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Analysis of Dependent Variables Following Bivariate J-shaped Distribution and Methods to Construct New Bivariate Classes

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ABSTRACT

A J-shaped distribution called inverted Topp-Leone (ITL) is discussed in this paper in both one and two dimensions. The bivariate inverted Topp-Leone (BITL) distribution is introduced as a shock Model. The mathematical properties of ITL and BITL distributions are discussed. All the characteristic functions for both ITL and BITL distributions are obtained explicitly in compact forms. For the unknown parameters maximum likelihood estimation is applied and the exact information matrix is derived. A new two methods for constructing bivariate distributions from BITL distribution are discussed in detail. Consequently, generalized and exponentiated ITL distributions are defined in both univariate and bivariate cases. An absolutely continuous BITL distribution is also discussed with its properties. A real data set is then re-analyzed for illustration.

Keywords: J-shaped distribution; Topp-Leone Distribution; Inverted Topp-Leone Distribution; Generalized Inverted Topp-Leone; Exponentiated Inverted Topp-Leone, Bivariate Inverted Topp-Leone Distribution.

1. Introduction

One of the most popular distributions in many research areas is the J-shaped distribution. Where the non-normal data's probability distributions' shapes display a J-shaped distribution. Inverted Topp-Leone (ITL) distribution is a J-shaped distribution that Muhammed (2019b) recently introduced with support $x > 1$. Which is useful for modeling lifetime Phenomena. The author talked about the mode, median, quantile function, and moments around zero among other statistical characteristics for the ITL distribution. Additionally, she obtained hazard, reliability, and reversed hazard functions for this distribution as reliability measures. Expressions for order statistics such as moments and product moments can also be obtained in closed forms. The MLE and confidence intervals are considered for the reliability, hazard, and reversed hazard functions. In the next section, a new ITL distribution is defined with wider support that is $x > 0$.

The analysis of dependent variables is crucial. For instance, in economic studies, investigate the relationship between years of education and personal income, personal income and expenditure, inflation, and unemployment; in biological studies, investigate the relationship between blood pressure and body weight for a patient and the time until kidney failure in the left and right kidneys; In engineering studies, the lifespan of a twin-engine plane is examined, along with warranty policies based on failure times and warranty servicing times, and various applications, such as the shock model, competing risks model, stress model, maintenance model, and longevity model.

Because it discusses all possible outcomes for the random variables (i.e., the first random variable is smaller, greater, or equal to the second random variable), the bivariate Marshal-Olkin

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family is crucial for understanding and analyzing the failure time of two variables interacting with one another. The introduction of a bivariate extension of the ITL distribution with marginals that match univariate ITL distributions is one of the goals of this paper. It is demonstrated that the proposed bivariate models have a structure with a single part.

The paper is structured as follows: in Section 2, an ITL distribution and its properties are introduced, and Section 3 provides a detailed discussion of the ITL's bivariate extension. In Section 4, it is observed how to obtain MLE and the exact information matrix for the BITL distribution parameters. In Section 5, application for a real data set is evaluated. Section 6, talks about a simulation study. In Section 7, new techniques for creating bivariate distributions resulting from BITL distributions are observed. Section 8 lists the conclusion and unresolved issues.

2. Univariate ITL Distribution

Following are the pdf and cdf that define the Topp-Leone distribution, respectively:

$$F(t;\beta) = t^{\beta}(2-t)^{\beta} \tag{2.1}$$

And

$$f(t;\beta) = \beta(2-2t)(2t-t^2)^{\beta-1} \tag{2.2}$$

For $0 < t < 1$ and $\beta > 0$.

Assume $X = \frac{1}{T} - 1$, then X's pdf and cdf are provided as follows, respectively.

$$f_{ITL}(x;\beta) = 2\beta x(x+1)^{-2\beta-1}(2x+1)^{\beta-1} \tag{2.3}$$

And

$$F_{ITL}(x;\beta) = 1-(x+1)^{-2\beta}(2x+1)^{\beta} \tag{2.4}$$

For $0 < x < \infty$ and $\beta > 0$.

The term "inverted Topp-Leone distribution" refers to the distribution of X in this situation and denoted by $ITL(\beta)$.

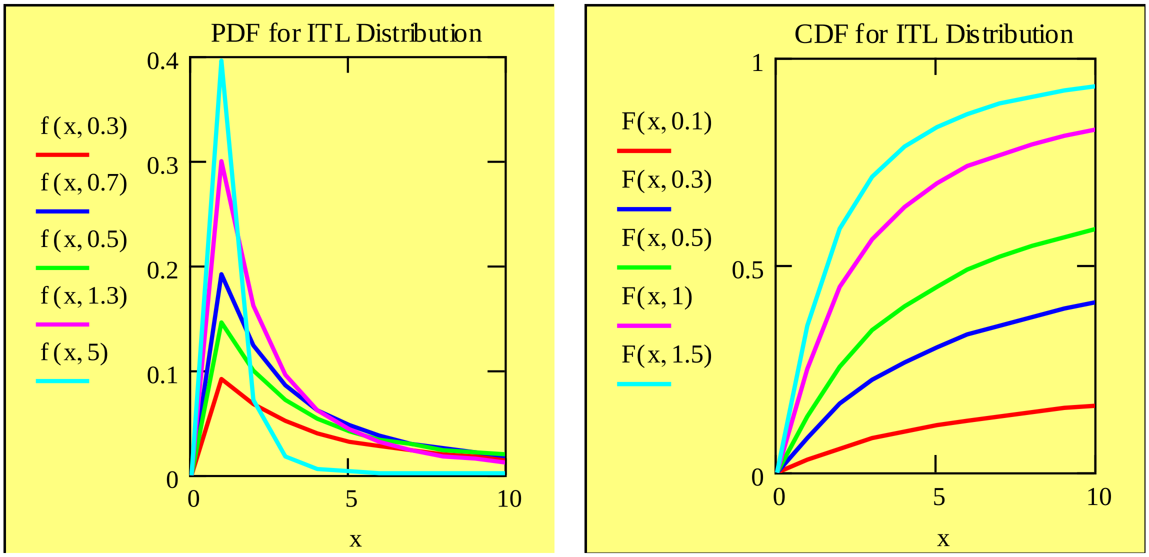

Figure 1: The pdf and cdf of the ITL Distribution for Some Values of β

The pdf (2.3) can be demonstrated to satisfy the following generalized Pearson differential equation system

$$\frac{f'(x)}{f(x)} = \frac{a_0+a_1x+a_2x^2}{b_0+b_1x+b_2x^2+b_3x^3}.$$

Where $a_0 = 1$, $a_1 = 1$, $a_2 = -2\beta - 2$, $b_0 = 0$, $b_1 = 1$, $b_2 = 3$ and $b_3 = 2$.

The IT(β) distribution may be considered as a J- shaped because $f(x) > 0$, $\frac{df(x)}{dx}<0$ and for some values $\frac{d^2f(x)}{dx^2} > 0$.

The $ITL(\beta)$ distribution's mode is as follows

$$\sqrt{\frac{1}{2\beta+2}}.$$

The $ITL(\beta)$ distribution's quantiles is as follows

$$x_q = (1-q)^{\frac{-1}{\beta}} \sqrt{1-(1-q)^{\frac{1}{\beta}}}\,.\left(1+\sqrt{1-(1-q)^{\frac{1}{\beta}}}\right), 0 < q < 1.$$

The $ITL(\beta)$ distribution's median is a special case from the quantile function, when $q = \frac{1}{2}$,

$$x_{0.5} = (0.5)^{\frac{-1}{\beta}} \sqrt{1-(0.5)^{\frac{1}{\beta}}}.\left(1+\sqrt{1-(0.5)^{\frac{1}{\beta}}}\right).$$

Proposition 1: The k^{th}- moment about zero denoted by $\acute{\mu}_k$ for $ITL(\beta)$ distribution is given by

$$\acute{\mu}_k = \beta \sum_{j=0}^{\infty} (-1)^j c(k,j) \mathrm{B}(\mathrm{j}-\mathrm{k}+1, \frac{k}{2}+1).$$

For $k = 1 \ldots n$ and $\beta \neq 1$.
such that

$c(k,0) = 2^k, c(k,1) = k\ 2^{k-1}$ and $c(k,j) = \frac{k\ 2^{k-2j}}{j!} \prod_{i=1}^{j-1}(k-j-i),\ j \geq 2$

and B(.,.)is a beta function.

Proof. Start with $\acute{\mu}_k = E(X^k) = \int_0^{\infty} x^k\ dF(x)$

Use the transformation $U = F(X)$ such that $x = F^{-1}(u)$,

So,

$$\acute{\mu}_k = \int_0^1 [u^{\frac{-1}{\beta}} \sqrt{1-u^{\frac{1}{\beta}}}\,.\left(1+\sqrt{1-u^{\frac{1}{\beta}}}\right)]^k\ du$$

Using the power series expansion for $[1+\sqrt{1-u^{\frac{1}{\beta}}})]^k$ [see p.49 of Gradshteyn and. Ryzhik(1980)7th edition]

$$[1+\sqrt{1+u})]^q = 2^q(1+\frac{q}{i!}\left(\frac{u}{2}\right)+\frac{q(q-3)}{2!}\left(\frac{u}{2}\right)^2+\frac{q(q-4)(q-5)}{3!}\left(\frac{u}{2}\right)^3+\cdots) \quad (2.5)$$

For any real number q; $|u| < 1$.

This completes the proof.

The $ITL(\beta)$ distribution's survival function is defined as

$$S_{ITL}(x;\beta) = P(X > x) = (x+1)^{-2\beta}(2x+1)^{\beta}$$

The probability of surviving until time x is indicated for fixed x.

The $ITL(\beta)$ distribution's hazard function is instantly derived as

$$h_{ITL}(x;\beta) \ = 2\beta\frac{x}{(x+1)(2x+1)}.$$

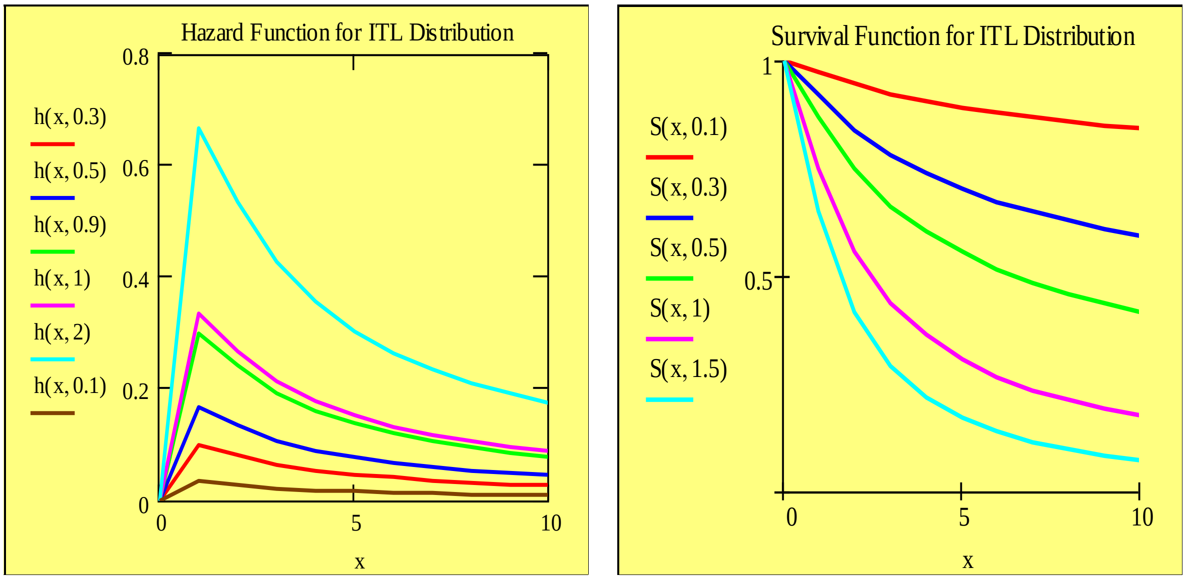

Figure 2: The Hazard and Survival Functions of the ITL Distribution for some values of β

3. Bivariate Inverted Topp-Leone Distribution

Suppose that $U_i \sim ITL(\beta_i), i = 1,2,3$ such that $U_i's$ are mutually independent random variables and define $X_j = min\left(U_j, U_3\right), j = 1,2$. Such that; $X_j's$ are dependent random variables. Consequently, $S_{BITL}(x_1, x_2)$ is used to denote the joint survival function of the vector (X_1, X_2) and it is given as

$$S_{BITL}(x_1,x_2) = S_{ITL}(x_1;\ \beta_1)S_{ITL}(x_2;\ \beta_2)S_{ITL}(x_3;\ \beta_3). \tag{3.1}$$

$$= (x_1+1)^{-2\beta_1}(2x_1+1)^{\beta_1}.(x_2+1)^{-2\beta_2}(2x_2+1)^{\beta_2}.(x_3+1)^{-2\beta_3}(2x_3+1)^{\beta_3}$$

where $x_3 = \max{(x_1, x_2)}$.

The following form can be used to stretch the joint survival function of the BITL distribution:

$$S_{BITL}(x_1,x_2) = \begin{cases} S_{ITL}(x_1;\ \beta_1)S_{ITL}(x_2;\ \beta_{23}), & x_1 < x_2 \\ S_{ITL}(x_1;\ \beta_{13})S_{ITL}(x_2;\ \beta_2), & x_1 > x_2\ . \\ S_{ITL}(x;\ \beta_{123}), & x_1 = x_2 = x \end{cases} \tag{3.2}$$

Where $\beta_{13} = \beta_1 + \beta_3$, $\beta_{23} = \beta_2 + \beta_3$ and $\beta_{123} = \beta_1 + \beta_2 + \beta_3$.

The joint pdf of the BITL distribution can therefore be obtained as

$$f_{BITL}(x_1, x_2) = \begin{cases} f_{ITL}(x_1; \beta_1) f_{ITL}(x_2; \beta_{23}), & x_1 < x_2 \\ f_{ITL}(x_1; \beta_{13}) f_{ITL}(x_2; \beta_2), & x_1 > x_2 \\ \frac{\beta_3}{\beta_{123}} f_{ITL}(x;\ \beta_{123}), & x_1 = x_2 \end{cases} . \tag{3.3}$$

3.1 Joint CDF, Joint Hazard and Reversed Hazard Functions

The BITL distribution's joint cdf is provided as

$$F_{BITL}(x_1, x_2) = \begin{cases} F_{ITL}(x_1; \beta_{13}) - F_{ITL}(x_1; \beta_1)[1 - F_{ITL}(x_2; \beta_{23})], & x_1 < x_2 \\ F_{ITL}(x_2; \beta_{23}) - F_{ITL}(x_2; \beta_2)[1 - F_{ITL}(x_1; \beta_{13})], & x_2 < x_1 \\ 1 - F_{ITL}(x;\ \beta_{123}), & x_1 = x_2 = x. \end{cases} \tag{3.4}$$

The BITL distribution's joint hazard function is given as

$$h_{BITL}(x_1, x_2) = \begin{cases} h_{ITL}(x_1;\ \beta_1) h_{ITL}(x_2;\ \beta_{23}), & x_1 < x_2 \\ h_{ITL}(x_1;\ \beta_{13}) h_{ITL}(x_2;\ \beta_2), & x_1 > x_2 \\ \frac{\beta_3}{\beta_{123}} h_{ITL}(x;\ \beta_{123}), & x_1 = x_2 = x. \end{cases} \tag{3.5}$$

3.2 Factorization Property

Both an absolutely continuous part and a singular part constitute the BITL distribution. It is possible to factorize the joint survival function of the BITL distribution into an absolutely continuous part and a singular part in the manner shown below.

$$S_{BITL}(x_1, x_2) = \frac{\beta_{12}}{\beta_{123}} S_a(x_1, x_2) + \frac{\beta_3}{\beta_{123}} S_s(x_3) \tag{3.6}$$

Where $x_3 = \max(x_1, x_2)$, $S_s(x_3) = S_{ITL}(x; \alpha)$, $\beta_{123} = \beta_1 + \beta_2 + \beta_3$

and $S_a(x_1, x_2) = \frac{\beta_{123}}{\beta_{12}} S_{ITL}(x_1; \beta_1) S_{ITL}(x_2; \beta_2) S_{ITL}(x_3; \beta_3) - \frac{\beta_3}{\beta_{12}} S_{ITL}(x; \beta_{123})$.

It is obvious that $S_s(.,.)$ and $S_a(.,.)$ represent the parts that are singular and absolutely continuous, respectively.

As a result, the BITL model's pdf can be factored into two parts: an absolutely continuous part and a singular part as follows

$$f_{BITL}(x_1, x_2) = \frac{\beta_{12}}{\beta_{123}} f_a(x_1, x_2) + \frac{\beta_3}{\beta_{123}} f_s(x_3) \tag{3.7}$$

Where

$$f_a(x_1, x_2) = \frac{\beta_{123}}{\beta_{12}} \begin{cases} f_{ITL}(x_1; \beta_{13}) f_{ITL}(x_2; \beta_2), & x_1 < x_2 \\ f_{ITL}(x_1; \beta_1) f_{ITL}(x_2; \beta_{23}), & x_1 < x_2 \end{cases}$$

and $f_s(x_3) = f_{ITL}(x; \beta_{123})$.

It is obvious that in this case $f_a(x_1, x_2)$ and $f_s(x_3)$ are the singular and absolutely continuous parts, respectively.

Depending on the values of β_1, β_2 and β_3, the absolutely continuous component of the BITL density may be unimodal, that is $f_a(x_1, x_2)$ is unimodal and the respective modes are

$$[(2\beta_1 + 2)^{-1/2}, (2\beta_{23} + 2)^{-1/2}] \text{and } [(2\beta_{13} + 2)^{-1/2}, (2\beta_2 + 2)^{-1/2}].$$

The absolutely continuous BITL distribution's median is provided as follows.

$$(0.5)^{\frac{-1}{\beta_{123}}} \sqrt{1 - (0.5)^{\frac{1}{\beta_{123}}}}.\left(1 + \sqrt{1 - (0.5)^{\frac{1}{\beta_{123}}}}\right).$$

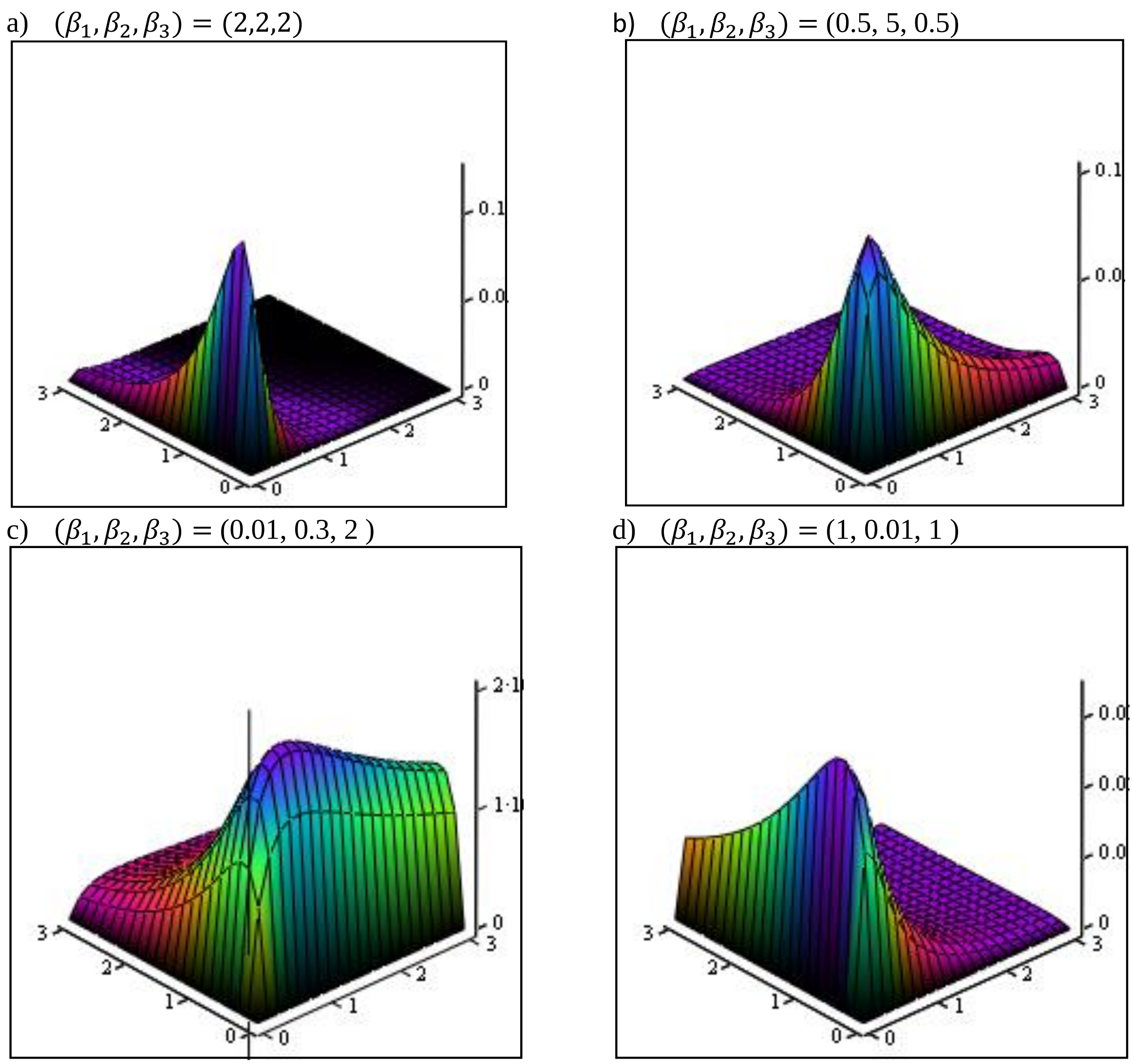

Figure 3: Surface plots of the absolutely continuous part of the joint pdf of the BITL model for different values of $(\beta_1, \beta_2, \beta_3)$

3.3 Marginal and Conditional Densities

It is significant to note that the BITL marginal distributions are univariate ITL with the corresponding survival and density functions as follows:

$$S_{X_i}(x_i) = S_{ITL}(x_i;\ \beta_{i3}) = (x_i+1)^{-2\beta_{i3}}(2x_i+1)^{\beta_{i3}},\ i=1,2 \tag{3.8}$$

$$f_{X_i}(x_i) = f_{ITL}(x_i;\ \beta_{i3}) = 2\beta_{i3}\, x_i\,(x_i+1)^{-2\beta_{i3}-1}(2x_i+1)^{\beta_{i3}-1},\ i=1,2 \tag{3.9}$$

Such that $\beta_{i3} = \beta_i + \beta_3,\ i = 1,2.$

Moreover, the distribution of the minimum of $(X_1, X_2) \sim BITL(\beta_1, \beta_2, \beta_3)$is also univariate ITL with shape parameter β_{123}. And the survival and density functions are given as follows

$$S_{\min\,(X_1,X_2)}(x) = S_{ITL}(x;\ \beta_{123}) = (x+1)^{-2\beta_{123}}(2x+1)^{\beta_{123}} \tag{3.10}$$

and

$$f_{\min\,(X_1,X_2)}(x) = f_{ITL}(x;\ \beta_{123}) = 2\beta_{123}\, x\,(x+1)^{-2\beta_{123}-1}(2x+1)^{\beta_{123}-1}$$
(3.11)

where $x = \min\,(x_1, x_2)$ and $\beta_{123} = \beta_1+\beta_2+\beta_3$.

The conditional density of X_i given $X_j = x_j, i \neq j$ is calculated as follows. This is because the marginal distributions of the vector ($(X_1, X_2) \sim BITL(\beta_1, \beta_2, \beta_3)$ are univariate ITL distributions.

$$f_{i/j}(x_i/x_j) = \begin{cases} f_{i/j}^{(1)}(x_i/x_j), & x_i < x_j \\ f_{i/j}^{(2)}(x_i/x_j), & x_i > x_j \\ f_i^{(3)}(x_i), & x_i = x_j \end{cases} \tag{3.12}$$

Where

$$f_{i/j}^{(1)}(x_i/x_j) = 2\beta_1\, x_i\,(x_i+1)^{-2\beta_1-1}(2x_i+1)^{\beta_1-1},$$

$$f_{i/j}^{(2)}(x_i/x_j) = 2\frac{\beta_{13}\beta_2}{\beta_{23}} x_i\,(x_i+1)^{-2\beta_{13}-1}(2x_i+1)^{\beta_{13}}(x_j+1)^{-2\beta_3}\left(2x_j+1\right)^{-\beta_3},$$

and $f_i^{(3)}(x_i) = \frac{\beta_3\, x_i\,(x_i+1)^{-2\beta_{123}-1}(2x_i+1)^{\beta_{123}-1}}{\beta_{23}\, x_j\,(x_j+1)^{-2\beta_{23}-1}(2x_j+1)^{\beta_{23}-1}}$.

3.4 Product Moments

Since the vectors (X_1, X_2) ' marginal distributions are univariate ITL distributions, the moments of X_1 and X_2 can be obtained directly from the marginals shown below.

$E(X_1^k) = \beta_{13}\sum_{j=0}^{\infty} (-1)^j\, c(k,j)\, B\left(j-k+\beta_{13}\,,\ \frac{k}{2}+1\right)$ and

$E(X_2^k) = \beta_{23}\sum_{j=0}^{\infty} (-1)^j\, c(k,j)\, B\left(j-k+\beta_{23}\,,\frac{k}{2}+1\right)$

For $k = 1\ldots n$

Where $c(k,0) = 2^k, c(k,1) = k\,2^{k-1}$ and $c(k,j) = \frac{k\,2^{k-2j}}{j!}\prod_{i=1}^{j-1}(k-j-i),\ j \geq 2.$

$B(.,.)$ is the beta function

Now, The r^{th} and s^{th} joint moments of $(X_1, X_2) \sim BITL(\beta_1, \beta_2, \beta_3)$, denoted by $\mu'_{r,s}$ can be given by the following theorem

Proposition 2: If $(X_1, X_2) \sim BITL(\beta_1, \beta_2, \beta_3)$ .Then, the r^{th} and s^{th} joint moments of (X_1, X_2) are given as follows

$$\acute{\mu}_{r,s} = E(X_1{}^r X_2{}^s) = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} K_{j_1 j_2} B\left(j_1 + j_2 - s - r + \beta_{123}\ , \frac{s}{2} + 1\right)$$
$$\cdot\ {}_3F_2\left(j_1 + j_2 - s - r + \beta_{123}\ , j_1 - r + \beta_1\ , \frac{-r}{2}; j_1 - r + \beta_1 + 1, j_1 + j_2 - r - \frac{s}{2} + \beta_{123}\ ; 1\right)$$
$$+ \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} \breve{K}_{j_1 j_2} B\left(j_1 + j_2 - s - r + \beta_{123}\ , \frac{r}{2} + 1\right)$$
$$\cdot\ {}_3F_2\left(j_1 + j_2 - s - r + \beta_{123}\ , j_1 - s + \beta_2\ , \frac{-s}{2}; j_1 - s + \beta_2 + 1, j_1 + j_2 - s - \frac{r}{2} + \beta_{123}\ ; 1\right)$$
$$+ \beta_{123} \sum_{j=0}^{\infty} (-1)^j\, c(r+s, j)\, B\left(j - s - r + \beta_{123}\ , \frac{r+s}{2} + 1\right). \qquad (3.13)$$

Where

$K_{j_1 j_2} = (-1)^{j_1 + j_2} \frac{c(r,j_1)c(s,j_1)}{j_1 - r + \beta_1}$, $\breve{K}_{j_1 j_2} = (-1)^{j_1 + j_2} \frac{c(s,j_1)c(r,j_1)}{j_1 - s + \beta_2}$,

$c(k,0) = 2^k, c(k,1) = k\ 2^{k-1}$ and $c(k,j) = \frac{k\ 2^{k-2j}}{j!} \prod_{i=1}^{j-1}(k - j - i),\ j \geq 2,$

${}_pF_q(b_1,...,b_P; c_1,...,c_q; u) = \sum_{i=0}^{\infty} \frac{(b_1)_i ... (b_P)_i}{(c_1)_i ... (c_q)_i} \frac{u^i}{i!}$ is a hypergeometric function,

$(b)_i = b(b+1)...(b+i-1) = \frac{\Gamma(b+i)}{\Gamma(b)} (b \neq 0, i = 1,2,...)$. and p,q are nonnegative integers,

And $B(.,.)$ is the beta function.

Proof. beginning with $E(X_1^r, X_2^s) = \int_0^{\infty}\int_0^{\infty} x_1^r x_2^s f(x_1, x_2) dx_1 dx_2$ $\quad r, s = 1,2,3,...$ and substituting for $f(x_1, x_2)$ from (3.2).

Take the following transformations

$U_1 = F(X_1)$ and $U_2 = F(X_2)$ such that $x_1 = F^{-1}(u_1)$ and $x_2 = F^{-1}(u_2)$.

And use the power series expansion for $[1 + \sqrt{1 - u^{\frac{1}{\beta}}}]^k$ given in Equation (2.5)

Then, by using the following relationship

$$\mathrm{B}_x(\alpha, \beta) = \int_0^x u^{\alpha-1}(1-u)^{\beta-1} du = \frac{x^\alpha}{\alpha}\ {}_2F_1(\alpha, 1-\beta; \alpha+1; x),$$

where $\mathrm{B}_x(\alpha, \beta)$ is an incomplete beta function

and the identity $\int u^{\alpha-1}(1-u)^{\beta-1}\ {}_2F_1(c, d; \rho; u) du = \mathrm{B}(\alpha, \beta)\ {}_3F_2(\alpha, c, d; \rho, \alpha+\beta; 1)$

for $\alpha, \beta > 0$ and $d + \beta - \alpha - c > 0$,

Then, the expression for $E(X_1^r, X_2^s)$ can be derived.

3.5 Measures of Correlation and the Copula

This section discusses how copulas can be used to obtain the BITL distribution. Let H be a bivariate distribution function with continuous marginals F_1 and F_2, as per Sklar (1959). Moreover, if $\bar{F}_1$, $\bar{F}_2$ and $\bar{H}$ are the survival functions that correspond to F_1, F_2 and H, respectively, it is evident that the copula function $C: [0,1]^2 \rightarrow [0,1]$, which has the property that $H(x_1, x_2) = C\left(F_1(x_1), F_2(x_2)\right)$ exists. It follows from Sklar's theorem that there is a unique function $\hat{C}: [0,1]^2 \rightarrow [0,1]$ known as a survival copula such that $\bar{H}(x_1, x_2) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2))$.

The (X_1, X_2) Marshall Olkin bivariate survival copula is provided by

$$\hat{C}(u_1, u_2) = u_1 u_2 \min({u_1}^{-\vartheta_1}, {u_2}^{-\vartheta_2})$$

This fulfills the relationship below

$$S_{BITL}(x_1, x_2) = \hat{C}\big(S_{ITL}(x_1;\ \beta_{13}), S_{ITL}(x_2;\ \beta_{23})\big).$$

This structure results in the copula family indicated by

$$C_{\vartheta_1,\vartheta_2}(u_1, u_2) = \min({u_1}^{1-\vartheta_1} u_2,\ u_1 {u_2}^{1-\vartheta_2})$$

$$= \begin{cases} {u_1}^{1-\vartheta_1}\ u_2 & {u_1}^{\vartheta_1} > {u_2}^{\vartheta_2} \\ u_1 {u_2}^{1-\vartheta_2} & {u_1}^{\vartheta_1} < {u_2}^{\vartheta_2} \end{cases}$$

where $\vartheta_1 = \frac{\beta_3}{\beta_1 + \beta_3}$, $\vartheta_2 = \frac{\beta_3}{\beta_2 + \beta_3}$, $u_1 = F_1(x_1)$ and $u_2 = F_2(x_2)$

It should be mentioned that Marshall-Olkin copulas have both singular and absolutely continuous elements.

$$\frac{\partial^2}{\partial u_1 \partial u_2} C_{\vartheta_1,\vartheta_2}(u_1, u_2) = \begin{cases} {u_1}^{-\vartheta_1} & {u_1}^{\vartheta_1} > {u_2}^{\vartheta_2} \\ {u_2}^{-\vartheta_2} & {u_1}^{\vartheta_1} < {u_2}^{\vartheta_2} \end{cases}.$$

The curve ${u_1}^{\vartheta_1} = {u_2}^{\vartheta_2}$ in $[0,1]^2$ is where the mass of the singular component is concentrated.

A copula offers a practical way to examine and quantify the interdependence of random variables. For the Marshall-Olkin BITL model, Spearman's rho $\rho S\big(C_{\vartheta_1,\vartheta_2}\big)$ and Kendall's tau $\tau\big(C_{\vartheta_1,\vartheta_2}\big)$ are both fairly simple to evaluate as follows

$$\rho S\big(C_{\vartheta_1,\vartheta_2}\big) = 12 \int_0^1 \int_0^1 C_{\vartheta_1,\vartheta_2}(u, v)\, du dv - 3$$

$$= \frac{3\vartheta_1\vartheta_2}{2\vartheta_1 + 2\vartheta_2 - \vartheta_1\vartheta_2} = \frac{3\beta_3}{2\beta_1 + 2\beta_2 + 3\beta_3}, \tag{3.14}$$

and

$$\tau\big(C_{\vartheta_1,\vartheta_2}\big) = 4 \int_0^1 \int_0^1 C_{\vartheta_1,\vartheta_2}(u, v)\, dC_{\vartheta_1,\vartheta_2}(u, v) - 1$$

$$= \frac{\vartheta_1\vartheta_2}{\vartheta_1 + \vartheta_2 - \vartheta_1\vartheta_2} = \frac{\beta_3}{\beta_1 + \beta_2 + \beta_3}. \tag{3.15}$$

Moreover, Marshall-Olkin copulas have upper tail dependence. Without loss of generality assume that $\vartheta_1 > \vartheta_2$, gets

$$\lambda_U = \lim_{u \to 1} \frac{1 - 2u + C(u, u)}{1 - u} = \lim_{u \to 1} \frac{1 - 2u + u^2 \min(u^{-\vartheta_1}, u^{-\vartheta_2})}{1 - u} = \vartheta_2 = \frac{\beta_3}{\beta_2 + \beta_3}.$$

and hence $\lambda_U = \min(\frac{\beta_3}{\beta_1+\beta_3}, \frac{\beta_3}{\beta_2+\beta_3})$ is the BITL model's upper tail dependence coefficient.

3.6 Absolute Continuous BITL Model

The Marshal-Olkin BITL distribution will be modified by an absolutely continuous bivariate inverted Topp-Leone ($BITL_{ac}$) distribution by removing the singular part and leaving only the absolutely continuous part, according to Block and Basu's concepts from 1974.

The $BITL_{ac}$ distribution's pdf is given below for the random vector (Y_1, Y_2)

$$f_{Y_1,Y_2}(y_1, y_2) = c.\begin{cases} f_{BITL}(y_1;\beta_1)\cdot f_{BITL}(y_2;\beta_2+\beta_3) & if \quad y_1 < y_2 \\ f_{BITL}(y_1;\beta_1+\beta_3)\cdot f_{BITL}(y_2;\beta_2) & if \quad y_1 > y_2 \end{cases}, \tag{3.16}$$

Such that the normalizing constant is $c = \frac{\beta_1+\beta_2}{\beta_1+\beta_2+\beta_3}$.

It is denoted that $(Y_1, Y_2) \sim BITL_{ac}(\beta_1, \beta_2, \beta_3)$ if (X_1, X_2) has a BITL$(\beta_1, \beta_2, \beta_3)$ distribution, then (X_1, X_2) given $X_1 \neq X_2$ has a $BITL_{ac}$ distribution.
The associated survival function of $(Y_1, Y_2) \sim BITL_{ac}(\beta_1, \beta_2, \beta_3)$is given by

$$S_{Y_1,Y_2}(y_1, y_2) = \frac{\beta_{123}}{\beta_{12}} S_{ITL}(y_1;\beta_1)\, S_{ITL}(y_2;\beta_2) S_{ITL}(y;\beta_3) - \frac{\beta_3}{\beta_{12}} S_{ITL}(y;\beta_{123}); \tag{3.17}$$

Where $y = \max(y_1, y_2)$ and $\beta_{123} = \beta_1 + \beta_2 + \beta_3$. Additionally, the marginal survival functions for Y_1 and Y_2 are provided respectively, as follows:

$$S_{Y_1}(y_1) = \frac{\beta_{123}}{\beta_{12}} S_{ITL}(y_1;\beta_{13}) - \frac{\beta_3}{\beta_{12}} S_{ITL}(y_1;\beta_{123})$$

$$S_{Y_2}(y_2) = \frac{\beta_{123}}{\beta_{12}} S_{ITL}(y_1;\beta_{23}) - \frac{\beta_3}{\beta_{12}} S_{ITL}(y_2;\beta_{123}).$$

The following are the corresponding marginal pdfs for Y_1 and Y_2 respectively

$$f_{Y_1}(y_1) = c f_{ITL}(y_1;\beta_{13}) - c\frac{\beta_3}{\beta_{123}} f_{ITL}(y_1;\beta_{123}),$$

and

$$f_{Y_2}(y_2) = c f_{ITL}(y_2;\beta_{23}) - c\frac{\beta_3}{\beta_{123}} f_{ITL}(y_2;\beta_{123}).$$

The marginals of the $\mathrm{BITL_{ac}}$ distribution are not ITL distributions, in contrast to those of the BITL distribution. If $\beta_3 \to 0^+$, then $\mathrm{Y_1}$ and $\mathrm{Y_2}$will follow ITL distributions and will then be independent.

The Stress- Strength parameter for $(Y_1, Y_2) \sim BITL_{ac}(\beta_1, \beta_2, \beta_3)$ is provided as;

$$R = P(Y_1 < Y_2) = \frac{\beta_1}{\beta_1 + \beta_2}$$

Moreover, $\min(Y_1, Y_2) \sim ITL(\beta_{123})$.

The product moments of $(Y_1, Y_2) \sim BITL_{ac}(\beta_1, \beta_2, \beta_3)$ denoted by $\acute{\mu}_{r,s}$ are given by

$$\acute{\mu}_{r,s} = \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} C_{j_1 j_2} B\left(j_1 + j_2 - s - r + \beta_{123}\,, \frac{s}{2} + 1\right)$$
$$\cdot\ {}_3F_2\left(j_1 + j_2 - s - r + \beta_{123}\,, j_1 - r + \beta_1\,, \frac{-r}{2}; j_1 - r + \beta_1 + 1, j_1 + j_2 - r - \frac{s}{2} + \beta_{123}\,; 1\right)$$
$$+ \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty} \check{C}_{j_1 j_2} B\left(j_1 + j_2 - s - r + \beta_{123}\,, \frac{r}{2} + 1\right)$$
$$\cdot\ {}_3F_2\left(j_1 + j_2 - s - r + \beta_{123}\,, j_1 - s + \beta_2\,, \frac{-s}{2}; j_1 - s + \beta_2 + 1, j_1 + j_2 - s - \frac{r}{2} + \beta_{123}\,; 1\right)$$

Where $C_{j_1 j_2} = \frac{\beta_{12}}{\beta_{123}} (-1)^{j_1+j_2} \frac{c(r,j_1)c(s,j_1)}{j_1-r+\beta_1}$ and $\check{C}_{j_1 j_2} = \frac{\beta_{12}}{\beta_{123}} (-1)^{j_1+j_2} \frac{c(s,j_1)c(r,j_1)}{j_1-s+\beta_2}$.

4 Estimation and Exact Information Matrix

The maximum likelihood estimators (MLEs) of the BITL distribution's unknown parameters will be discussed in this section. Assume that $\{(x_{11}, x_{21}), \ldots, (x_{1n}, x_{2n})\}$ is a sample drawn at random from the $BITL(\beta_1, \beta_2, \beta_3)$ distribution. Take into account the notation below.

$$I_1 = \{i; x_{1i} < x_{2i}\}, \quad I_2 = \{i; x_{1i} > x_{2i}\}, \quad I_3 = \{x_{1i} = x_{2i} = x_i\}, \quad I = I_1 \cup I_2 \cup I_3,$$
$$|I_1| = n_1, \quad |I_2| = n_2, \quad |I_3| = n_3, \quad \text{and} \quad n_1 + n_2 + n_3 = n.$$

The log-likelihood function [$l = ln\, L\left(\underline{\beta}\right)$] of the sample of size n is given by

$$l = \ln L(\underline{\beta}) = \sum_{i\in I_1} \ln\, f_1(x_{1i}, x_{2i}) + \sum_{i\in I_2} \ln\, f_2(x_{1i}, x_{2i}) + \sum_{i\in I_3} \ln\, f_3(x_i)$$

$$\begin{aligned} ln\, L\left(\underline{\beta}\right) \propto\; & n_1 \ln\beta_1 + n_2 \ln\beta_2 + n_3 \ln\beta_3 + n_1 \ln\beta_{23} + n_2 \ln\beta_{13} - (2\beta_1 + 1)\textstyle\sum_{i\in I_1} \ln(x_{1i} + 1) \\ & -(\beta_{23} + 1)\textstyle\sum_{i\in I_1} \ln(2x_{1i} + 1) + (\beta_{23} - 1)\sum_{i\in I_1} \ln(2x_{1i} + 1) - (2\beta_{13} - 1)\sum_{i\in I_2} \ln(x_{1i} + 1) \\ & +(\beta_{13} - 1)\textstyle\sum_{i\in I_2} \ln(2x_{1i} + 1) - (2\beta_2 + 1)\sum_{i\in I_2} \ln(x_{2i} + 1) + (\beta_2 - 1)\sum_{i\in I_2} \ln(2x_{2i} + 1) \\ & -(2\beta_{123} - 1)\textstyle\sum_{i\in I_3} \ln(x_i + 1) + (\beta_{123} - 1)\sum_{i\in I_3} \ln(2x_i + 1). \end{aligned} \tag{4.1}$$

The first derivatives are as follows

$$\frac{\partial l}{\partial \beta_1} = \frac{n_1}{\beta_1} + \frac{n_2}{\beta_{13}} - 2\left[\sum_{i\in I_1\cup I_2} \ln(x_{1i} + 1) + \sum_{i\in I_3} \ln(x_i + 1)\right] + \sum_{i\in I_2} \ln(2x_{1i} + 1) + \sum_{i\in I_3} \ln(2x_i + 1),$$

$$\frac{\partial l}{\partial \beta_2} = \frac{n_2}{\beta_2} + \frac{n_1}{\beta_{23}} - 2\left[\sum_{i\in I_1} \ln(x_{1i} + 1) + \sum_{i\in I_2} \ln(x_{2i} + 1) + \sum_{i\in I_3} \ln(x_i + 1)\right]$$

$$+\sum_{i\in I_1} \ln(2x_{2i} + 1) \sum_{i\in I_2} \ln(2x_{2i} + 1) + \sum_{i\in I_3} \ln(2x_i + 1),$$

and

$$\frac{\partial l}{\partial \beta_3} = \frac{n_3}{\beta_3} + \frac{n_1}{\beta_{23}} + \frac{n_2}{\beta_{13}} - 2\left[\sum_{i\in I_1} \ln(x_{1i} + 1) + \sum_{i\in I_2} \ln(x_{1i} + 1) + \sum_{i\in I_3} \ln(x_i + 1)\right]$$

$$+\sum_{i\in I_1} \ln(2x_{2i} + 1) \sum_{i\in I_2} \ln(2x_{1i} + 1) + \sum_{i\in I_3} \ln(2x_i + 1).$$

It is obvious that these three equations cannot be expressed explicitly, so their solutions must be numerically obtained using the Newton-Raphson method, as will be seen in Section 6. The three equations are simultaneously solved to produce MLEs.

4.1 Exact Fisher Information Matrix

This section determines the exact Fisher information matrix for the BITL distribution. And utilized for the asymptotic distribution of maximum likelihood estimators (MLEs) and variance-covariance matrix derivation.

The information matrix matrix $I(\underline{\beta})$in this instance is a 3×3 symmetric matrix with elements.

$$I_{ij}\left(\underline{\beta}\right) = -E[\frac{\partial^2 \log l\ (\underline{\beta})}{\partial\beta_i \partial\beta_j}]$$

The $BITL(\beta_1, \beta_2, \beta_3)$ distribution's exact Fisher information matrix is obtained to be

$I_{11}\left(\underline{\beta}\right) = \frac{n_1}{{\beta_1}^2} + \frac{n_2}{(\beta_1+\beta_3)^2}$, $I_{22}\left(\underline{\beta}\right) = \frac{n_2}{{\beta_2}^2} + \frac{n_1}{(\beta_2+\beta_3)^2}$, $I_{33}\left(\underline{\beta}\right) = \frac{n_1}{(\beta_2+\beta_3)^2} + \frac{n_2}{(\beta_1+\beta_3)^2} + \frac{n_3}{{\beta_3}^2}$

$I_{13}\left(\underline{\beta}\right) = \frac{n_2}{(\beta_1+\beta_3)^2}$ $I_{23}\left(\underline{\beta}\right) = \frac{n_1}{(\beta_2+\beta_3)^2}$ and $I_{12}\left(\underline{\beta}\right) = I_{21}\left(\underline{\beta}\right) = 0.$

4.2 Confidence Intervals

We'll present the asymptotic confidence intervals for the BITL distribution's parameters. Let $\underline{\hat{\beta}} = (\widehat{\beta_1}, \widehat{\beta_2}, \widehat{\beta_3})$ be the MLEs of $\underline{\beta} = (\beta_1, \beta_2, \beta_3)$. When the parameters are inside the parameter space rather than on the boundary and the regularity conditions are met, we have:

$$\sqrt{n}\left(\underline{\hat{\beta}} - \underline{\beta}\right) \xrightarrow{d} N_3(\underline{0}, I^{-1}(\underline{\beta})). \tag{4.2}$$

where $I\left(\underline{\beta}\right)$ is the Fisher information matrix. In order to create confidence intervals for the model parameters, the multivariate normal distribution with mean vector $\underline{0} = (0,0,0)$ and variance-covariance matrix $I^{-1}(\underline{\beta})$ can be used. In other words, a $(1-\alpha)\%$ two-sided confidence interval can be introduced for values of β_1, β_2 and β_3 as follows:

$$\widehat{\beta_1} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\beta_1\beta_1}(\underline{\beta})},\ \widehat{\beta_2} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\beta_2\beta_2}(\underline{\beta})} \text{ and } \widehat{\beta_3} \pm Z_{\frac{\alpha}{2}}\sqrt{I^{-1}_{\beta_3\beta_3}(\underline{\beta})}. \tag{4.3}$$

Respectively, where $I^{-1}_{\beta_1\beta_1}(\underline{\beta})$, $I^{-1}_{\beta_2\beta_2}(\underline{\beta})$ and $I^{-1}_{\beta_3\beta_3}(\underline{\beta})$are diagonal elements of the variance-covariance matrix and $Z_{\frac{\alpha}{2}}$ is the $(\frac{\alpha}{2})^{th}$percentile of a standard normal distribution.

5. The BITL Model with Real Data

In this section, one data set will be used to illustrate how the BITL model performs in practical situations. Meintanis (2007) provided the data set, which was used in this study. According to him, the data represent football (soccer) games in which at least one goal was scored by the home team and at least one goal was scored by any team directly from a penalty kick, foul kick, or other direct kick (collectively referred to as "kick goals"). Here, X_1 stands for the first goal of any kind scored by any team's first kick in minutes, and X_2 stands for the first goal of any kind scored by the visiting team. In this situation, all options are open, such as $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$.

Meintanis (2007) used the Marshall-Olkin bivariate exponential model to analyze these data, and many other authors have since used it, including Kundu and Gupta (2009), Muhammed (2016,2019,2020), who also considered the bivariate generalized exponential model, the bivariate inverse Weibull model, the bivariate generalized Burr model and the bivariate generalized inverted Kumaraswamy model, respectively. Here, these data will be applied to the BITL distribution and compared to some other bivariate models.

First, for X_1 , X_2 and $\min(X_1, X_2)$ with ITL(4.635), ITL(1.647) and ITL(5.543), the Kolmogorov-Smirnov distances between the fitted distribution and the empirical distribution function are (0.383), (0.412), and (0.419), respectively. This might suggest that the BITL model could be applied to fit the given data set.

The following is the asymptotic variance-covariance matrix for the BITL model parameters

$$I^{-1}(\underline{\beta}) = \left\langle \begin{array}{c|c|c} 0.091 & 0.00044 & -0.036 \\ 0.00044 & 0.046 & -0.00631 \\ -0.036 & 0.00631 & 0.523 \end{array} \right\rangle$$

Moreover, a 95% confidence intervals of $\beta_1, \beta_2, \beta_3$are computed and they are as follows; (0.659, 0.854), (0.959, 0.821), (3.663,4.129). With correspondence lengths (0.195, 0.138 , 0.466)

The BITL model can be compared to other bivariate distributions using the Akaike information criterion (AIC), Bayesian information criterion (BIC), Consistent Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC). These bivariate distributions include the bivariate generalized inverted Kumaraswamy (BGIKum) model [Muhammed (2020)], bivariate generalized Burr (BGB) model [Muhammed (2019a)], bivariate generalized exponential (BGE) model [Kundu and Gupta (2009)] and bivariate exponential (BVE) model [Meintanis (2007)] as shown in Table 2.

It is now obvious that the BITL model can be representing this case based on the confidence intervals, log-likelihood values, and Kolmogorov-Smirnov distances.

6 Simulation Study

The outcomes of a Monte Carlo simulation study testing the effectiveness of MLE of the model Parameters will be presented in this section. For each sample size, the following measurements were used to evaluate the MLEs: Average Estimates (AE), Mean Squared Error (MSE), Relative Absolute Bias (RAB), and Confidence Interval Length (CIL) are estimated from $R = 1000$ replications for $\widehat{\beta_1}, \widehat{\beta_2}$ and $\widehat{\beta_3}$ the sample size has been considered at $n = 50, 100, 150, 200$ and 300, and some values for β_1, β_2 and β_3 have been considered.

Algorithm to generate from BITL distribution

Step 1. Generate U_1, U_2 and U_3 from U(0,1).

Step 2. Compute $Z_1 = (1-U_1)^{\frac{-1}{\beta_1}} \sqrt{1-(1-U_1)^{\frac{1}{\beta_1}}}.\left(1+\sqrt{1-(1-U_1)^{\frac{1}{\beta_1}}}\right)$

$Z_2 = (1-U_2)^{\frac{-1}{\beta_2}} \sqrt{1-(1-U_2)^{\frac{1}{\beta_2}}}.\left(1+\sqrt{1-(1-U_2)^{\frac{1}{\beta_2}}}\right)$ and $Z_3 = (1-U_3)^{\frac{-1}{\beta_3}} \sqrt{1-(1-U_3)^{\frac{1}{\beta_3}}}.\left(1+\sqrt{1-(1-U_3)^{\frac{1}{\beta_3}}}\right)$

Step3. Obtain $X_1 = \min(Z_1, Z_3)$ and $X_2 = \min(Z_2, Z_3)$.

Step4. Define the indicator functions as

$\delta_{1i} = \begin{cases} 1; & x_{1i} < x_{1i} \\ 0; & otherwise \end{cases}$, $\delta_{2i} = \begin{cases} 1; & x_{1i} > x_{1i} \\ 0; & otherwise \end{cases}$ and $\delta_{3i} = \begin{cases} 1; & x_{1i} = x_{1i} \\ 0; & otherwise \end{cases}$.

Step5. The corresponding sample size n must satisfy $n = n_1 + n_2 + n_3$

Such that $n_1 = \sum_{i=1}^{n} \delta_{1i}$, $n_2 = \sum_{i=1}^{n} \delta_{2i}$ and $n_3 = \sum_{i=1}^{n} \delta_{3i}$.

The MATHCAD program is used to generate 1000 data sets for various sample size selections in order to solve the nonlinear likelihood equations. Tables 3 through 7 show that the estimates are accurate and that MSE and RAB go down as sample size goes up.

7. Bivariate Distributions Arising from BITL Distribution

7.1 Bivariate Generating Families

The logarithm of both the cdf $G(x)$ and survival function $\overline{G}(x)$ of the baseline distribution is consequently added to the survival function of the BITL model to produce the joint survival function of the new class of distributions, known as the *BITL-G* class of distributions. Let G be the continuous distribution function of a continuous random variable with support $A \subseteq R$.

Now, define the two logits $W_1(x) = -\log G(x)$ and $W_2(x) = -\log(1-G(x))$. Thus, a new class of bivariate distributions can be obtained by defining the following joint survival function:

$$s_{BITL-G}(x,y) = s_{BITL}(W(x), W(y)) \quad \forall (x,y) \in A \times A$$

$$s_{BITL-G}(x,y) = S_{ITL}(W(x);\, \beta_1) S_{ITL}(W(y);\, \beta_2) S_{ITL}(W(z);\, \beta_3) \tag{7.1}$$

Where $z = \max(x,y)$

Moreover,

$$S_{BITL-G}\,(x,y)=\begin{cases} S_{ITL}(W(x);\ \beta_1)S_{ITL}(W(y);\ \beta_{23}), & x<y \\ S_{ITL}(W(x);\ \beta_{13})S_{ITL}(W(y);\ \beta_2), & x>y. \\ S_{ITL}(W(z);\ \beta_{123}), & x=y=z \end{cases}$$

The joint pdf of the new *BITL-G* class of distributions can be written as

$$f_{BITL-G}\,(x,y)=\begin{cases} w(x)w(y)f_{ITL}(W(x);\beta_1)f_{ITL}(W(y);\beta_{23}), & x<y \\ w(x)w(y)f_{ITL}(W(x);\beta_{13})f_{ITL}(W(y;\beta_2), & x>y \\ \frac{\beta_3}{\beta_{123}}w(z)f_{ITL}(W(z);\ \beta_{123}), & x=y=z \end{cases}.$$

Where $w(x)=\frac{dW(x)}{dx}$.

Logit 1: when $W_1(x)=-\log G(x)$ then Equation (7.1) can be written as

$$s_{BITL-G}(x,y)=S_{ITL}(-\log G(x)\,;\ \beta_1)S_{ITL}(-\log G(y)\,;\ \beta_2)S_{ITL}(-\log G(z)\,;\ \beta_3).$$

$$s_{BITL-G}(x,y)=(1-\log G(x))^{-2\beta_1}.(1-\log G(y))^{-2\beta_2}(1-\log G(z))^{-2\beta_3}$$

$$(1-2\log G(x))^{\beta_1}(1-2\log G(y))^{\beta_2}(1-2\log G(z))^{\beta_3}\ . \qquad (7.2)$$

Logit 2: when $W_1(x)=-\log \overline{G}(x)$ then Equation (7.1) can be written as

$$s_{BITL-G}(x,y)=S_{ITL}\big(-\log \overline{G}(x)\,;\ \beta_1\big)S_{ITL}\big(-\log \overline{G}(y)\,;\ \beta_2\big)S_{ITL}\big(-\log \overline{G}(z)\,;\ \beta_3\big)$$

$$s_{BITL-G}(x,y)=(1-\log \overline{G}(x))^{-2\beta_1}.(1-\log \overline{G}(y))^{-2\beta_2}(1-\log \overline{G}(z))^{-2\beta_3}$$

$$\big(1-2\log \overline{G}(x)\big)^{\beta_1}\big(1-2\log \overline{G}(y)\big)^{\beta_2}(1-2\log \overline{G}(z))^{\beta_3}\ . \qquad (7.3)$$

Now, some special bivariate distributions created with the suggested generators are shown.

7.1.1 Bivariate Exponential Distribution

Let $G(x)=1-e^{-\lambda x}$, hence $\overline{G}(x)=e^{-\lambda x}$ be the cdf and survival function of the exponential distribution respectively, then by taking the exponential distribution as a base distribution in Equation (7.3), a new bivariate exponential distribution is produced in the following form

$$s_{BITLE}(x,y)=(1+\lambda x)^{-2\beta_1}.(1+\lambda y)^{-2\beta_2}(1+\lambda z)^{-2\beta_3}$$

$$(1+2\lambda x)^{\beta_1}(1+2\lambda y)^{\beta_2}(1+2\lambda z)^{\beta_3}.$$

Where $\lambda>0$. This version of the bivariate exponential distribution is called bivariate inverted Topp-Leone exponential (BITL-E).

7.1.2 Bivariate Weibull Distribution

Let $\overline{G}(x)=e^{-\lambda x^{\alpha}}$ be the survival function of the Weibull distribution, which will be used as a base distribution in Equation (7.3) to produce a new version of the bivariate Weibull distribution in the following form

$$s_{BITLW}(x,y)=(1+\lambda x^{\alpha})^{-2\beta_1}.(1+\lambda y^{\alpha})^{-2\beta_2}(1+\lambda z^{\alpha})^{-2\beta_3}$$

$$(1+2\lambda x^{\alpha})^{\beta_1}(1+2\lambda y^{\alpha})^{\beta_2}(1+2\lambda z^{\alpha})^{\beta_3}.$$

Where $\lambda > 0$ and $\alpha > 0$. This new Weibull distribution is called bivariate inverted Topp-Leone Weibull (BITL-W).

7.1.3 Bivariate Gumbel Distribution

Let $\overline{G}(x) = 1 - \exp(-\exp(x-\mu))$ for $x \in R$ where $\mu \in R$ is a location parameter, is the survival function of the Gumbel distribution, then by using the logit in Equation (7.3) a new bivariate Gumbel distribution is produced and called bivariate inverted Topp-Leone Gumbel (BITL-G) and can be written in the following form.

$$s_{BITL-G}(x,y) = (1+\exp(x-\mu))^{-2\beta_1}.(1+\exp(y-\mu))^{-2\beta_2}(1+\exp(z-\mu))^{-2\beta_3}$$
$$(1+2\exp(x-\mu))^{\beta_1}(1+2\exp(y-\mu))^{\beta_2}(1+2\exp(z-\mu))^{\beta_3}.$$

7.1.4 Bivariate Pareto Distribution

Let $\overline{G}(x) = \frac{1}{x^{\alpha}}$ for $x > 1$ and $\alpha > 0$ is the survival function of the Pareto distribution, then by using the Pareto distribution as a base distribution in Equation (7.3) another bivariate Pareto distribution is obtained as follows

$$s_{BITL-P}(x,y) = (1+\alpha\log x)^{-2\beta_1}.(1+\alpha\log y)^{-2\beta_2}(1+\alpha\log z)^{-2\beta_3}$$
$$(1+2\alpha\log x)^{\beta_1}(1+2\alpha\log y)^{\beta_2}(1+2\alpha\log z)^{\beta_3}.$$

This new distribution is called bivariate inverted Topp-Leone Pareto (BITL-P).

7.1.5 Bivariate Linear Failure Distribution

Assume the linear failure rate distribution is taken as a base distribution with the survival function

$$\overline{G}(x) = \exp(-ax - \tfrac{b}{2}x^2) \text{ for } x > 0 \text{ where } a, b > 0$$

Then, by using Equation (7.3) a new bivariate linear failure rate distribution is gotten as follows

$$s_{BITL-LFR}(x,y) = (1+ax+\frac{b}{2}x^2)^{-2\beta_1}(1+ay+\frac{b}{2}y^2)^{-2\beta_2}(1+az+\frac{b}{2}z^2)^{-2\beta_3}$$
$$(1+2ax+bx^2)^{\beta_1}(1+2ay+by^2)^{\beta_2}(1+2az+bz^2)^{\beta_3}$$

This new distribution is called bivariate inverted Topp-Leone linear failure rate (BITL-LFR).

7.1.6 Bivariate Lomax Distribution

If the Lomax distribution with the survival function $\overline{G}(x) = (1+\frac{x}{\lambda})^{-\alpha}, x > 0, , \lambda > 0$. Is taken as a base distribution in Equation (7.3), then a new Lomax distribution is produced as follows

$$s_{BITL-L}(x,y) = (1+\alpha\log(1+\frac{x}{\lambda}))^{-2\beta_1}.(1+\alpha\log(1+\frac{y}{\lambda}))^{-2\beta_2}(1+\alpha\log(1+\frac{z}{\lambda}))^{-2\beta_3}$$
$$\left(1+2\alpha\log(1+\tfrac{x}{\lambda})\right)^{\beta_1}\left(1+2\alpha\log(1+\tfrac{y}{\lambda})\right)^{\beta_2}(1+2\alpha\log(1+\tfrac{z}{\lambda}))^{\beta_3}.$$

This new extension of Lomax distribution is called bivariate inverted Topp-Leone Lomax (BITL-L).

7.1.7 Bivariate Kumaraswamy Distribution

Let $\overline{G}(x) = (1 - x^{\alpha})^{\theta}$ for $0 < x < 1$ and $\alpha, \theta > 0$ is the survival function of the Kumaraswamy distribution, then by using the Kumaraswamy distribution as a base distribution in Equation (7.3) another bivariate Kumaraswamy distribution is obtained as follows

$$s_{BITL-K}(x, y) = (1 - \theta \log(1 - x^{\alpha}))^{-2\beta_1}.(1-\theta \log(1 - y^{\alpha}))^{-2\beta_2}(1 - \theta \log(1 - z^{\alpha}))^{-2\beta_3}$$

$$(1 - 2\theta \log(1 - x^{\alpha}))^{\beta_1}(1 - 2\theta \log(1 - y^{\alpha}))^{\beta_2}(1 - 2\theta \log(1 - z^{\alpha}))^{\beta_3}.$$

This new distribution is called bivariate inverted Topp-Leone Kumaraswamy (BITL-K).

7.1.8 Bivariate Fréchet Distribution

Let $G(x) = \exp\left(-\left(\frac{x}{\alpha}\right)^{-\lambda}\right), x > 0$ be the cdf of the Fréchet distribution, then by taking the Fréchet distribution as a base distribution in Equation (7.2), a new bivariate Fréchet distribution is produced in the following form

$$s_{BITL-F}(x, y) = (1 + \left(\frac{x}{\alpha}\right)^{-\lambda})^{-2\beta_1}.(1+\left(\frac{y}{\alpha}\right)^{-\lambda})^{-2\beta_2}(1 + \left(\frac{z}{\alpha}\right)^{-\lambda})^{-2\beta_3}$$

$$\left(1 + 2\left(\frac{x}{\alpha}\right)^{-\lambda}\right)^{\beta_1}\left(1 + 2\left(\frac{y}{\alpha}\right)^{-\lambda}\right)^{\beta_2}(1 + 2\left(\frac{z}{\alpha}\right)^{-\lambda})^{\beta_3}.$$

Where $\lambda, \alpha > 0$. This version of bivariate Fréchet distribution is called bivariate inverted Topp-Leone Fréchet (BITL-F).

7.1.9 Bivariate Generalized Exponential Distribution

Let $G(x) = (1 - e^{-\lambda x})^{\theta}, x > 0$ be the cdf of the generalized exponential distribution, then by taking the generalized exponential distribution as a base distribution in Equation (7.2), a new bivariate generalized exponential distribution is produced in the following form

$$s_{BITL-GE}(x, y) = (1 + \theta \log(1 - e^{-\lambda x}))^{-2\beta_1}.(1+\theta \log(1 - e^{-\lambda y}))^{-2\beta_2}$$

$$(1 + \theta \log(1 - e^{-\lambda z}))^{-2\beta_3}\ \left(1 + 2\theta \log(1 - e^{-\lambda x})\right)^{\beta_1}$$

$$\left(1 + 2\theta \log(1 - e^{-\lambda y})\right)^{\beta_2}(1 + 2\theta \log(1 - e^{-\lambda z}))^{\beta_3}.$$

Where $\theta, \lambda > 0$. This version of the bivariate generalized exponential distribution is called bivariate inverted Topp-Leone generalized exponential (BITL-GE).

7.2. Bivariate Semi- parametric families

In the next sub-sections, a second parameter is added to the ITL distribution by two ways of semi-parametric families and hence bivariate extensions are obtained for both ways.

7.2.1 Bivariate Generalized Inverted Topp-Leone Model

A generalized inverted Topp-Leone distribution is obtained by adding a power parameter α as follows: Assume $T \sim ITL(\beta)$, take $X = T^{\frac{1}{\alpha}}$, then X is distributed as generalized inverted Topp-Leone (GITL) with parameters (α, β) and has a survival function given by

$$S_{GITL}(x, \alpha, \beta) = (x^{\alpha} + 1)^{-2\beta}(2x^{\alpha} + 1)^{\beta} \quad , \alpha, \beta > 0$$

The GITL distribution is denoted as $GITL(\alpha, \beta)$. Now, a bivariate generalized inverted Topp-Leone (BGITL) distribution is defined as follows:

Assume that $U_i \sim GITL(\alpha, \beta_i), i = 1,2,3$ such that $U_i's$ are mutually independent random variables and define $X_j = min\left(U_j, U_3\right), j = 1,2$. Such that; $X_j's$ are dependent random variables. Hence the joint survival function of the vector (X_1, X_2) denoted by $S_{GBITL}(x_1, x_2)$ is given as

$$S_{BGITL}(x_1, x_2) = S_{GITL}(x_1; \alpha, \beta_1) S_{GITL}(x_2; \alpha, \beta_2) S_{GITL}(x_3; \alpha, \beta_3).$$

$$= (x_1^{\alpha} + 1)^{-2\beta_1}(2x_1^{\alpha} + 1)^{\beta_1} . (x_2^{\alpha} + 1)^{-2\beta_2}(2x_2^{\alpha} + 1)^{\beta_2} . (x_3^{\alpha} + 1)^{-2\beta_3}(2x_3^{\alpha} + 1)^{\beta_3} \quad (7.4)$$

where $x_3 = \max(x_1, x_2)$.

The following form can be used to rewrite the joint survival function of the BGITL distribution:

$$S_{BGITL}(x_1, x_2) = \begin{cases} (x_1^{\alpha} + 1)^{-2\beta_1}(2x_1^{\alpha} + 1)^{\beta_1} . (x_2^{\alpha} + 1)^{-2\beta_{23}}(2x_2^{\alpha} + 1)^{\beta_{23}}, & x_1 < x_2 \\ (x_1^{\alpha} + 1)^{-2\beta_{13}}(2x_1^{\alpha} + 1)^{\beta_{13}} . (x_2^{\alpha} + 1)^{-2\beta_2}(2x_2^{\alpha} + 1)^{\beta_2}, & x_1 > x_2 \\ (x^{\alpha} + 1)^{-2\beta_{123}}(2x^{\alpha} + 1)^{\beta_{123}}, & x_1 = x_2 = x \end{cases}.$$

Where $\beta_{13} = \beta_1 + \beta_3$, $\beta_{23} = \beta_2 + \beta_3$ and $\beta_{123} = \beta_1 + \beta_2 + \beta_3$.

We can obtain the joint pdf of the BGITL distribution as $f_{BITL}(x_1, x_2) =$

$$\begin{cases} f_{GITL}(x_1; \alpha, \beta_1) f_{GITL}(x_2; \alpha, \beta_{23}), & x_1 < x_2 \\ f_{GITL}(x_1; \alpha, \beta_{13}) f_{GITL}(x_2; \alpha, \beta_2), & x_1 > x_2 \\ \frac{\beta_3}{\beta_{123}} f_{GITL}(x; \alpha, \beta_{123}), & x_1 = x_2 \end{cases}. \quad (7.5)$$

7.2.2 Bivariate Exponentiated Inverted Topp-Leone Model

The exponentiated inverted Topp-leone (EITL) is obtained by adding a resilience parameter to the cdf of the inverted Topp-Leone distribution as follows

$$F_{EITL}(x; \alpha, \beta) = [F_{ITL}(x, \beta)]^{\alpha}, \qquad \alpha > 0$$

$$= [1 - (x + 1)^{-2\beta}(2x + 1)^{\beta}]^{\alpha}.$$

Then,the pdf of the EITL distribution is given as

$$f_{EITL}(x; \alpha, \beta) = \alpha\, [F_{ITL}(x, \beta)]^{\alpha - 1} f_{ITL}(x, \beta), \qquad \alpha, \beta > 0$$

Based on the same bivariate idea in the previous sections and by using the maximization process a bivariate exponentiated inverted Topp-leone is introduced with the joint cdf and pdf respectively, as follows

$$F_{BEITL}(x_1, x_2) = [F_{ITL}(x_1, \beta)]^{\alpha_1}[F_{ITL}(x_2, \beta)]^{\alpha_2}[F_{ITL}(x_3, \beta)]^{\alpha_3}, \quad (7.6)$$

such that $x_3 = \min(x_1, x_2)$

$$f_{BEITL}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & x_1 < x_2 \\ f_2(x_1, x_2), & x_1 > x_2, \\ f_3(x), & x_1 = x_2 = x \end{cases} \quad (7.7)$$

where

$$f_1(x_1, x_2) = (\alpha_1 + \alpha_3)\alpha_2 f_{ITL}(x_1, \beta)\, f_{ITL}(x_2, \beta)[F_{ITL}(x_1, \beta)]^{\alpha_1+\alpha_3-1}[F_{ITL}(x_2, \beta)]^{\alpha_2-1},$$

$$f_2(x_1, x_2) = \alpha_1(\alpha_2 + \alpha_3) f_{ITL}(x_1, \beta)\, f_{ITL}(x_2, \beta)[F_{ITL}(x_1, \beta)]^{\alpha_1-1}[F_{ITL}(x_2, \beta)]^{\alpha_2+\alpha_3-1},$$

and $f_3(x) = \alpha_3 f_{ITL}(x)\, [F_{ITL}(x, \beta)]^{\alpha_1+\alpha_2+\alpha_3-1}$.

8 Conclusions

In this paper, a new j-shaped distribution called ITL is discussed for both univariate and bivariate cases. In the bivariate case, the distribution is called BITL and whose marginals are univariate ITL distributions. There are absolute continuous and singular parts to the BITL distribution. This distribution can be used in practice for dependent and non-negative random variables because both the joint distribution function and the joint density function are in closed forms. Three of the model's parameters are unknown. To test the effectiveness of the MLEs, simulations were run using the three unknown parameters' exact information matrix and maximum likelihood estimates. One set of data has been investigated for illustration's purpose. An absolute continuous version of the BITL was also obtained based on Block and Basu (1974), and several of its properties are presented. It is demonstrated that the Marshall and Olkin survival copula is used to derive the BITL distribution, and a tail dependence measure is discussed. A new two methods for constructing bivariate distributions from BITL distribution are discussed in details. Consequently, generalized and exponentiated ITL distributions are also defined in both univariate and bivariate cases. The distribution of sum, product and ratio for two dependent variables follow BITL distribution is in progress as a future work. Moreover, the estimation of BITL distribution in presence of censored samples is under study and it will be done soon as possible.

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Table1: UEFA Champions League Data

No.	x_1	x_2	No.	x_1	x_2
1	26	20	20	34	34
2	63	18	21	53	39
3	19	19	22	54	7
4	66	85	23	51	28
5	4	4	24	76	64
6	49	49	25	64	15
7	8	8	26	26	48
8	69	71	27	16	16
9	39	39	28	44	6
10	82	48	29	25	14
11	72	72	30	55	11
12	66	62	31	49	49
13	25	9	32	24	24
14	41	3	33	44	30
15	16	75	34	42	3
16	18	18	35	27	47
17	22	14	36	28	28
18	42	42	37	2	2
19	36	0.52			

Table 2. logL, AIC, BIC,CAIC and HQIC for different bivariate models

Model	log L	AIC	BIC	CAIC	HQIC
BITL	-138.482	144.482	149.315	145.21	146.186
BGIKum	-157.043	167.043	175.098	168.978	169.883
BGB	-43.509	97.018	105.073	98.954	99.858
BGE	-20.59	49.18	48.40		
BE	-44.56	96.12	95.46	96.11	97.62

Table 3: The AE, MSE,RAB and CIL of β_1, β_2 and β_3For BITL Distribution with True Parameter Values $(\beta_1, \beta_2, \beta_3) =$ (0.2,0.2,0.9)

N	Parameter	AE	MSE	RAB	CIL
50	β_1	0.123	0.0061	0.387	0.024
	β_2	0.164	0.0013	0.179	0.032
	β_3	0.757	0.020	0.159	0.062
100	β_1	0.121	0.0063	0.396	0.012
	β_2	0.163	0.0014	0.187	0.016
	β_3	0.799	0.010	0.112	0.033
150	β_1	0.119	0.0065	0.405	0.0078
	β_2	0.159	0.0016	0.203	0.010
	β_3	0.789	0.012	0.123	0.0220
200	β_1	0.116	0.0070	0.420	0.0058
	β_2	0.154	0.0021	0.228	0.0077
	β_3	0.811	0.0079	0.099	0.0170
300	β_1	0.115	0.0073	0.427	0.0038
	β_2	0.153	0.0022	0.237	0.0050
	β_3	0.811	0.0079	0.099	0.011

Table 4: The AE, MSE,RAB and CIL of β_1, β_2 and β_3For BITL Distribution with True Parameter Values $(\beta_1, \beta_2, \beta_3) =$ (0.5, 0.5, 0.9)

N	Parameter	AE	MSE	RAB	CIL
50	β_1	0.249	0.063	0.502	0.038
	β_2	0.439	0.00376	0.123	0.064
	β_3	0.829	0.00501	0.079	0.076
100	β_1	0.121	0.00627	0.396	0.012
	β_2	0.163	0.0014	0.187	0.016
	β_3	0.799	0.01	0.112	0.033
150	β_1	0.119	0.00656	0.405	0.0078
	β_2	0.159	0.00646	0.203	0.01
	β_3	0.789	0.012	0.123	0.022
200	β_1	0.116	0.00706	0.42	0.0058
	β_2	0.154	0.00208	0.228	0.00771
	β_3	0.811	0.00794	0.099	0.017
300	β_1	0.115	0.0073	0.427	0.00381
	β_2	0.153	0.0022	0.237	0.0050
	β_3	0.811	0.0079	0.099	0.011

Table 5: The AE, MSE,RAB and CIL of β_1, β_2 and β_3For BITL Distribution with True Parameter Values $(\beta_1, \beta_2, \beta_3) =$ (1.5, 1.5, 0.9)

N	Parameter	AE	MSE	RAB	CIL
50	β_1	0.554	0.896	0.631	0.067
	β_2	1.381	0.014	0.08	0.152
	β_3	0.938	0.0014	0.042	0.115
100	β_1	0.566	0.872	0.623	0.034
	β_2	1.378	0.015	0.081	0.077
	β_3	0.952	0.0027	0.058	0.058
150	β_1	0.53	0.94	0.646	0.021
	β_2	1.225	0.076	0.184	0.046
	β_3	0.973	0.00538	0.081	0.038
200	β_1	0.516	0.968	0.656	0.016
	β_2	1.202	0.089	0.198	0.034
	β_3	0.954	0.00293	0.06	0.028
300	β_1	0.534	0.934	0.644	0.011
	β_2	1.256	0.06	0.163	0.023
	β_3	0.949	0.00237	0.054	0.019

Table 6: The AE, MSE,RAB and CIL of β_1, β_2 and β_3For BITL Distribution with True Parameter Values $(\beta_1, \beta_2, \beta_3) =$ (1.5, 1.5, 1.5)

N	Parameter	AE	MSE	RAB	CIL
50	β_1	0.663	0.701	0.558	0.084
	β_2	1.617	0.014	0.078	0.188
	β_3	0.91	0.348	0.393	0.105
100	β_1	0.65	0.722	0.567	0.041
	β_2	1.471	0.000855	0.02	0.086
	β_3	0.83	0.449	0.447	0.048
150	β_1	0.66	0.706	0.56	0.028
	β_2	1.481	0.00038	0.013	0.057
	β_3	0.859	0.411	0.427	0.033
200	β_1	0.644	0.733	0.571	0.02
	β_2	1.427	0.00529	0.048	0.042
	β_3	0.837	0.439	0.442	0.024
300	β_1	0.635	0.748	0.577	0.013
	β_2	1.411	0.00793	0.059	0.027
	β_3	0.839	0.437	0.441	0.016

Table 7: The AE, MSE,RAB and CIL of β_1, β_2 and β_3For BITL Distribution with True Parameter Values $(\beta_1, \beta_2, \beta_3) =$ (1, 2, 1)

N	parameter	AE	MSE	RAB	CIL
50	β_1	0.444	0.309	0.556	0.066
	β_2	1.2	0.641	0.4	0.124
	β_3	0.901	0.0098	0.099	0.103
100	β_1	0.444	0.309	0.556	0.033
	β_2	1.179	0.674	0.41	0.061
	β_3	0.937	0.0041	0.063	0.053
150	β_1	0.424	0.332	0.576	0.021
	β_2	1.138	0.743	0.431	0.039
	β_3	0.943	0.0032	0.057	0.035
200	β_1	0.43	0.324	0.57	0.016
	β_2	1.161	0.704	0.419	0.03
	β_3	0.963	0.00135	0.037	0.027
300	β_1	0.411	0.347	0.589	0.01
	β_2	1.096	0.817	0.452	0.019
	β_3	0.98	0.000395	0.02	0.018