with fewer remaining bandages are more likely to be chosen [4, 9].

A variation on the problem is to have more than two bandages fastened together, such as them being sold in triples. Another variation could entail the selection of two bandages at the same time daily.

Another potential setting for the bandages problem arises because it describes a type of decay in which the number of items that are leaving the system is constant. Aging in some biological or mechanical systems might be similar. Another circumstance is to reverse the process by adding an item each day into a system in which pairing occurs.

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Appendix A

This proof of Theorem 1 uses mathematical induction on N for $N \geq 2$ [2, pp. 53–66, 8, pp. 38–47]. Equation (1) is true for $N = 2$, because

$$P(1;2{,}0) = 0,\ P(2;2{,}0) = 1,\ P(1;2{,}2) = 1, \text{ and } P(2;2{,}2) = 1$$

from (1) and from the physical setup as well. For the induction hypothesis, assume that (1) is correct for $N - 1$ bandages. An expression for $P(n; N, s)$ in terms of functions of $N - 1$ is obtained by conditioning on the selection for day 1 as follows.

$$\begin{aligned} &P(\text{Single on day } n) \\ &\quad = P(\text{Single on day } n|\text{Single on day 1})P(\text{Single on day 1}) \\ &\qquad + P(\text{Single on day } n|\text{From pair on day 1})P(\text{From pair on day 1}). \end{aligned}$$

Then,

$$\begin{aligned} P(n; N, s) &= P(n-1; N-1, s-1)\frac{s}{N} + P(n-1; N-1, s+1)\frac{N-s}{N} \\ &= \left(\frac{(N-1)-(s-1)}{(N-1)((N-1)-1)}(n-1) + \frac{(s-1)-1}{(N-1)-1}\right)\frac{s}{N} + \left(\frac{(N-1)-(s+1)}{(N-1)((N-1)-1)}(n-1) + \frac{(s+1)-1}{(N-1)-1}\right)\frac{N-s}{N} \\ &= \frac{s(N-s)}{N(N-1)(N-2)}(n-1) + \frac{s(s-2)}{NN-2)} + \frac{(N-s)(N-s-2)}{N(N-1)(N-2)}(n-1) + \frac{s(N-s)}{N(N-2)} \\ &= \frac{(N-s)(s+N-s-2)}{N(N-1)(N-2)}(n-1) + \frac{s(s-2+N-s)}{N(N-2)} \\ &= \frac{N-s}{N(N-1)}n - \frac{N-s}{N(N-1)} + \frac{s}{N} \\ &= \frac{N-s}{N(N-1)}n + \frac{s-1}{N-1}, \end{aligned}$$

which is (1). The following observation was used to justify the first equation:

$$P(\text{Single on day } n|\text{Single on day 1}) = P(n-1; N-1, s-1),$$

because under the conditional statement that on day 1 a single bandage was selected, start over on day 2 with one less single, hence $s - 1$. There are $n - 1$ days remaining until day n with just $N - 1$ bandages available on day 2, which is the time that the clock is figuratively reset for the probability with $N - 1$ bandages. Similarly,

$$P(\text{Single on day } n|\text{From pair on day 1}) = P(n-1; N-1, s+1),$$

where the clock is reset on day 2 with $s + 1$ singles, because a pair had been selected on the original day 1.