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A New Three-Parameter Generalized Lindley Probability Model With Engineering Applications

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ABSTRACT

During recent decades several lifetime distributions of one parameter, two-parameter and three-parameter have been proposed to model engineering data but due to either theoretical nature or the stochastic nature of the distribution, these available distributions are not suitable to model the data. In this paper, an attempt has been made to propose a new three-parameter generalized Lindley distribution which includes several one parameter and two-parameter continuous distributions including exponential, Lindley, Weibull, Power-Lindley and new two-parameter Lindley as particular cases. Descriptive statistical properties including hazard function, mean residual life function, moments and order statistics have been derived and discussed. Fisher's information matrix and confidence interval of the proposed distribution have been derived. Numerical simulation study has been carried out to know the consistency of maximum likelihood estimators. Maximum likelihood estimation for estimating parameters has been explained. Two under-dispersed real lifetime datasets from the field of engineering have been presented to test the goodness of fit of the proposed distribution over other one parameter, two-parameter and three-parameter lifetime distributions.

Keywords: Lindley distribution, Power Lindley distribution, Weibull distribution, Statistical properties, Maximum likelihood estimation, Fisher's information matrix, Applications.

1. Introduction

Several statistical distributions have been extensively used for the modelling and analysis of survival times (time to event) data, also known as reliability data in engineering and biomedical sciences. It has been observed that the datasets from engineering and biomedical sciences are either under-dispersed (mean greater than variance) or over-dispersed (mean less than variance). The distributions which are derived using the compound of positively skewed continuous distributions are in general over-dispersed and the examples includes gamma - Lindley distribution by Abdi et al. (2019), gamma-Shanker distribution by Ray and Shanker (2023), gamma-Sujatha distribution by Ray and Shanker (2024), some among others, On the other hand the distributions derived by compounding the Poisson distribution with positively skewed continuous distributions are, in general, over-dispersed and the examples includes negative binomial distribution, Poisson-Lindley distribution by Sankaran (1970), quasi Poisson-Sujatha distribution by Shanker and Shukla (2019), are some among others.

The probability density function (pdf) and the cumulative density function (cdf) of Lindley distribution introduced by Lindley (1958) are given by

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$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};\ \ x>0,\theta>0 \tag{1.1}$$

$$F\left(x;\theta\right)=1-\left[1+\frac{\theta x}{\theta+1}\right]e^{-\theta x}\quad ;x>0,\theta>0\ ; \tag{1.2}$$

Lindley distribution, being a convex combination of exponential and gamma distribution, gives better fit than exponential distribution and it is more flexible than the exponential distribution. The Lindley distribution exhibits the character of a rapid drop-off at zero and a long tail to the right, which means that the Lindley distribution is a suitable probability model for non-negative right skewed data having monotonically increasing hazard rate. The one of the interesting and useful advantages of the Lindley distribution is its ability to capture the effects of covariates or explanatory variables meaning that the Lindley distribution can be used in regression modelling where it can account for the influence of one or more predictor variables on the distribution of the response variable and due to this the Lindley distribution has a lot of applications in engineering, epidemiology and finance. Ghitany et al. (2008) have studied many interesting properties, estimation of parameter using both the method of moments and the method of maximum likelihood, and application of Lindley distribution. Nadarajah et al. (2011) proposed a two-parameter generalized Lindley distribution using exponentiated technique and discussed its various descriptive and inferential properties, estimation of parameters and applications. Bakouch et al. (2012) derived two-parameter extended Lindley distribution and discussed its statistical properties, estimation of parameters and applications. Recently, several two-parameter Lindley distributions have been introduced by different researchers which are presented in tables 1 and 2 along with its pdf and introducer.

Table 1: Two-parameter Lindley distributions with their pdf and introducers (year)

Name of the distributions	probability density function(pdf)	Introducers (years)
Two-parameter Lindley distribution-1 (TPLD-1)	$f\left(x;\theta,\alpha\right)=\frac{\theta^2}{\theta\alpha+1}\left(\alpha+x\right)e^{-\theta x};x>0,\theta>0,\theta\alpha>-1$	Shanker and Mishra (2013 a)
Two-parameter Lindley distribution-2 (TPLD-II)	$f\left(x;\theta,\alpha\right)=\frac{\theta^2}{\theta+\alpha}\left(1+\alpha x\right)e^{-\theta x};x>0,\theta>0,\alpha>0$	Shanker *et al* (2013)
Quasi Lindley distribution (QLD)	$f\left(x;\theta,\alpha\right)=\frac{\theta}{\alpha+1}\left(\alpha+\theta x\right)e^{-\theta x};x>0,\theta>0,\alpha>-1$	Shanker and Mishra (2013 b)
New Quasi Lindley distribution (NQLD)	$f\left(x;\theta,\alpha\right)=\frac{\theta^2}{\theta^2+\alpha}\left(\theta+\alpha x\right)e^{-\theta x};x>0,\theta>0,\alpha>0$	Shanker and Amanuel (2013)

The statistical properties, estimation of parameters and applications of these two-parameter Lindley distributions are available in the respective papers. Further, during a short span of time several three-parameter generalizations of Lindley distribution by different researchers have been suggested, which are presented in the following table-2 along with its pdf and introducers.

Table 2: Three-parameter Lindley distributions with their pdf and introducer (year)

Name of the distributions	probability density function(pdf)	Introducers (years)
A three-parameter Lindley distribution (ATPLD)	$f(x;\theta,\alpha,\beta)=\frac{\theta^2}{\theta\alpha+\beta}(\alpha+\beta x)e^{-\theta x}$ $;x>0,\theta>0,\beta>0,\theta\alpha+\beta>0$	Shanker et al (2017)
Generalized Lindley distribution (GLD)	$f(x;\theta,\alpha,\beta)=\frac{\theta^{\alpha+1}}{\theta+\beta}\frac{x^{\alpha-1}}{\Gamma(\alpha+1)}(\alpha+\beta x)e^{-\theta x}$ $;x>0,\theta>0,\alpha>0,\beta>0$	Zakerzadeh and Dolati (2009)
A three-parameter Generalized Lindley Distribution (TPGLD)	$f(x;\theta,\alpha,\beta)=\frac{\beta\theta^2}{\theta\alpha+1}(\alpha+x^{\beta})x^{\beta-1}e^{-\theta x^{\beta}}$ $;x>0,\theta>0,\alpha>0,\beta>0$	Nosakhare and Festus (2018)
New generalized Lindley distribution (NGLD)	$f(x;\theta,\alpha,\beta)=\frac{\theta^{\alpha+1}\Gamma(\beta)x^{\alpha-1}+\theta^{\beta}\Gamma(\alpha)x^{\beta-1}}{(\theta+1)\Gamma(\alpha)\Gamma(\beta)}$ $;x>0,\theta>0,\alpha>0,\beta>0$	Ibrahim et al (2013)

The statistical properties, estimation of parameters and applications of these three-parameter Lindley distributions are available in the respective papers.

Recently, Shanker and Rahman (2020) introduced a new two-parameter Lindley distribution defined (NTPLD) by its pdf and cdf

$$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}(\alpha+x^{\alpha})e^{-\theta x}}{\alpha\theta^{\alpha}+\Gamma(\alpha+1)};x>0,\theta>0,\alpha>0 \tag{1.3}$$

$$F(x;\theta,\alpha)=\frac{\alpha\theta^{\alpha}(1-e^{-\theta x})+\gamma(\alpha+1,\theta x)}{\alpha\theta^{\alpha}+\Gamma(\alpha+1)};x>0,\theta>0,\alpha>0 \tag{1.4}$$

where α is the shape parameter and θ is the scale parameter and $\gamma(\alpha, z) = \int_0^z e^{-t} t^{\alpha-1} dt$ is the lower incomplete gamma function. It can be easily verified that Lindley distribution and exponential distribution are the particular cases of NTPLD for $\alpha = 1$ and $\alpha = 0$ respectively

The main reason behind proposing a new three-parameter generalized Lindley distribution is that it contains several one parameter and two-parameter classical lifetime distributions available in statistics literature and it is expected that the proposed distribution would give much better fit than these classical distributions. Statistical properties including hazard function, mean residual life function, moments and order statistics have been discussed. Maximum likelihood estimation has been discussed for estimating parameters of the proposed distribution. Fisher's information matrix with confidence interval has also been derived. To know the consistency of maximum likelihood estimators of parameters, a numerical simulation study has been presented. At last, goodness of fit of the proposed distribution and its comparative fit with other well-known one parameter, two-parameter and three-parameter lifetime distributions are presented with two under-dispersed real lifetime datasets from the engineering field.

2. A NEW THREE-PARAMETER GENERALIZED LINDLEY DISTRIBUTION

Taking the power transformation $x = y^{1/\beta}$ and following the approach of obtaining the pdf of power-Lindley distribution by Ghitany et al. (2013), the pdf and the cdf of the new three-parameter generalized Lindley distribution (NTPGLD) can be expressed as

$$f(x;\theta,\alpha,\beta) = \frac{\beta\,\theta^{\alpha+1} x^{\beta-1}\left(\alpha + x^{\alpha\beta}\right) e^{-\theta x^{\beta}}}{\alpha\theta^{\alpha} + \Gamma(\alpha+1)}; x > 0, (\theta,\alpha,\beta) > 0 \tag{2.1}$$

$$F(x;\theta,\alpha,\beta) = \frac{\alpha\theta^{\alpha}\left(1 - e^{-\theta x^{\beta}}\right) + \gamma\left(\alpha+1, \theta x^{\beta}\right)}{\alpha\theta^{\alpha} + \Gamma(\alpha+1)}; x > 0, (\theta,\alpha,\beta) > 0 \tag{2.2}$$

where α and β are the shape parameters and θ is the scale parameter, $\gamma(\alpha, z) = \int_0^z e^{-t} t^{\alpha-1} dt$ is the lower incomplete gamma function, $\Gamma(\alpha, z) = \int_z^\infty e^{-t} t^{\alpha-1} dt$ is the upper incomplete gamma function and $\Gamma(\alpha+1)$ is the complete gamma function. The particular distributions of NTPGLD for different values of parameters are presented in table 3.

Table 3: Particular distributions of NTPGLD

Parameter values	Distribution	Pdf of the distribution	Introducers (year)
$\alpha=0,\beta=1$	**Exponential**	$f\left(x,\theta\right)=\theta e^{-\theta x};x>0,\theta>0$	
$\alpha=1,\beta=1$	**Lindley**	$f\left(x,\theta\right)=\dfrac{\theta^2}{\theta+1}(1+x)e^{-\theta x};x>0,\theta>0$	**Lindley (1958)**
$\alpha=0$	**Weibull**	$f\left(x;\theta,\beta\right)=\theta\,\beta\,x^{\beta-1}e^{-\theta x^{\beta}};x>0,\theta>0,\beta>0$	**Weibull (1951)**
$\alpha=1$	**Power Lindley**	$f\left(x;\theta,\beta\right)=\dfrac{\beta\,\theta^2x^{\beta-1}\left(1+x^{\beta}\right)e^{-\theta x^{\beta}}}{\theta+1};x>0,\theta>0,\beta>0$	**Ghitany et al (2013)**
$\beta=1$	**NTPLD**	$f\left(x;\theta,\alpha\right)=\dfrac{\theta^{\alpha+1}\left(\alpha+x^{\alpha}\right)e^{-\theta x}}{\alpha\theta^{\alpha}+\Gamma\left(\alpha+1\right)};x>0,\alpha>0,\theta>0$	**Shanker and Rahman (2020)**

The NTPGLD has a proper density function since

$$\lim_{x\to\infty}F\left(x;\theta,\alpha,\beta\right)=1 \qquad \text{and} \qquad \lim_{x\to-\infty}F\left(x;\theta,\alpha,\beta\right)=0$$

Graphs of the pdf and cdf of NTPGLD has been shown in figures 1 and 2 respectively for varying values of the parameters θ,α and β. From the figure 1, it is clear that for different values of the parameters, the proposed distribution has unimodal, bimodal, positively skewed, negatively skewed, leptokurtic, mesokurtic and platykurtic natures. Further, it is also clear that NTPGLD exhibits a rapid-off at zero and a long tail to the right. This means that it would be a suitable probability model for non-negative right skewed dataset having under-dispersion and monotonic increasing hazard rate.

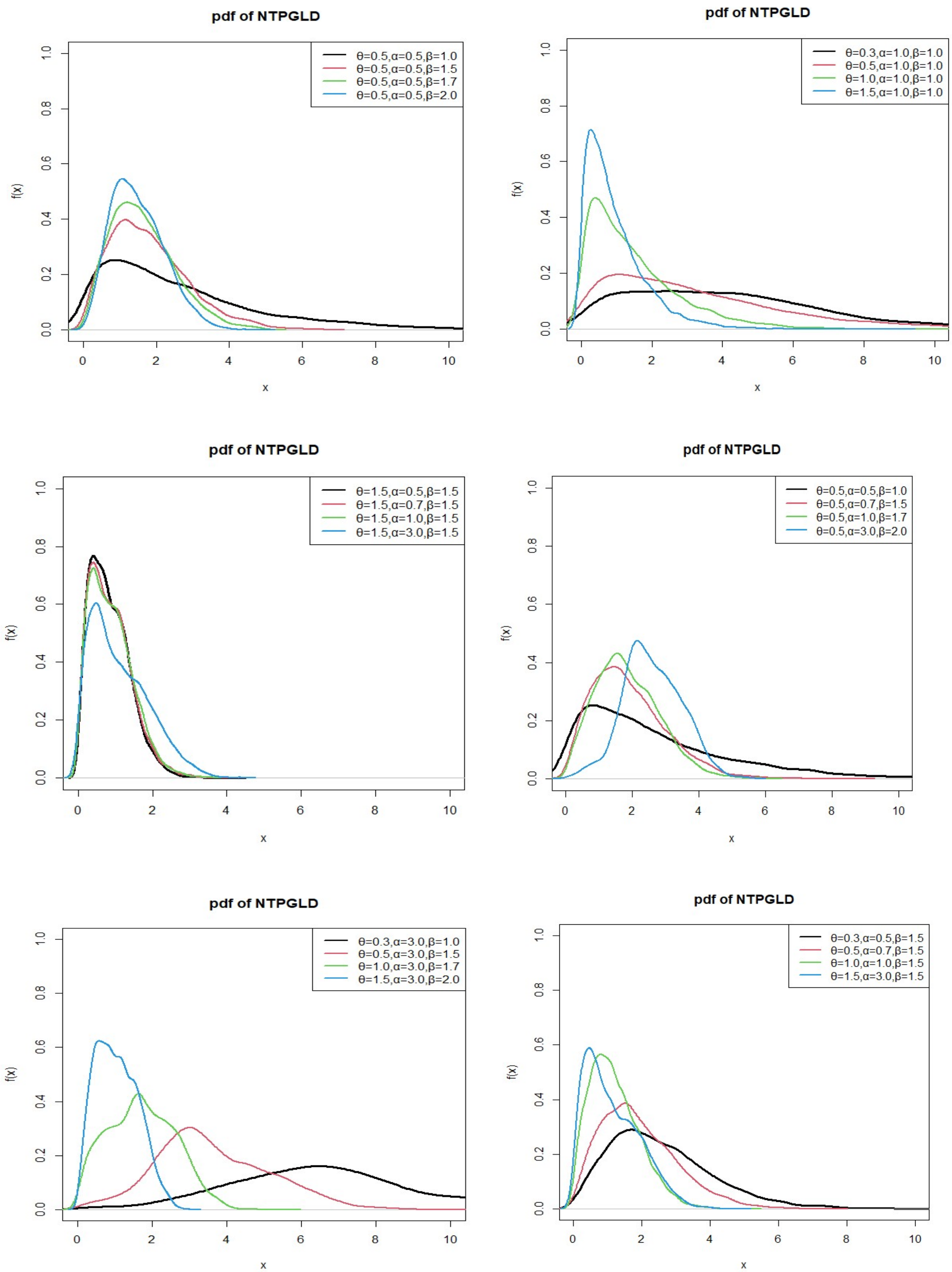


Figure 1: pdf plot of NTPGLD for varying values of parameters θ, α and β

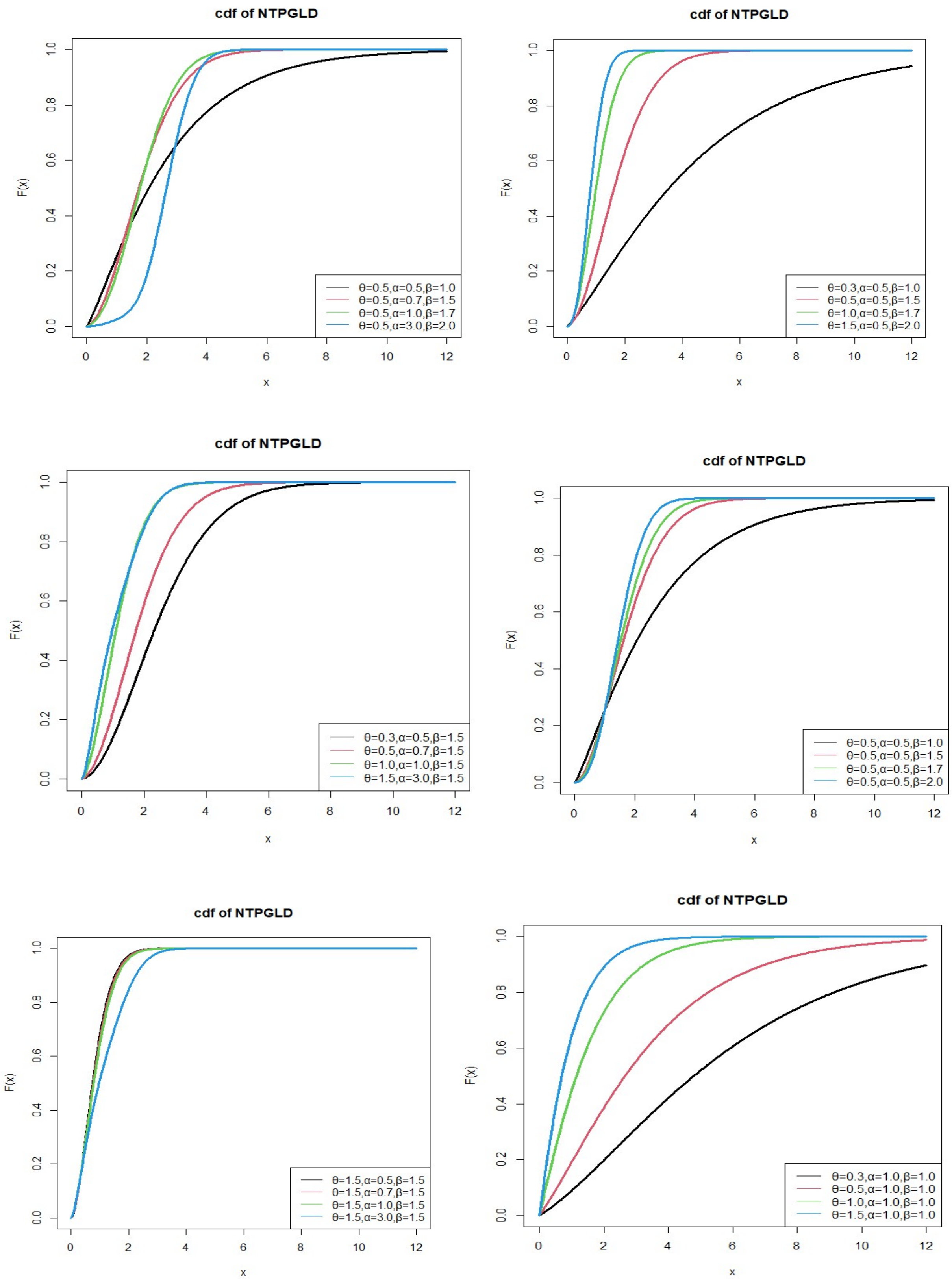


Figure 2: cdf plot of NTPGLD for varying values of parameters θ, α and β

3. STATISTICAL PROPERTIES

In this section, statistical properties including asymptotic behaviour, reliability analysis and mean residual life function of NTPGLD has been studied.

3.1. Asymptotic Behaviour

The asymptotic behaviour of NTPGLD for $x \to 0$ and $x \to \infty$ are

$$\lim_{x\to 0} f\left(x;\theta,\alpha,\beta\right)=\lim_{x\to 0}\left[\frac{\beta\,\theta^{\alpha+1}x^{\beta-1}\left(\alpha+x^{\alpha\beta}\right)e^{-\theta x^{\beta}}}{\alpha\theta^{\alpha}+\Gamma\left(\alpha+1\right)}\right]=0$$

$$\lim_{x\to \infty} f\left(x;\theta,\alpha,\beta\right)=\lim_{x\to \infty}\left[\frac{\beta\,\theta^{\alpha+1}x^{\beta-1}\left(\alpha+x^{\alpha\beta}\right)e^{-\theta x^{\beta}}}{\alpha\theta^{\alpha}+\Gamma\left(\alpha+1\right)}\right]=0.$$

These results confirm that the proposed distribution has a mode.

3.2. Reliability Analysis

The survival function (or the reliability function) is the probability that a subject survives longer than the expected time. The survival function of NTPGLD is given by

$$S\left(x;\theta,\alpha,\beta\right)=1-F\left(x;\theta,\alpha,\beta\right)=\frac{\alpha\theta^{\alpha}e^{-\theta x^{\beta}}-\gamma\left(\alpha+1,\theta x^{\beta}\right)+\Gamma\left(\alpha+1\right)}{\alpha\theta^{\alpha}+\Gamma\left(\alpha+1\right)};x>0,\left(\theta,\alpha,\beta\right)>0$$

The hazard function (also known as the hazard rate, instantaneous failure rate or force of mortality) is the probability to measure the instant death rate of a subject. Suppose X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. The hazard function of X is defined as

$$h(x)=\lim_{\Delta x\to 0}\frac{P(X<x+\Delta x/X>x)}{\Delta x}=\frac{f(x)}{1-F(x)}$$

The corresponding $h(x)$ of NTPGLD can be obtained as

$$h\left(x;\theta,\alpha,\beta\right)=\frac{\beta\,\theta^{\alpha+1}x^{\beta-1}\left(\alpha+x^{\alpha\beta}\right)e^{-\theta x^{\beta}}}{\alpha\theta^{\alpha}e^{-\theta x^{\beta}}-\gamma\left(\alpha+1,\theta x^{\beta}\right)+\Gamma\left(\alpha+1\right)};x>0,\left(\theta,\alpha,\beta\right)>0$$

The natures of survival function and the hazard function of NTPGLD for varying values of parameters are shown graphically in figures 3 and 4, respectively.
From the figure 3, it is clear that for all values of the parameters, survival function is monotonically decreasing over the time. From figure 4, it is clear that for any values of the parameters, hazard function has increasing natures.

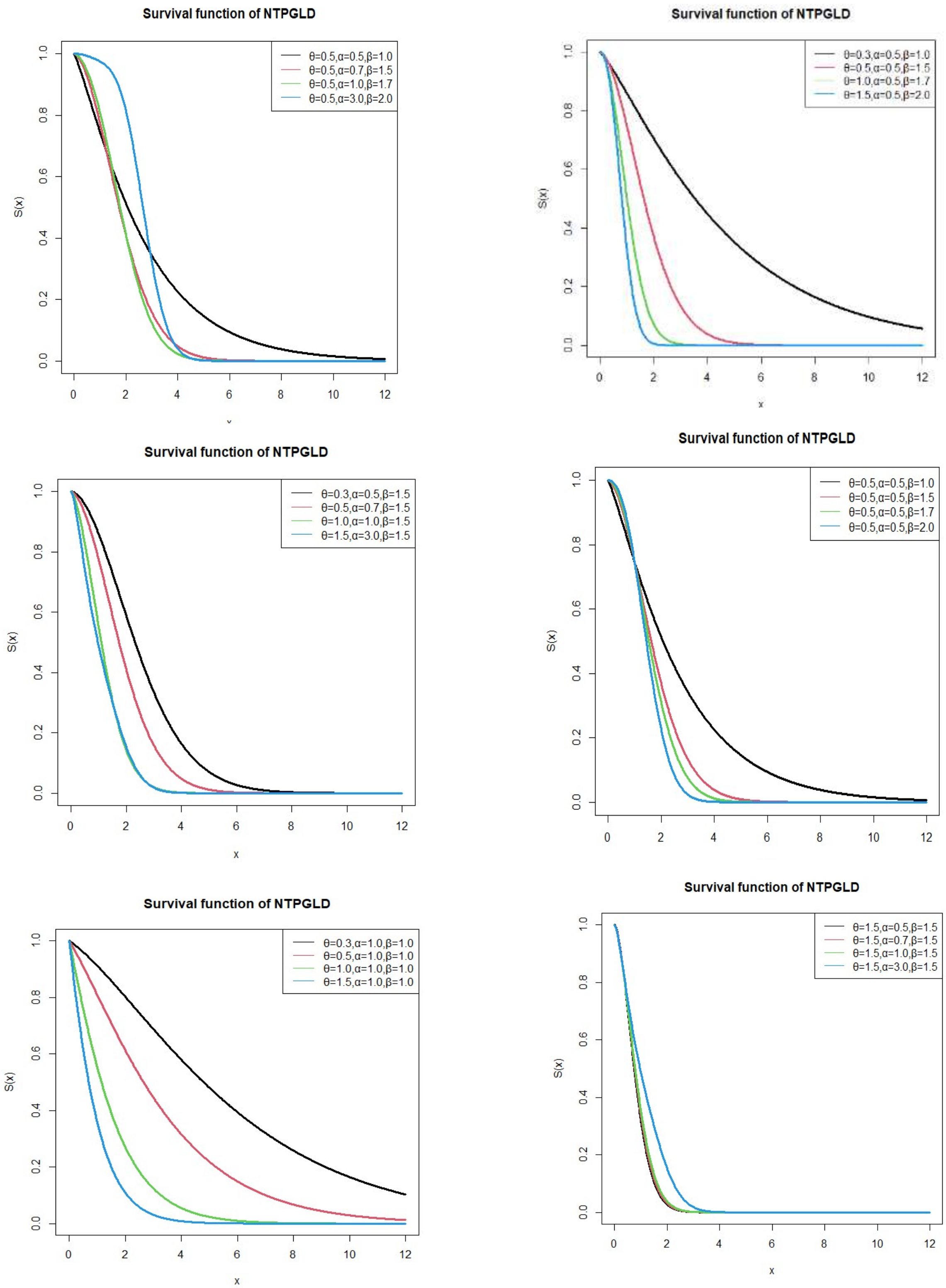


Figure 3: Survival function of NTPGLD for varying values of parameters θ, α and β.

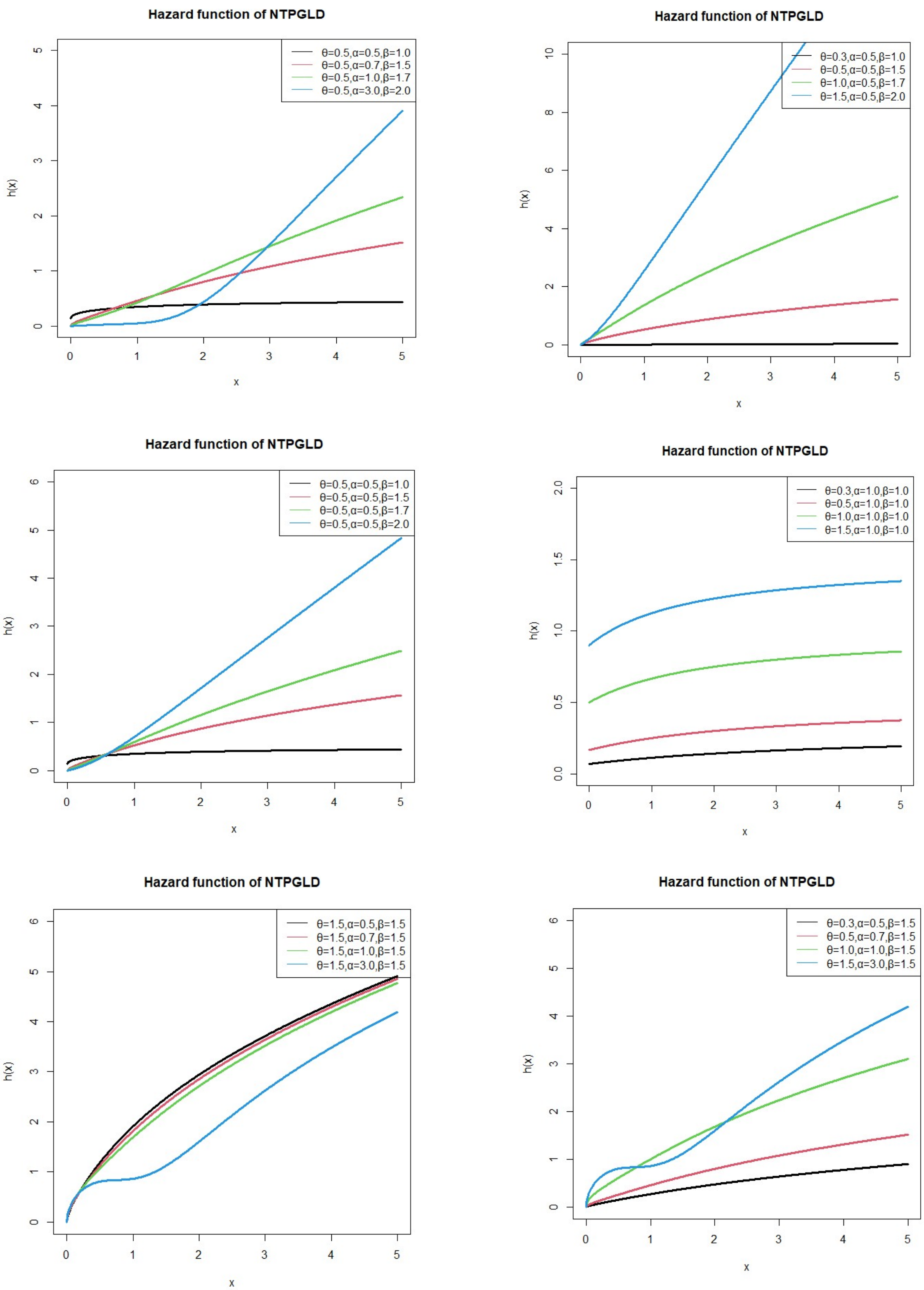


Figure 4: Hazard function of NTPGLD for varying values of parameters θ, α and β.

3.3. Mean Residual Life Function

The mean residual life function of the NTPGLD can be obtained as

$$E\left[X-x\mid X>x\right]=\frac{1}{1-F\left(x;\theta,\alpha,\beta\right)}\int_x^\infty\left[1-F\left(t;\theta,\alpha,\beta\right)\right]dt$$

$$=\frac{1}{S\left(x;\theta,\alpha,\beta\right)}\int_x^\infty t\,f\left(t;\theta,\alpha,\beta\right)dt-x$$

$$=\frac{\alpha\theta^{\alpha}\Gamma\left(\frac{1}{\beta}+1,\theta x^{\beta}\right)+\Gamma\left(\alpha+\frac{1}{\beta}+1,\theta x^{\beta}\right)}{\theta^{\frac{1}{\beta}}\left[\alpha\theta^{\alpha}e^{-\theta x^{\beta}}-\gamma\left(\alpha+1,\theta x^{\beta}\right)+\Gamma\left(\alpha+1\right)\right]}-x$$

,

where $\gamma\left(\alpha,z\right)=\int_0^z e^{-y}y^{\alpha-1}dy;\,y>0,\alpha>0,z\geq 0$ is the lower incomplete gamma function.

The behaviours of the mean residual life function of NTPGLD for varying values of parameters are shown in the figure 5

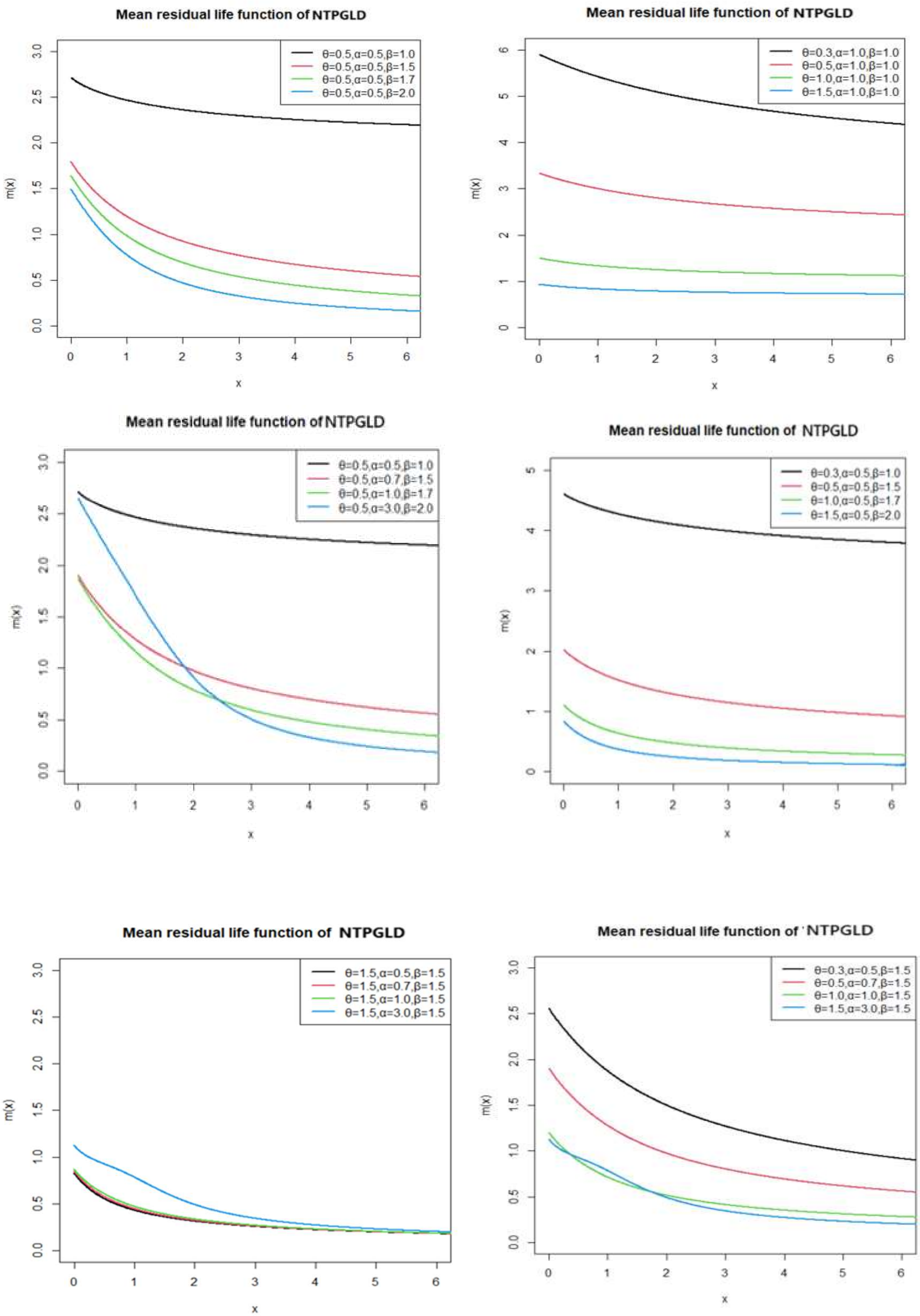


Figure 5: Mean residual life function of NTPGLD for varying values of parameters θ, α and β.

From the figure 5, it is clear that for all values of the parameters mean residual life function of NTPGLD has decreasing natures.

3.4. Moments and related measures

Theorem: If μ_r' be the r^{th} moment about origin of NTPGLD then

$$\mu_r' = \frac{\alpha\,\theta^{\alpha}\Gamma\left(\frac{r}{\beta}+1\right)+\theta\,\Gamma\left(\alpha+\frac{r}{\beta}+1\right)}{\theta^{\frac{r}{\beta}}\left[\alpha\,\theta^{\alpha}+\Gamma(\alpha+1)\right]};r=1,2,3,... \qquad (3.4.1)$$

Proof: We have

$$\mu_r' = E\left(X^r\right) = \frac{\beta\,\theta^{\alpha+1}}{\alpha\theta^{\alpha}+\Gamma(\alpha+1)}\int_0^{\infty} x^r x^{\beta-1}\left(\alpha+x^{\alpha\beta}\right)e^{-\theta x^{\beta}}\,dx$$

$$= \frac{\beta\,\theta^{\alpha+1}}{\alpha\theta^{\alpha}+\Gamma(\alpha+1)}\left[\alpha\int_0^{\infty} e^{-\theta x^{\beta}}x^{\beta+r-1}dx+\int_0^{\infty} e^{-\theta x^{\beta}}x^{\alpha\beta+\beta+r-1}dx\right].$$

Taking $u=\theta\,x^{\beta}$ which gives $x=\left(\frac{u}{\theta}\right)^{1/\beta}$ and $dx=\frac{1}{\beta}\left(\frac{u}{\theta}\right)^{\frac{1-\beta}{\beta}}\frac{1}{\theta}du$, and after simplification, we get

$$\mu_r' = \frac{\theta^{\alpha+1}}{\alpha\theta^{\alpha}+\Gamma(\alpha+1)}\left[\frac{\alpha}{\theta^{\frac{r}{\beta}+1}}\int_0^{\infty} e^{-u}u^{\frac{r}{\beta}+1-1}du+\frac{1}{\theta^{\frac{r}{\beta}+\alpha}}\int_0^{\infty} e^{-u}x^{\alpha+\frac{r}{\beta}+1-1}du\right]$$

$$= \frac{\theta^{\alpha+1}}{\alpha\theta^{\alpha}+\Gamma(\alpha+1)}\left[\frac{\alpha\,\Gamma\left(\frac{r}{\beta}+1\right)}{\theta^{\frac{r}{\beta}+1}}+\frac{\Gamma\left(\alpha+\frac{r}{\beta}+1\right)}{\theta^{\frac{r}{\beta}+\alpha}}\right]$$

$$= \frac{\alpha\,\theta^{\alpha}\Gamma\left(\frac{r}{\beta}+1\right)+\theta\,\Gamma\left(\alpha+\frac{r}{\beta}+1\right)}{\theta^{\frac{r}{\beta}}\left[\alpha\,\theta^{\alpha}+\Gamma(\alpha+1)\right]};r=1,2,3,...$$

For $\beta =`1$, it reduces to μ_r' of NTPLD given by

$$\mu_r' = \frac{\alpha\,\theta^{\alpha}\,\Gamma(r+1)+\theta\,\Gamma(\alpha+r+1)}{\theta^{\frac{r}{\beta}}\left[\alpha\,\theta^{\alpha}+\Gamma(\alpha+1)\right]};r=1,2,3,...$$

Taking $r=1,2,3$ and 4 in (3.4.1), the first four moments about origin are obtained as

$$\mu_1' = \frac{\alpha\,\theta^{\alpha}\,\Gamma\left(\frac{1}{\beta}+1\right)+\theta\,\Gamma\left(\alpha+\frac{1}{\beta}+1\right)}{\theta^{\frac{1}{\beta}}\left[\alpha\,\theta^{\alpha}+\Gamma(\alpha+1)\right]}$$

$$\mu_2' = \frac{\alpha\theta^\alpha\Gamma\left(\frac{2}{\beta}+1\right)+\theta\Gamma\left(\alpha+\frac{2}{\beta}+1\right)}{\theta^{\frac{2}{\beta}}\left[\alpha\theta^\alpha+\Gamma(\alpha+1)\right]}$$

$$\mu_3' = \frac{\alpha\theta^\alpha\Gamma\left(\frac{3}{\beta}+1\right)+\theta\Gamma\left(\alpha+\frac{3}{\beta}+1\right)}{\theta^{\frac{3}{\beta}}\left[\alpha\theta^\alpha+\Gamma(\alpha+1)\right]}$$

$$\mu_4' = \frac{\alpha\theta^\alpha\Gamma\left(\frac{4}{\beta}+1\right)+\theta\Gamma\left(\alpha+\frac{4}{\beta}+1\right)}{\theta^{\frac{4}{\beta}}\left[\alpha\theta^\alpha+\Gamma(\alpha+1)\right]}$$

Thus, the variance of NTPGLD can be expressed as

$$\mu_2 = \mu_2' - \left(\mu_1'\right)^2 = \left(\frac{\alpha\theta^\alpha\Gamma\left(\frac{2}{\beta}+1\right)+\theta\Gamma\left(\alpha+\frac{2}{\beta}+1\right)}{\theta^{\frac{2}{\beta}}\left[\alpha\theta^\alpha+\Gamma(\alpha+1)\right]}\right)-\left(\frac{\alpha\theta^\alpha\Gamma\left(\frac{1}{\beta}+1\right)+\theta\Gamma\left(\alpha+\frac{1}{\beta}+1\right)}{\theta^{\frac{1}{\beta}}\left[\alpha\theta^\alpha+\Gamma(\alpha+1)\right]}\right)^2$$

Similarly, using the relationship between central moments and raw moments, other central moments can be obtained and hence other statistical constants including coefficient of variation, skewness, kurtosis and index of dispersion.

The conditional mean and variance of NTPGLD can be obtained as

$$E\left[X \mid X > x\right] = \frac{1}{S(x;\theta,\alpha,\beta)}\int_x^\infty t\, f(t;\theta,\alpha,\beta)\,dt = \frac{\alpha\theta^\alpha\Gamma\left(\frac{1}{\beta}+1,\theta x^\beta\right)+\Gamma\left(\alpha+\frac{1}{\beta}+1,\theta x^\beta\right)}{\theta^{\frac{1}{\beta}}\left[\alpha\theta^\alpha e^{-\theta x^\beta}-\gamma\left(\alpha+1,\theta x^\beta\right)+\Gamma(\alpha+1)\right]}$$

$$E\left[X^2 \mid X > x\right] = \frac{1}{S(x;\theta,\alpha,\beta)}\int_x^\infty t^2 f(t;\theta,\alpha,\beta)\,dt = \frac{\alpha\theta^\alpha\Gamma\left(\frac{2}{\beta}+1,\theta x^\beta\right)+\Gamma\left(\alpha+\frac{2}{\beta}+1,\theta x^\beta\right)}{\theta^{\frac{2}{\beta}}\left[\alpha\theta^\alpha e^{-\theta x^\beta}-\gamma\left(\alpha+1,\theta x^\beta\right)+\Gamma(\alpha+1)\right]}$$

Now, $Var\left[X \mid X > x\right] = E\left[X^2 \mid X > x\right] - \left[E\left[X \mid X > x\right]\right]^2$

$$= \left[\frac{\alpha\theta^\alpha\Gamma\left(\frac{2}{\beta}+1,\theta x^\beta\right)+\Gamma\left(\alpha+\frac{2}{\beta}+1,\theta x^\beta\right)}{\theta^{\frac{2}{\beta}}\left[\alpha\theta^\alpha e^{-\theta x^\beta}-\gamma\left(\alpha+1,\theta x^\beta\right)+\Gamma(\alpha+1)\right]}\right]-\left[\frac{\alpha\theta^\alpha\Gamma\left(\frac{1}{\beta}+1,\theta x^\beta\right)+\Gamma\left(\alpha+\frac{1}{\beta}+1,\theta x^\beta\right)}{\theta^{\frac{1}{\beta}}\left[\alpha\theta^\alpha e^{-\theta x^\beta}-\gamma\left(\alpha+1,\theta x^\beta\right)+\Gamma(\alpha+1)\right]}\right]^2$$

4. DISTRIBUTION OF ORDER STATISTICS

Let $x_1, x_2, ..., x_n$ be the random samples from NTPGLD (θ, α, β). The pdf of i^{th} order statistics is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} f_X(x)\left[F_X(x)\right]^{i-1}\left[1-F_X(x)\right]^{n-i}$$

The pdf of i^{th} order statistics $X_{(i)}$ of NTGPLD is given by

$$f_{i:n}(x) = \frac{n!}{(i-1)!(n-i)!} \frac{\beta\,\theta^{\alpha+1} x^{\beta-1}\left(\alpha + x^{\alpha\beta}\right) e^{-\theta x^{\beta}}}{\alpha\theta^{\alpha} + \Gamma(\alpha+1)} \left[\frac{\alpha\theta^{\alpha}\left(1-e^{-\theta x^{\beta}}\right) + \gamma\left(\alpha+1, \theta x^{\beta}\right)}{\alpha\theta^{\alpha} + \Gamma(\alpha+1)}\right]^{i-1}$$

$$\times\left[1 - \frac{\alpha\theta^{\alpha}\left(1-e^{-\theta x^{\beta}}\right) + \gamma\left(\alpha+1, \theta x^{\beta}\right)}{\alpha\theta^{\alpha} + \Gamma(\alpha+1)}\right]^{n-i}$$

The pdf of the first order statistic $X_{(1)}$ can be expressed as

$$f_{1:n}(x) = n\frac{\beta\,\theta^{\alpha+1} x^{\beta-1}\left(\alpha + x^{\alpha\beta}\right) e^{-\theta x^{\beta}}}{\alpha\theta^{\alpha} + \Gamma(\alpha+1)} \left[1 - \frac{\alpha\theta^{\alpha}\left(1-e^{-\theta x^{\beta}}\right) + \gamma\left(\alpha+1, \theta x^{\beta}\right)}{\alpha\theta^{\alpha} + \Gamma(\alpha+1)}\right]^{n-1}$$

The pdf of the largest order statistic $X_{(n)}$ can be expressed as

$$f_{n:n}(x) = n\frac{\beta\,\theta^{\alpha+1} x^{\beta-1}\left(\alpha + x^{\alpha\beta}\right) e^{-\theta x^{\beta}}}{\alpha\theta^{\alpha} + \Gamma(\alpha+1)} \left[\frac{\alpha\theta^{\alpha}\left(1-e^{-\theta x^{\beta}}\right) + \gamma\left(\alpha+1, \theta x^{\beta}\right)}{\alpha\theta^{\alpha} + \Gamma(\alpha+1)}\right]^{n-1}$$

5. MAXIMUM LIKELIHOOD ESTIMATION

Let $x_1, x_2, ..., x_n$ be a random sample of size n from a NTPGLD (θ, α, β). The log-likelihood function L can be expressed as

$$L = \sum_{i=1}^{n} \ln f\left(x_i; \theta, \alpha, \beta\right) = n\left[\ln\beta + (\alpha+1)\ln\theta - \ln\left\{\alpha\theta^{\alpha} + \Gamma(\alpha+1)\right\}\right]$$
$$+(\beta-1)\sum_{i=1}^{n}\ln x_i + \sum_{i=1}^{n}\ln\left(\alpha + x_i^{\alpha\beta}\right) - \theta\sum_{i=1}^{n} x_i^{\beta}$$

The maximum likelihood estimates (MLE) $\left(\hat{\theta}, \hat{\alpha}, \hat{\beta}\right)$ of parameters (θ, α, β) of NTPGLD are the solutions of the following log- likelihood equations

$$\frac{\partial L}{\partial\theta} = \frac{n(\alpha+1)}{\theta} - \frac{n\,\alpha^{2}\theta^{\alpha-1}}{\alpha\theta^{\alpha} + \Gamma(\alpha+1)} - \sum_{i=1}^{n} x_i^{\beta} = 0$$

$$\frac{\partial L}{\partial \alpha}=n\ln\theta-\frac{n\left\{\theta^{\alpha}+\alpha^{2}\theta^{\alpha-1}+\Psi\left(\alpha+1\right)\right\}}{\alpha\theta^{\alpha}+\Gamma\left(\alpha+1\right)}+\sum_{i=1}^{n}\frac{1+\beta x_i^{\beta-1}\ln x_i}{\alpha+x_i^{\alpha\beta}}=0$$

$$\frac{\partial L}{\partial \beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}\frac{\alpha x_i^{\alpha\beta}\ln x_i}{\alpha+x_i^{\alpha\beta}}-\theta\sum_{i=1}^{n}x_i^{\beta}\ln x_i=0,$$

Where, $\Psi(\alpha+1)=\frac{\partial}{\partial\alpha}\Gamma(\alpha+1)$ is the digamma function.

These log-likelihood equations seem difficult to solve analytically and thus requires statistical software available in R to find the solutions.

The observed (3×3) matrix of NTPGLD can be expressed as

$$\begin{pmatrix}\hat{\theta}\\ \hat{\alpha}\\ \hat{\beta}\end{pmatrix}\sim\left[\begin{pmatrix}\theta\\ \alpha\\ \beta\end{pmatrix},\begin{pmatrix}\frac{\partial^2 L}{\partial\theta^2} & \frac{\partial^2 L}{\partial\theta\,\partial\alpha} & \frac{\partial^2 L}{\partial\theta\,\partial\beta}\\ \frac{\partial^2 L}{\partial\alpha\,\partial\theta} & \frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\,\partial\beta}\\ \frac{\partial^2 L}{\partial\beta\,\partial\theta} & \frac{\partial^2 L}{\partial\beta\,\partial\alpha} & \frac{\partial^2 L}{\partial\beta^2}\end{pmatrix}\right]$$

The inverse of the information matrix results in the well-known variance-covariance matrix. The 3×3 approximate Fisher's information matrix corresponding to the above observed information matrix is given by

$$I^{-1}=-E\begin{bmatrix}\frac{\partial^2 L}{\partial\theta^2} & \frac{\partial^2 L}{\partial\theta\,\partial\alpha} & \frac{\partial^2 L}{\partial\theta\,\partial\beta}\\ \frac{\partial^2 L}{\partial\alpha\,\partial\theta} & \frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\,\partial\beta}\\ \frac{\partial^2 L}{\partial\beta\,\partial\theta} & \frac{\partial^2 L}{\partial\beta\,\partial\alpha} & \frac{\partial^2 L}{\partial\beta^2}\end{bmatrix}$$

The solution of the Fisher's information matrix will yield asymptotic variance and covariance of the ML estimators for $(\hat{\theta},\hat{\alpha},\hat{\beta})$. The approximate $100(1-\alpha)\%$ confidence intervals for (θ,α,β) respectively are $\hat{\theta}\pm Z_{\frac{\alpha}{2}}\frac{\sigma_{\theta\theta}}{n}$, $\hat{\alpha}\pm Z_{\frac{\alpha}{2}}\frac{\sigma_{\alpha\alpha}}{n}$ and $\hat{\beta}\pm Z_{\frac{\alpha}{2}}\frac{\sigma_{\beta\beta}}{n}$ where Z_{α} is the upper $100\alpha^{th}$ percentile of the standard normal distribution.

6. NUMERICAL SIMULATION STUDY

To assess the effectiveness of maximum likelihood estimators for NTPGLD, a simulation study has been conducted. The investigation involved examining mean estimates, biases (B), mean square errors (MSEs), and variances of the maximum likelihood estimates (MLEs) for NTPGLD, utilizing the specified formulas.

$$Mean = \frac{1}{n}\sum_{i=1}^{n}\hat{H}_i,\ B = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{H}_i - H\right),\ MSE = \frac{1}{n}\sum_{i=1}^{n}\left(\hat{H}_i - H\right)^2,\ Variance = MSE - B^2$$

where $H = (\theta,\alpha,\beta)$ and $\hat{H} = (\hat{\theta},\hat{\alpha},\hat{\beta})$.

The simulation results for various parameter values of NTPGLD are outlined in tables 4 and 5, respectively. The acceptance-rejection method of simulation has been employed to generate data. This method is commonly used in simulation studies to produce random samples from a target distribution. The acceptance-rejection method of simulation for generating random samples from the NTPGLD involves the following steps:

a. Generate a random variable Y from exp (θ) distribution

b. Generates U from Uniform$(0,1)$ distribution

c. If $U \le \frac{f(y)}{M\,g(y)}$, then set $X = Y$ ("accept the sample"); otherwise ("reject the sample") and if reject then repeat the process: step (a-c) until getting the required samples. Where M is a constant.

Here the sample sizes $n = 20, 40, 60, 80, 100$ and the parameter values taken are $\theta = 0.5, \alpha = 1.7, \beta = 1.5$ and $\theta = 0.9, \alpha = 2.6, \beta = 1.4$ and each sample size has been replicated 10000 times

The biases, MSEs, and variances of the MLEs of the parameters are decreasing for increasing sample size as evident in Tables 4 and 5. This supports the first-order asymptotic theory of MLEs.

Table 4. Descriptive constants of NTPGLD for $\theta = 0.5, \alpha = 1.7, \beta = 1.5$

Parameters	n	**Mean**	**Bias**	**MSE**	**Variance**
$\theta = 0.5$	20	0.48305	-0.01694	0.00058	0.00029
	40	0.48621	-0.01378	0.00041	0.00022
	60	0.49028	-0.00971	0.00032	0.00022
	80	0.49353	-0.00646	0.00026	0.00021
	100	0.49503	-0.00496	0.00021	0.00019

$\alpha = 1.7$	20	1.68947	-0.01052	0.00043	0.00032
	40	1.69442	-0.00557	0.00030	0.00027
	60	1.69628	-0.00371	0.00024	0.00022
	80	1.69789	-0.00210	0.00021	0.00020
	100	1.69795	-0.00204	0.00019	0.00019
$\beta = 1.5$	20	1.48951	-0.01048	0.00037	0.00026
	40	1.49473	-0.00526	0.00029	0.00026
	60	1.49681	-0.00318	0.00022	0.00021
	80	1.49725	-0.00274	0.00019	0.00018
	100	1.49815	-0.00184	0.00017	0.00016

Table 5. Descriptive constants of NTPGLD for $\theta = 0.9, \alpha = 2.6, \beta = 1.4$

Parameters	n	Mean	Bias	MSE	Variance
$\theta = 0.9$	20	0.90306	0.00306	0.00202	0.00201
	40	0.90304	0.00304	0.00195	0.00194
	60	0.90180	0.00180	0.00174	0.00174
	80	0.90150	0.00150	0.00169	0.00168
	100	0.90126	0.00126	0.00124	0.00124
$\alpha = 2.6$	20	2.64101	0.04101	0.01336	0.01168
	40	2.63309	0.03309	0.01251	0.01141
	60	2.61724	0.01724	0.01080	0.01050
	80	2.61112	0.01112	0.00915	0.00903
	100	2.60499	0.00499	0.00751	0.00749
$\beta = 1.4$	20	1.37591	-0.02408	0.00138	0.00080
	40	1.38588	-0.01411	0.00090	0.00070
	60	1.39447	-0.00395	0.00089	0.00088
	80	1.39604	-0.00328	0.00083	0.00081
	100	1.396711	-0.00092	0.00048	0.00047

7. GOODNESS OF FIT

A new three-parameter generalized Lindley distribution (NTPGLD) has been fitted to two under-dispersed real lifetime datasets from engineering. We present the goodness of fit of NTPGLD and compared its goodness of fit with three-parameter generalized Lindley distribution(TPGLD), a three-parameter Lindley distribution (ATPLD), new generalized Lindley distribution (NGLD), generalized Lindley distribution (GLD), Weibull distribution (WD), two parameter Lindley-1 (TPLD-1), two parameter Lindley-2 (TPLD-2), Quasi-Lindley distribution (QLD), new Quasi-Lindley distribution (NQLD), Lindley distribution (LD) and exponential distribution (ED). The following dataset has been considered. The descriptive summary of the datasets 1 and 2 are given in table 6. The descriptive summary of the datasets show that the datasets are 1 and 2 are under-dispesed (mean greater than variance).

Dataset-1: The following symmetric data, discussed by Murthy et al (2004), studies the failure times of wind shields and the values are as follows

0.04, 0.3, 0.31, 0.557, 0.943, 1.07, 1.124, 1.248, 1.281, 1.281, 1.303, 1.432, 1.48, 1.51, 1.51, 1.568, 1.615, 1.619, 1.652, 1.652, 1.757, 1.795, 1.866, 1.876, 1.899, 1.911, 1.912, 1.9141, 0.981, 2.010, 2.038, 2.085, 2.089, 2.097, 2.135, 2.154, 2.190, 2.194, 2.223, 2.224, 2.23, 2.3, 2.324, 2.349, 2.385, 2.481, 2.610, 2.625, 2.632, 2.646, 2.661, 2.688, 2.823, 2.89, 2.9, 2.934, 2.962, 2.964, 3, 3.1, 3.114, 3.117, 3.166, 3.344, 3.376, 3.385, 3.443, 3.467, 3.478, 3.578, 3.595, 3.699, 3.779, 3.924, 4.035, 4.121, 4.167, 4.240, 4.255, 4.278, 4.305, 4.376, 4.449, 4.485, 4.570, 4.602, 4.663, 4.694.

Dataset-2: The following moderately skewed to right a set of data, presented by Tahir et al (2015), relates to the service times of 63 Aircraft Windshield (the unit for measurement is 1000 hours). The values are:

0.046, 1.436, 2.592, 0.140, 1.492, 2.600, 0.150, 1.580, 2.670, 0.248, 1.719, 2.717, 0.280, 1.794, 2.819, 0.313, 1.915, 2.820, 0.389, 1.920, 2.878, 0.487, 1.963, 2.950, 0.622, 1.978, 3.003, 0.900, 2.053, 3.102, 0.952, 2.065, 3.304, 0.996, 2.117, 3.483, 1.003, 2.137, 3.500, 1.010, 2.141, 3.622, 1.085, 2.163, 3.665, 1.092, 2.183, 3.695, 1.152, 2.240, 4.015, 1.183, 2.341, 4.628, 1.244, 2.435, 4.806, 1.249, 2.464, 4.881, 1.262, 2.543, 5.140.

The total time on test (TTT) plot of the datasets 1 and 2 and the simulated dataset of NTPGLD are presented in the figure 6. The upward pattern of the curve indicates that the system has decreasing hazard rate (or infant mortality) over the time while the downward pattern of the curve indicates that the system has increasing hazard rate over the time.

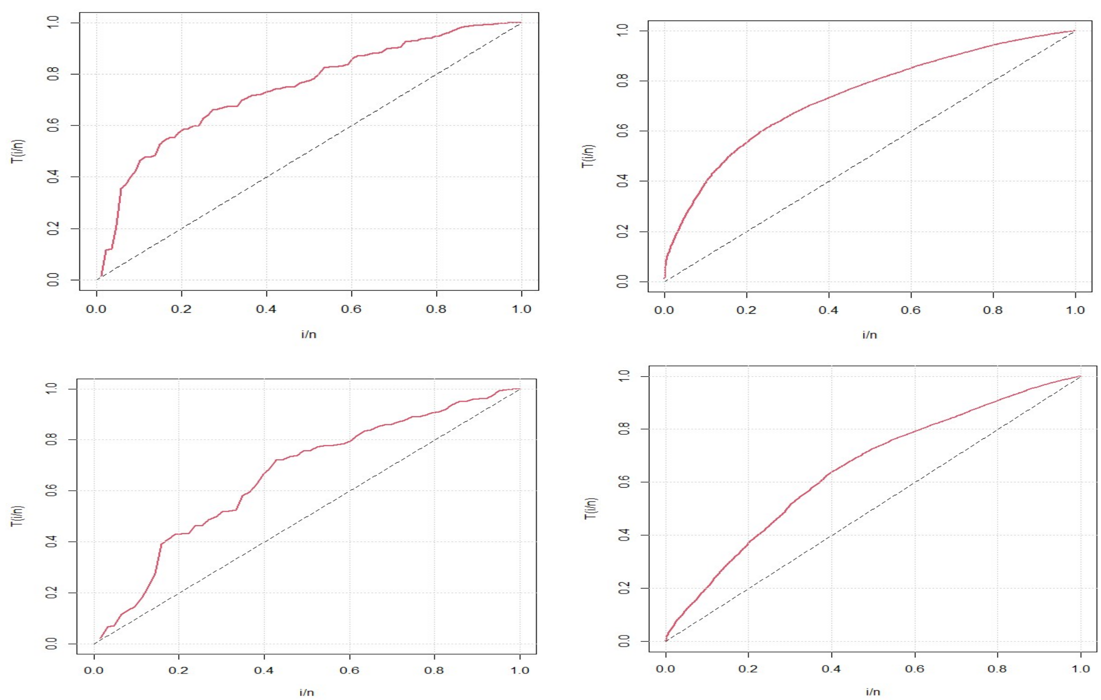


Figure 6: TTT-plot of considered datasets and simulated datasets

Table 6. Summary of the dataset 1 and 2

Datasets	Minimum	1st Quartiles	Median	Mean	3rd Quartiles	Maximum	Variance
1	0.040	1.786	2.367	2.569	3.400	4.694	1.286
2	0.046	1.122	2.065	2.085	2.820	5.140	1.550

In order to compare lifetime distributions, values of $-2\log L$, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Consistent Akaike Information Criterion (CAIC), Hannan-Quinn Information Criterion (HQIC), Kolmogorov-Smirnov (K-S) statistics and the corresponding probability value (P-value) for the above datasets has been computed. The formulae for computing AIC, BIC, CAIC, HQIC and K-S are as follows:

$$AIC = -2\log L + 2p\,,\; BIC = -2\log L + p\log(n)\,,\; CAIC = -2\log L + \frac{2pn}{n-p-1}$$

$$HQIC = -2\log L + 2p\log\left[\log(n)\right],\; K\text{-}S = \underset{x}{Sup}\,|\,F_m(x) - F_o(x)\,|$$

where, p = number of parameters, n = sample size, $F_m(x)$ = empirical cdf of considered distribution, $F_o(x)$ = cdf of considered distribution and *-2 logL* is the maximized value of log likelihood function. The ML estimates of the parameters with standard error (SE) of the considered distributions for datasets 1 and 2 are presented in the Table 7 and the values of $-2\log L$, AIC, BIC, CAIC, HQIC, K-S and P-value for the datasets 1 and 2 are presented in

Table 8 and 9. The confidence interval of the estimated parameters values of NTPGLD are presented in the table 10. The fitted plot of the considered distributions, Quantiles-Quantiles (Q-Q) plot and Probability-Probability (P-P) plot of the datasets 1 and 2 are presented in the figure 7.

Table 7. MLEs of the parameters of the considered distributions along with the standard deviation for dataset 1 and 2

Distributions	MLE of the dataset-1			MLE of the dataset-2		
	$\hat{\theta}$ $SE(\hat{\theta})$	$\hat{\alpha}$ $SE(\hat{\alpha})$	$\hat{\beta}$ $SE(\hat{\beta})$	$\hat{\theta}$ $SE(\hat{\theta})$	$\hat{\alpha}$ $SE(\hat{\alpha})$	$\hat{\beta}$ $SE(\hat{\beta})$
NTPGLD	0.4641 (0.2059)	1.8823 (0.8272)	1.6773 (0.1672)	0.9091 (0.2736)	2.6101 (0.9874)	1.3616 (0.0978)
TPGLD	0.2589 (0.1026)	1.2242 (1.7491)	1.8746 (0.2541)	0.4935 (0.1805)	1.5000 (2.0556)	1.4343 (0.2142)
NGLD	1.4809 (0.2600)	3.3086 (0.5340)	4.5336 (1.0751)	1.2034 (0.2769)	1.7409 (0.3668)	3.4015 (1.0243)
GLD	1.3556 (0.2101)	2.4824 (0.5018)	38057.6159 (3424.6348)	0.9151 (0.1725)	0.9084 (0.3148)	13.1304 (18.2595)
ATPLD	0.7775 (0.0658)	0.1000 (1.2975)	27.3745 (34900.2300)	0.9093 (0.0958)	0.2639 (58.5030)	2.0742 (459.7216)
NTPLD	0.7045 (0.1385)	1.3727 (0.6145)	…	0.7851 (0.1564)	1.1544 (0.6706)	…
WD	0.1000 (0.0281)	2.2263 (0.2098)	…	0.2556 (0.0560)	1.6290 (0.1683)	…
TPLD-1	0.7513 (0.0776)	0.1000 (0.2168)	…	0.9093 (0.0958)	0.1272 (0.1525)	…
TPLD-2	0.7772 (0.0596)	232.6230 (2012.7725)	…	0.9093 (0.0958)	7.8577 (9.4216)	…
QLD	0.7749 (0.0604)	0.0100 (0.0427)	…	0.9093 (0.0958)	0.1157 (0.1326)	…
NQLD	0.7774 (0.0606)	216.0758 (3071.3260)	…	0.9093 (0.0958)	7.1459 (8.9909)	…
LD	0.6283 (0.0492)	…	…	(0.4795) (0.0604)	…	…
ED	0.3892 (0.0414)	…	…	0.7531 (0.0704)	…	…

Table 8. Goodness of fit measures for the datasets-1

Distributions	$-2\log L$	AIC	BIC	CAIC	HQIC	K-S	P-Value
NTPGLD	270.51	276.51	279.49	278.01	277.09	0.05	0.97

TPGLD	271.82	277.82	280.80	279.32	278.40	0.06	0.90
NGLD	286.72	292.72	295.70	294.22	293.30	0.20	0.00
GLD	287.89	293.89	296.87	295.39	294.47	0.09	0.37
ATPLD	300.52	306.52	309.50	308.02	307.10	0.17	0.01
NTPLD	318.94	324.94	327.92	326.44	325.52	0.22	0.00
WD	274.68	278.68	280.67	279.38	279.06	0.97	0.00
TPLD-1	302.29	306.29	308.28	306.99	306.67	0.46	0.00
TPLD-2	300.52	304.52	306.51	305.22	304.90	0.38	0.00
QLD	300.55	304.55	306.54	305.25	304.93	0.21	0.00
NQLD	300.52	304.52	306.51	305.22	304.90	0.16	0.04
LD	319.45	321.45	322.44	321.67	321.64	0.24	0.00
ED	342.04	344.04	345.03	344.26	344.2	0.29	0.00

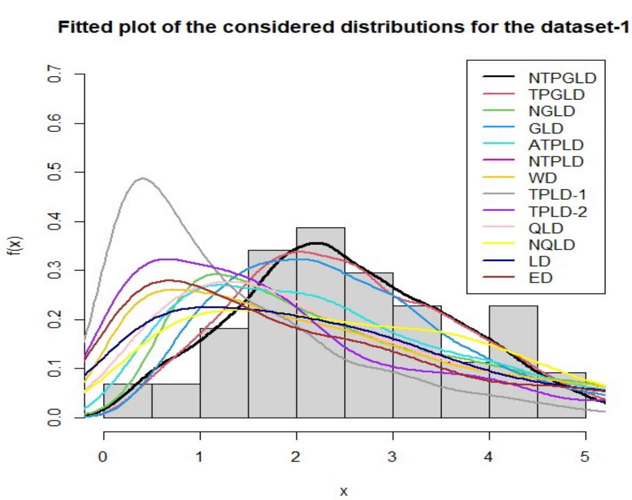
Table 9. Goodness of fit measures for the datasets-2

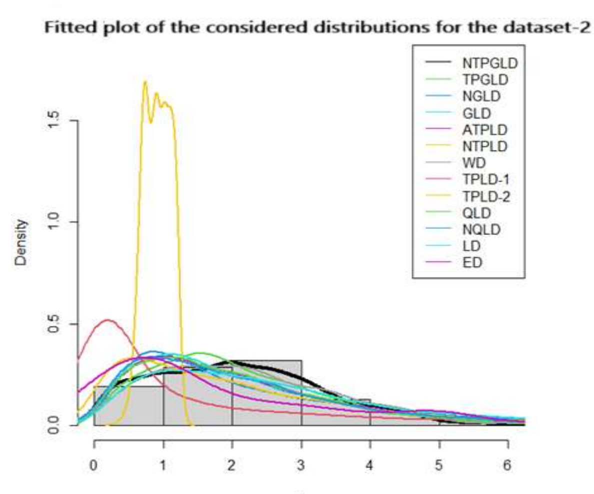
Distributions	$-2\log L$	**AIC**	**BIC**	**CAIC**	**HQIC**	**K-S**	**P-Value**
NTPGLD	196.26	202.26	208.68	202.66	204.78	0.06	0.97
TPGLD	199.07	205.07	211.49	205.47	207.59	0.09	0.63
NGLD	203.39	209.39	215.81	209.79	211.91	0.12	0.30
GLD	204.76	210.76	217.18	211.16	213.28	0.166	0.06
ATPLD	204.20	210.20	216.62	210.60	212.72	0.13	0.19
NTPLD	209.09	213.09	217.37	213.29	214.77	0.15	0.08
WD	200.63	204.63	208.91	204.83	206.31	0.10	0.52
TPLD-1	204.20	208.20	212.48	208.40	209.88	0.15	0.12
TPLD-2	204.20	208.20	212.48	208.40	209.88	0.38	0.00
QLD	204.20	208.20	212.48	208.40	209.88	0.13	0.23
NQLD	204.20	208.20	212.48	208.40	209.88	0.15	0.10
LD	218.59	220.59	222.73	220.65	221.43	0.27	0.00
ED	209.15	211.15	213.29	211.21	211.99	0.14	0.17

It is obvious from the goodness of fit of the considered distributions in table 8 and 9 that NTPGLD provides much better fit than other three-parmeter distributions, two-parameter distributions and one-parameter distributions. Figure 7 also justifies that NTPGLD provides a better fit.

Table 10: Confidence interval of the parameters of the NTPGLD for the considered datasets

Datasets	Parameters	90% Confidence interval (Lower limit, Upper limit)	95% Confidence interval (Lower limit, Upper limit)
1	θ	(0.2041, 0.8863)	(0.1632, 0.9865)
	α	(0.6606, 3.4111)	(0.3959, 3.7441)
	β	(1.4452, 2.0145)	(1.4091, 2.0995)
2	θ	(0.5334, 1.4209)	(0.4582, 1.5322)
	α	(1.0559, 4.3173)	(0.7655, 4.6659)
	β	(1.2090, 1.5387)	(1.1800, 1.5772)





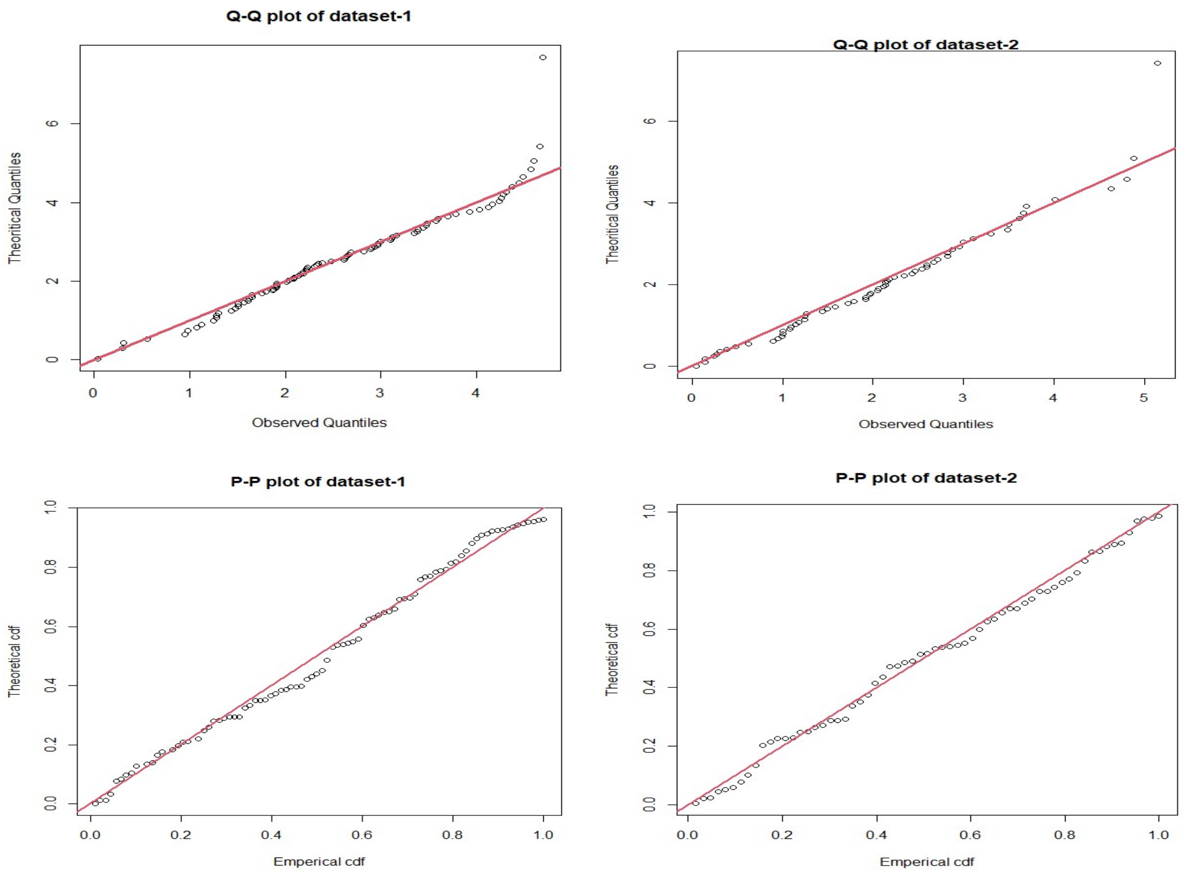


Figure 7: Fitted plot of considered distributions, Q-Q plot and P-P plot of NTPGLD

8. CONCLUSIONS

A new three-parameter generalized Lindley distribution (NTPGLD) of which one parameter Lindley and exponential distributions and two-parameter distributions namely Weibull, power Lindley and NTPLD are particular cases, has been proposed. The nature of pdf, cdf, survival function, hazard function, mean residual life function has been studied with varying values of parameters. Statistical properties including moments based measures and order statistics have been studied. The method of maximum likelihood has been discussed for estimating parameters. Fisher's information matrix and confidence intervals of the parameters of the proposed distribution have been presented. The goodness of fit of NTPGLD has been discussed with two real under-dispersed lifetime datasets from engineering and the fit has been found quite satisfactory over the considered one-parameter, two-parameter and three-parameter lifetime distributions. Therefore, the proposed distribution can be a suitable probability model for modelling under-dispersed datasets from engineering.

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