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A Stochastic Model of Mixture Distribution Properties and its Applications

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ABSTRACT

In this paper, specific statistical considerations are typically required in order to select the best model for fitting survival data. The new Mixture of Gompertz and Gamma Distribution (MGGD), which gets its name from the particular mixing of two distributions, Gompertz and Gamma, is proposed in this study. Also, the reliability analysis, statistical properties like stochastic ordering, moments, order statistics, and the estimation of parameters using the method of maximum likelihood estimation. Finally, an application of the goodness-of-fit criteria to a real cancer data set is shown. It is contrasted with the fit and demonstrates that the mixture of the Gompertz and gamma distributions has greater flexibility than the other distributions.

Keywords: Gamma and Gompertz distribution, Mixture model, Moments, Order Statistics, Maximum likelihood Estimation.

1. Introduction

Medical research is mostly interested in studying the survival of cancer patients, as applied to statistical research. The statistical distributions have been extensively utilized for analyzing time-to-event data, also referred to as survival or reliability data, in different areas of applicability, including medical science. In recent years, an impressive set of new statistical distributions has been explored by statisticians. The necessity of developing an extended class of classical distribution is analysis, biomedicine, reliability, insurance, and finance. Recently, many researchers have been working in this area and have proposed new methods to develop improved probability distributions with utility.

A Gompertz [16] distribution RV X with a parameter $\eta > 0$ is described by its pdf is defined as

$$f(x,\eta,\beta) = \eta e^{\beta x} e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}, x > 0, \eta, \beta > 0$$

Considering the gamma [12] distribution with parameters $\beta = (3,\eta)$ the pdf can be defined as

$$f(\eta, x) = \frac{1}{2}\eta^3 x^2\, e^{-\eta x} \quad , x > 0, \eta > 0$$

The concept of a finite mixture of probabilities was pioneered by Newcomb [10] as a model for outliers. Weldon [19] provided a mixture technique for analyzing crab morphometric data.

Pearson [9] introduced a statistical model using finite mixtures of normal distributions and also estimated the parameters of the mixture. Fisher [5] introduced the concept of a weighted mixture of outcomes and developed the Sib method. He also obtained the properties of the new distribution. Lindley [8] introduced the fiducial distribution and Bayes theorem. Rama Shanker [14], has introduced a mixture of exponential (θ) and gamma (2, θ) distributions and proposed a shanker distribution. Akash distribution is a two-component mixture of an exponential distribution

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and gamma distribution with their mixing proportions $\frac{\theta^2}{\theta^2+2}$ and $\frac{2}{\theta^2+2}$, Shanker [15]. Proposed a Komal distribution with applications in survival analysis, Ramma Shanker [16], the combination of exponential (θ) and gamma (2, θ) distribution with mixing proportions $\frac{\theta(\theta+1)}{\theta^2+\theta+1}$ and $\frac{1}{\theta^2+\theta+1}$.

This article is based on a mixture of Gompertz and Gamma distributions in order to create the (MGGD) that was proposed. For the remainder of this research work, the presented (pdf) and (cdf) functions of the proposed distribution, together with some of its properties, provide an approach to the maximum likelihood estimators for estimating the model parameters. Finally, the results of fitting the caner survival data with (MGGD) also show the other well-known distributions. Throughout this research, the statistical programming language R was used for all computations.

2. New Mixture Distribution

This section introduces the (MGGD) distribution, which is a new distribution created by combining two existing distributions. Let X be a random variable with a mixed distribution. Its density function (pdf), $f(x)$ is expressed as follows:

$$f(x) = \sum_{i=1}^{k} \omega_i f_i(x)$$

$f_i(x)$probability density function for all i

$\omega_i,\ i = 1, \dots, n$ denote mixing proportions that are non-negative and $\sum_{i=1}^{k} \omega_i = 1$. The $f_1(x) \sim$ gamma (η, β) with parameters $\beta = 3$, η and $f_2(x) \sim$ Gompertz (η, β) two independents random variables with $\frac{\beta}{\beta+1}$ *and* $\frac{1}{\beta+1}$respectively. Now the density function of X is given by.

$$f(x;\ \eta, \beta) = \frac{\eta}{1+\beta}\left(e^{\beta x} e^{-\frac{\eta}{\beta}(e^{\beta x}-1)} + \frac{\eta^2 \beta}{2} x^2 e^{-\eta}\ \right) \tag{1}$$

The function defined in (1) represents a probability distribution function $f(x;\ \eta, \beta) for\ all\ x > 0$

$$f(x;\ \eta, \beta) = \frac{\eta}{1+\beta}\int_0^\infty \left(e^{\beta x} e^{-\frac{\eta}{\beta}(e^{\beta x}-1)} + \frac{\eta^2 \beta}{2} x^2 e^{-\eta x}\right) dx$$

$$= \frac{\eta}{1+\beta}\left[\frac{1+\beta}{\eta}\right] = 1$$

The cumulative distribution function cdf is defined as

$$F(x;\ \eta, \beta) = \int_0^x \frac{\eta}{1+\beta}\left(e^{\beta z} e^{-\frac{\eta}{\beta}(e^{\beta z}-1)} + \frac{\eta^2 \beta}{2} z^2 e^{-\eta z}\right) dz$$

Simplifying the integration,

$$F(x;\ \eta, \beta) = \frac{1}{\beta+1}\left(1 - \left(e^{-\frac{\eta}{\beta}(e^{\beta x}-1)} + \beta\left(\left(\frac{\eta^2 x}{2} + 1\right)\eta x + 1\right)e^{-\eta x}\right) + \beta\right) \tag{2}$$

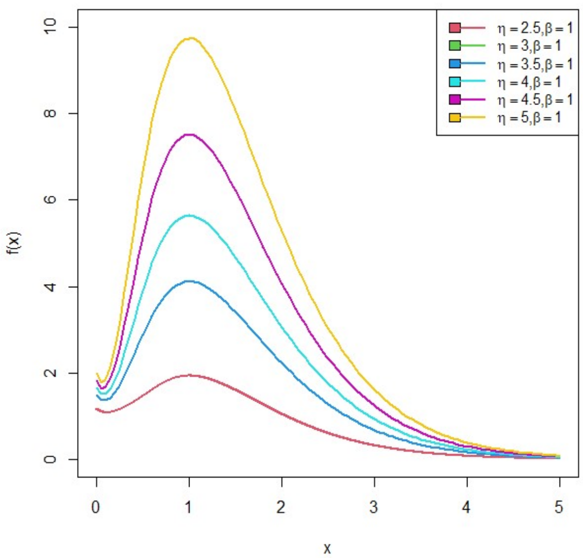

Figure.1:Pdf plot of Mixture of Gamma and Gompertz distribution

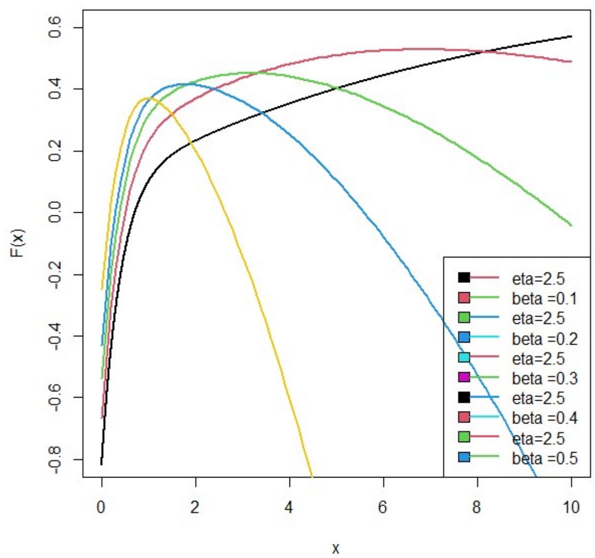

Figure.2 cdf plot of Mixture of Gamma and Gompertz distribution

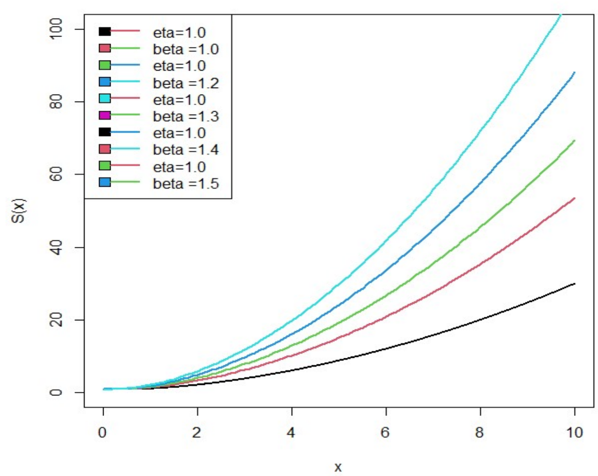

Figure.3 Survival plot of a Mixture of Gamma and Gompertz Distribution

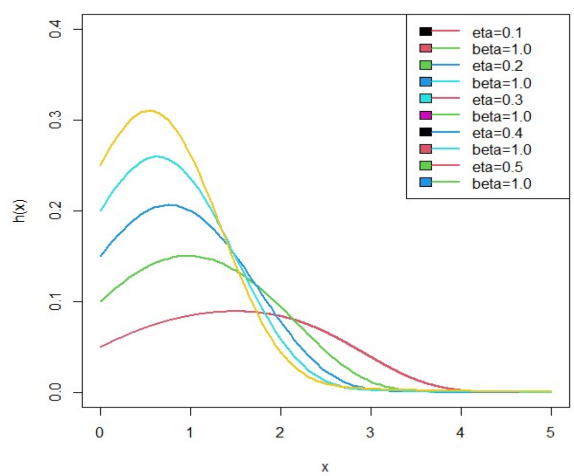

Figure.4 Hazard plot of a Mixture of Gompertz and Gamma Distribution

3. Reliability Analysis

This section will provide the reliability function, hazard function, reverse hazard function, cumulative hazard function, Odds rate and, Mean Residual function for the specified MGG distribution.

3.1 Survival Function

The survival function of MGG distribution is defined as

$$S(x) = 1 - F(x;\, \eta, \beta) = 1 - \frac{1}{\beta + 1}\left(1 - \left(e^{-\frac{\eta}{\beta}\left(e^{\beta x} - 1\right)} + \beta\left(\left(\frac{\eta^2 x}{2} + 1\right)\eta x + 1\right)e^{-\eta x}\right) + \beta\right)$$

3.2 Hazard Rate Function

An important metric for describing life phenomena is the hazard rate function of the MGG distribution, which is defined as

$$h(x)=\frac{f(x;\ \eta,\beta)}{\mathbf{1}-F(x;\ \eta,\beta)}=\frac{\eta\left(e^{\beta x}e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\frac{\eta^2\beta}{2}x^2e^{-\eta x}\right)}{(\beta+1)-\left(1-\left(e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\beta\left(\left(\frac{\eta^2 x}{2}+1\right)\eta x+1\right)e^{-\eta x}\right)+\beta\right)}$$

3.3 Revers hazard rate

The Revers hazard rate of MGG distribution is defined as

$$h_r(x)=\frac{f(x;\ \eta,\beta)}{F(x;\ \eta,\beta)}=\frac{\eta\left(e^{\beta x}e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\frac{\eta^2\beta}{2}x^2e^{-\eta x}\right)}{\left(1-\left(e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\beta\left(\left(\frac{\eta^2 x}{2}+1\right)\eta x+1\right)e^{-\eta x}\right)+\beta\right)}$$

3.4 Cumulative hazard function

The Cumulative hazard function of MGG distribution is defined as

$$H(x)=-\ln(1-F(x;\ \eta,\beta))$$

$$=\ln\left(\frac{1}{\beta+1}\left(1-\left(e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\beta\left(\left(\frac{\eta^2 x}{2}+1\right)\eta x+1\right)e^{-\eta x}\right)+\beta\right)-1\right)$$

3.5 Odds rate function

The Odds rate function of MGG distribution is defined as

$$O(x)=\frac{F(x;\ \eta,\beta)}{1-F(x;\ \eta,\beta)}=\frac{\left(1-\left(e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\beta\left(\left(\frac{\eta^2 x}{2}+1\right)\eta x+1\right)e^{-\eta x}\right)+\beta\right)}{(\beta+1)-\left(1-\left(e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\beta\left(\left(\frac{\eta^2 x}{2}+1\right)\eta x+1\right)e^{-\eta x}\right)+\beta\right)}$$

3.6 Mean Residual function

The mean residual function of MGG distribution is defined as

$$M(x)=\frac{1}{S(x)}\int_x^\infty t\ \frac{\eta}{1+\beta}\left(e^{\beta t}e^{-\frac{\eta}{\beta}\left(e^{\beta t}-1\right)}+\frac{\eta^2\beta}{2}t^2e^{-\eta t}\right)dt$$

Then, using the following geometric series expansion is defined as

$$(1+x)^{-n}=\sum_{j=0}^{\infty}\binom{-n}{j}x^j$$

Then, using the gamma function is defined as,

$$\Gamma(s,x)\ =\int_x^\infty t^{s-1}e^{-t}dt$$

Substitute the limits of integration and simplify the expression.

$$M(x)=\frac{1}{(1+\beta)}\left[\frac{1}{\beta}\sum_{j=0}^{\infty}\binom{-1}{j}\Gamma\left(j+1,\frac{\eta}{\beta}e^{\beta x}-1\right)+\frac{\beta\Gamma(4,\eta x\ )}{2\eta}\right]$$

4 STATISTICAL PROPERTIES

In this section, also derived the structural properties, moments, the moment generating function, and the Characteristic function for the MGG distribution of the random variable. Including the mean, and variance, investigated.

4.1 Moments

The r[th] moments of a RV X, is defined as

$$E(X^r) = \mu_r' = \int_0^\infty x^r\ f(x;\ \eta,\beta)\ dx$$

$$E(X^r) = \int_0^\infty x^r\ \frac{\eta}{1+\beta}\left(e^{\beta x}e^{-\frac{\eta}{\beta}(e^{\beta x}-1)} + \frac{\eta^2\beta}{2}x^2e^{-\eta x}\right) dx$$

Then, using the following power series method is defined as,

$$\ln(x+1) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$$

This series converges for $|x| > 1$

This process involves using the multinomial expansion for the power of a series defined as.

$$\ln(x+1) = \sum_{n=1}^{\infty} a_n x^n \quad where\ a_n = (-1)^{n+1}\frac{1}{n}$$

Then, $(\ln(x+1))^r$

$$(\ln(x+1))^r = \left(\sum_{n=1}^{\infty} a_n x^n\right)^r$$

Expanding $(\sum_{n=1}^{\infty} a_n x^n)^r$ can be done using the Cauchy product for power series.

Thus, we can express $(\ln(x+1))^r$

$$(\ln(x+1))^r = \sum_{m=0}^{\infty} c_m x^m$$

where c_m are the coefficients obtained from the Cauchy product of the series

$$\sum_{n=1}^{\infty} a_n x^n \quad for\ m \geq 1$$

Then, using the following gamma function is defined as,

$$\int_0^\infty x^{z-1}e^{-px}\,dx = \frac{\Gamma(z)}{p^z}$$

Substitute the limits of the integration, and the simplified expression becomes

$$E(X^r) = \frac{1}{1+\beta}\left(\sum_{m=0}^{\infty} c_m\ \frac{\eta\Gamma(m+1)}{\beta^{r+1}\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{\beta\Gamma(r+3)}{2\eta^{r+2}}\right) \tag{3}$$

Where $\Gamma(.)$ Is the gamma function. Subsequently, the mean and variance can be defined by substituting $r = 1,2,3,4$ in equation (3)

$$E(X) = \boldsymbol{Mean} = \frac{1}{1+\beta}\left(\sum_{m=0}^{\infty} c_m\ \frac{\eta\Gamma(m+1)}{\beta^2\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{3\beta}{\eta^3}\right) E(X^2) = \frac{1}{1+\beta}\left(\sum_{m=0}^{\infty} c_m\ \frac{\eta\Gamma(m+1)}{\beta^3\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{12\beta}{\eta^4}\right)$$

$$E(X^3) = \frac{1}{1+\beta}\left(\sum_{m=0}^{\infty} c_m \frac{\eta\Gamma(m+1)}{\beta^4\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{60\beta}{\eta^5}\right), E(X^4) = \frac{1}{1+\beta}\left(\sum_{m=0}^{\infty} c_m \frac{\eta\Gamma(m+1)}{\beta^5\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{360\beta}{\eta^6}\right)$$

$$\boldsymbol{Variance} = \sigma^2 = E(X^2) - \left(E(X)\right)^2$$

$$\sigma^2 = \left(\frac{1}{1+\beta}\left(\sum_{m=0}^{\infty} c_m \frac{\eta\Gamma(m+1)}{\beta^3\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{12\beta}{\eta^4}\right)\right) - \left(\frac{1}{1+\beta}\left(\sum_{m=0}^{\infty} c_m \frac{\eta\Gamma(m+1)}{\beta^2\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{3\beta}{\eta^3}\right)\right)^2$$

After simplification,

$$\sigma^2 = \left(\frac{(1+\beta)\left(\sum_{m=0}^{\infty} c_m \frac{\eta\Gamma(m+1)}{\beta^3\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{12\beta}{\eta^4}\right) - \left(\sum_{m=0}^{\infty} c_m \frac{\eta\Gamma(m+1)}{\beta^2\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{3\beta}{\eta^3}\right)^2}{(1+\beta)^2}\right)$$

4.2 Moment Generating Function

The MGF of a RV X is denoted by $M_X(t)$ and is defined as

$$M_X(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x;\, \eta, \beta)\ \ dx \quad, t\epsilon\mathcal{R}$$

$$M_X(t) = \int_0^{\infty} e^{tx} \frac{\eta}{1+\beta}\left(e^{\beta x} e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)} + \frac{\eta^2\beta}{2} x^2 e^{-\eta x}\right) dx$$

To solve the expression, using the power series method is defined as,

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k$$

This series converges for |v|>1.

let's make a substitution to simply the integral

$$M_X(t) = \frac{\eta}{1+\beta}\left[\frac{1}{\beta}\sum_{k=0}^{\infty}\binom{\frac{t}{\beta}-1}{k}\frac{\Gamma(k+1)}{\left(\frac{\eta}{\beta}\right)^{k+1}} + \frac{\eta^2\beta}{(\eta-t)^3}\right] \tag{4}$$

The characteristics function (CF) of a RV X, it is denoted by $\phi_X(t)$ and is defined as

$$\phi_X(t) = \frac{\eta}{1+\beta}\left[\frac{1}{\beta}\sum_{k=0}^{\infty}\binom{\frac{it}{\beta}-1}{k}\frac{\Gamma(k+1)}{\left(\frac{\eta}{\beta}\right)^{k+1}} + \frac{\eta^2\beta}{(\eta-it)^3}\right] \tag{5}$$

5. Hormonic Mean

If H_X is the harmonic mean (HM) of the RV X, then

$$H_X = E\left[\frac{1}{X}\right]$$

$$H.M = \int_0^\infty \frac{1}{x}\frac{\eta}{1+\beta}\left(e^{\beta x}e^{-\frac{\eta}{\beta}(e^{\beta x}-1)} + \frac{\eta^2\beta}{2}x^2e^{-\eta x}\right)dx$$

Then, using the following power series expansion is defined as

$$(1+x)^{-n} = \sum_{j=0}^{\infty}\binom{-n}{j}x^j$$

Simplifying, the final results of the integral are

$$H.M = \frac{\eta}{1+\beta}\left[\frac{1}{\beta^2}\sum_{j=0}^{\infty}\binom{-2}{j}\frac{\Gamma(j+1)}{\left(\frac{\eta}{\beta}\right)^{j+1}} + \frac{\beta}{2}\right]$$

6. Mean

The Mean deviation (MD) of the RV X, is defined as

$$D(\mu) = E(|X-\mu|)$$

$$D(\mu) = \int_0^\infty |X-\mu| \; f(x;\,\eta,\beta)dx$$

$$D(\mu) = \int_0^\mu (\mu - x)\, f(x;\,\eta,\beta)\,dx + \int_\mu^\infty (x-\mu)\; f(x;\,\eta,\beta)\,dx$$

Simplifying the integration

$$D(\mu) = 2\mu F(\mu) - 2\int_0^\mu x\; f(x;\,\eta,\beta)\,dx$$

Then,

$$= \int_0^\mu x\; f(x;\,\eta,\beta)\,dx$$

$$= \int_0^\mu x\,\frac{\eta}{1+\beta}\left(e^{\beta x}e^{-\frac{\eta}{\beta}(e^{\beta x}-1)} + \frac{\eta^2\beta}{2}x^2e^{-\eta x}\right)dx$$

Then, using the following geometric Series expansion is defined as

$$(1+x)^{-n} = \sum_{j=0}^{\infty}\binom{-n}{j}x^j$$

So, the final expressions for the integral is

$$D(\mu) = 2\mu\frac{1}{\beta+1}\left(1 - \left(e^{-\frac{\eta}{\beta}(e^{\beta\mu}-1)} + \beta\left(\left(\frac{\eta^2\mu}{2}+1\right)\eta\mu + 1\right)e^{-\eta\mu}\right) + \beta\right)$$
$$- 2\frac{1}{1+\beta}\left[\frac{1}{\beta}\sum_{j=0}^{\infty}\binom{-1}{j}\gamma\left(j+1,\frac{\eta}{\beta}e^{\beta\mu}-1\right) + \eta\beta\left[-\left(\frac{\mu^2}{2}(\eta\mu+3) + \frac{3}{\eta}\left(\mu+\frac{1}{\eta}\right)\right)e^{-\eta\mu} + \frac{3}{\eta^2}\right]\right]$$

7. Median

The mean deviation from median of the RV X, is defined as

$$D(M) = E(|X - M|)$$

$$D(M) = \int_0^\infty |X - M| \ f(x;\ \eta, \beta) dx$$

$$D(M) = \int_0^M (M - x) f(x;\ \eta, \beta)\ dx + \int_M^\infty (x - M) f(x;\ \eta, \beta)\ dx$$

Simplifying the integration

$$D(M) = \mu - 2\int_0^M x\ f(x;\ \eta, \beta)\ dx$$

Then,

$$= \int_0^M x\ \ f(x;\ \eta, \beta)\ dx$$

$$= \int_0^M x\ \frac{\eta}{1 + \beta}\left(e^{\beta x} e^{-\frac{\eta}{\beta}(e^{\beta x} - 1)} + \frac{\eta^2 \beta}{2} x^2 e^{-\eta}\ \right) dx$$

Then, the final result of the integration is

$$D(\mu) = \mu - 2\frac{1}{1 + \beta}\left[\frac{1}{\beta}\sum_{j=0}^{\infty}\binom{-1}{j}\gamma\left(j + 1, \frac{\eta}{\beta}e^{\beta M} - 1\right) + \eta\beta\left[-\left(\frac{M^2}{2}(\eta M + 3) + \frac{3}{\eta}\left(M + \frac{1}{\eta}\right)\right)e^{-\eta M} + \frac{3}{\eta^2}\right]\right]$$

8. Order Statistics

The derived pdf of the i^{th} order statistics of the mixture of Gompertz and gamma distribution. Let $X_1, X_2, \dots, X_n$ be a simple random sample from MGG distribution with cdf and pdf given by (1) and (2), respectively. Let $X_{(1:n)} \leq X_{(2:n)} \leq \cdots \leq X_{(n:n)}$ denote the order statistics defined from this sample. We now given the pdf of $X_{r:n}$, say $f_{r;n}(x)$ of $X_{r:n}$, $i = 1,2, \dots, n$. The pdf of the r^{th} order statistics $X_{r:n}$, $r = 1, 2, \dots, n$ is defined as

$$f_{r:n}(x) = \frac{n!}{(r - 1)!\,(n - r)!}\big(F(x;\ \eta, \beta)\big)^{r-1}\big(1 - F(x;\ \eta, \beta)\big)^{n-r} f(x;\ \eta, \beta), x > 0, \eta, \beta > 0 \quad (6)$$

Where $F(.)$ and $f(.)$ are given by (1) and (2) respectively, *and* $W_{\ r:n} = \frac{n!}{(r-1)!(n-r)!}$

$$f_{r:n} =\ W_{\ r:n}\big(F(x;\ \eta, \beta)\big)^{r-1}\big(1 - F(x;\ \eta, \beta)\big)^{n-r} f(x;\ \eta, \beta)$$

Then, using the following binomial series expansion is defined as

$$(1 - z)^a = \sum_{j=0}^{\infty}(-1)^j \binom{a}{j}\ z^j$$

$$(a + b)^z = \sum_{j=0}^{\infty}\binom{Z}{j}\ (a)^j b^{z-j}$$

let’s make a substitution to simplify the expression,

$$f_{r:n} = W_{r:n}\frac{\eta}{1+\beta}\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n-r}{s}\binom{r+s-1}{j}\binom{j}{k}\binom{k}{l}\binom{k-l}{n}\binom{n}{q}(-1)^{s+k}\frac{\left(-\frac{\eta l}{\beta}\left(e^{\beta z}-1\right)\right)^{m}}{m!}$$
$$\times\left(\frac{1}{\beta+1}\right)^{r+s-1}\beta^{n}(\eta x)^{n}e^{-\eta(k-l)x}\ \frac{\eta}{1+\beta}\left(e^{\beta x}e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\frac{\eta^{2}\beta}{2}x^{2}e^{-\eta}\right)$$

First order statistics

$$f_{1:n} = W_{1:n}\frac{\eta}{1+\beta}\sum_{s=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n-1}{s}\binom{s}{j}\binom{j}{k}\binom{k}{l}\binom{k-l}{n}\binom{n}{q}(-1)^{s+k}\frac{\left(-\frac{\eta l}{\beta}\left(e^{\beta z}-1\right)\right)^{m}}{m!}$$
$$\times\left(\frac{1}{\beta+1}\right)^{s}\beta^{n}(\alpha x)^{n}e^{-\alpha(k-l)x}\ \frac{\eta}{1+\beta}\left(e^{\beta x}e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\frac{\eta^{2}\beta}{2}x^{2}e^{-\beta x}\right)$$

n^{th} order statistics

$$f_{n:n} = W_{n:n}\frac{\eta}{1+\beta}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+s-1}{j}\binom{j}{k}\binom{k}{l}\binom{k-l}{n}\binom{n}{q}(-1)^{s+k}\frac{\left(-\frac{\eta l}{\beta}\left(e^{\beta z}-1\right)\right)^{m}}{m!}$$
$$\times\left(\frac{1}{\beta+1}\right)^{n+s-1}\beta^{n}(\eta x)^{n}e^{-\eta(k-l)x}\ \frac{\eta}{1+\beta}\left(e^{\beta x}e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\frac{\eta^{2}\beta}{2}x^{2}e^{-\eta x}\right)$$

9. Entropies

In this section, we derive the Rényi entropy, and the Tsallis entropy from the MGG distribution. It is well known that entropy and information can be considered measures of uncertainty, or the randomness of a probability distribution. It is applied in many fields, such as engineering, finance, information theory, and biomedicine. The entropy functionals for probability distribution were derived on the basis of a variational definition of uncertainty measure.

9.1 Rényi Entropy

Entropy is defined as a random variable X is a measure of the variation of the uncertainty. It is used in many fields, such as engineering, statistical mechanics, finance, information theory, biomedicine, and economics. The entropy measure is the Rényi of order which is defined as

$$R_{\gamma} = \frac{1}{1-\gamma}\log\int_{0}^{\infty}[f(x;\ \eta,\beta)]^{\gamma}\ dx \qquad ;\gamma>0,\gamma\neq 1$$

$$R_{\gamma} = \frac{1}{1-\gamma}\log\int_{0}^{\infty}\left[\frac{\eta}{1+\beta}\left(e^{\beta x}e^{-\frac{\eta}{\beta}\left(e^{\beta x}-1\right)}+\frac{\eta^{2}\beta}{2}x^{2}e^{-\eta x}\right)\right]^{\gamma}\ dx$$

Using the following Binomial series expansion is defined as

$$(a+b)^{z} = \sum_{j=0}^{\infty}\binom{z}{j}(a)^{j}b^{z-j}$$

Then, using the following binomial Series expansion is defined as.

$$(1+x)^{n} = \sum_{k=0}^{\infty}\binom{n}{k}x^{k}$$

So, the final expressions of the integral are

$$R_\gamma = \frac{1}{1-\gamma}\log\left(\frac{\eta}{1+\beta}\right)^{\gamma}\frac{1}{\beta}\sum_{k=0}^{\infty}\binom{j-1}{k}\left(\frac{\eta^2\beta}{2}\right)^{\gamma-j}\frac{\Gamma(j+k)}{\left(\frac{\eta j}{\beta}\right)^{j+k+}}\frac{\Gamma(2(\gamma-j)+1)}{\left(\eta(j-\gamma)\right)^{2(\gamma-j)+1}}$$

9.2 Tsallis Entropy

The Boltzmann-Gibbs (B-G) statistical properties initiated by Tsallis have received a great deal of attention. This generalization of (B-G) statistics was first proposed by introducing the mathematical expression of Tsallis entropy (Tsallis, (1988) for continuous random variables, which is defined as

$$T_\gamma = \frac{1}{\gamma-1}\left[1-\int_0^\infty [\,f(x;\,\eta,\beta)]^{\gamma}\,dx\right] \qquad ;\gamma > 0, \gamma \neq 1$$

$$T_\gamma = \frac{1}{\gamma-1}\left[1-\int_0^\infty \left(\frac{\eta}{1+\beta}\left(e^{\beta x}e^{-\frac{\eta}{\beta}(e^{\beta x}-1)}+\frac{\eta^2\beta}{2}x^2e^{-\eta}\right)\right)^{\gamma}dx\right]$$

Then, solving the integration and the simplified expression becomes.

$$T_\gamma = \frac{1}{\gamma-1}\left(1-\left(\frac{\eta}{1+\beta}\right)^{\gamma}\frac{1}{\beta}\sum_{k=0}^{\infty}\binom{j-1}{k}\left(\frac{\eta^2\beta}{2}\right)^{\gamma-j}\frac{\Gamma(j+k)}{\left(\frac{\eta j}{\beta}\right)^{j+k+}}\frac{\Gamma(2(\gamma-j)+1)}{\left(\eta(j-\gamma)\right)^{2(\gamma-j)+1}}\right)$$

10. Stochastic Ordering

A crucial technique in reliability and finance for evaluating the relative performance of the models is stochastic ordering. Let X and Y be two random variables with pdf, cdf, and reliability functions $f(x), f(y), F(x), F(y). S(x) = 1 - F(x) and\ F(y)$.

1- Likelihood ratio order ($X \leq_{LR}\ Y$) if $\frac{f_X(x;\,\eta,\beta)}{f_Y(x;\,\eta,\beta)}$ decreases in x
2- Stochastic order ($X \leq_{ST}\ Y$) if $F_X(x; x;\,\eta,\beta) \geq F_Y(x;\,\eta,\beta)\ \forall\ x$
3- Hazard rate order ($X \leq_{HR}\ Y$) if $h_X(x;\,\eta,\beta) \geq F_Y(x;\,\eta,\beta)\ \forall\ x$
4- Mean residual life order ($X \leq_{MRL}\ Y$) if $MRL_X(X) \leq MRL_Y(X)\ \forall\ x$

Prove that the mixture of Gompertz and gamma distribution complies with the ordering with the highest likelihood (the likelihood ratio ordering).

Assume that X and Y are two independent Random variables with probability distribution function$f_x(x;\eta,\beta)$and $f_y(x;\psi,\delta)$If $\eta < \psi$ and $\beta < \delta$, then

$$\Lambda = \frac{f_x(x;\eta,\beta)}{f_y(x;\psi,\delta)} = \frac{\frac{\eta}{1+\beta}\left(e^{\beta x}e^{-\frac{\eta}{\beta}(e^{\beta x}-1)}+\frac{\eta^2\beta}{2}x^2e^{-\eta x}\right)}{\frac{\psi}{1+\delta}\left(e^{\delta x}e^{-\frac{\psi}{\delta}(e^{\delta x}-1)}+\frac{\psi^2\delta}{2}x^2e^{-\psi x}\right)}$$

Therefore,

$$\log[\Lambda] = \log\left[\frac{\eta(1+\delta)}{\psi(1+\beta)}\right]+\log\left[e^{\beta x}e^{-\frac{\eta}{\beta}(e^{\beta x}-1)}+\frac{\eta^2\beta}{2}x^2e^{-\eta x}\right]-\log\left[e^{\delta x}e^{-\frac{\psi}{\delta}(e^{\delta x}-1)}+\frac{\psi^2\delta}{2}x^2e^{-\psi x}\right]$$

Differentiating with respect to x,

$$\frac{\partial\log[\Lambda]}{\partial x} = \frac{\left[-\beta e^{\beta x}\,\eta e^{\beta x}e^{-\frac{\eta}{\beta}(e^{\beta x}-1)}+\eta^2\beta e^{-\eta x}-\frac{\eta^3\beta}{2}x^2e^{-\eta x}\right]}{\left[e^{\beta x}e^{-\frac{\eta}{\beta}(e^{\beta x}-1)}+\frac{\eta^2\beta}{2}x^2e^{-\eta x}\right]}-\frac{\left[-\delta e^{\delta x}\,\psi e^{\delta x}e^{-\frac{\psi}{\delta}(e^{\delta x}-1)}+\psi^2\delta e^{-\psi x}-\frac{\psi^3\delta}{2}x^2e^{-\psi x}\right]}{\left[e^{\delta x}e^{-\frac{\psi}{\delta}(e^{\delta x}-1)}+\frac{\psi^2\delta}{2}x^2e^{-\psi x}\right]}$$

Hence, $\frac{\partial \log[\Lambda]}{\partial x} < 0, \text{if } \eta < \psi, \beta < \delta.$

11. Bonferroni and Lorenz Curves

The Bonferroni and Lorenz curves have been obtained using the MGG distribution in this section. The Bonferroni and Lorenz curve is a powerful tool in the analysis of distributions and has applications in many fields, such as economies, insurance, income, reliability, and medicine. The Bonferroni and Lorenz cures for a X be the random variable of a unit and $f(x)$ be the probability density function of x. $f(x)dx$ will be represented by the probability that a unit selected at random is defined as

$$B(p) = \frac{1}{p\mu}\int_0^q x\, f(x;\, \eta, \beta)dx \quad and\ L(p) = \frac{1}{\mu}\int_0^q x\, f(x;\, \eta, \beta)dx$$

Where, $q = F^{-1}(p);\quad q\epsilon[0,1]$ and $\mu = E(X)$

$$\mu = E(X) = \frac{1}{1+\beta}\left(\sum_{m=0}^{\infty} c_m \frac{\eta\Gamma(m+1)}{\beta^2\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{3\beta}{\eta^3}\right)$$

$$B(p) = \frac{1}{p\mu}\int_0^q x\, \frac{\eta}{1+\beta}\left(e^{\beta x} e^{-\frac{\eta}{\beta}(e^{\beta x}-1)} + \frac{\eta^2\beta}{2} x^2 e^{-\eta x}\right) dx$$

Then, using the following geometric Series expansion is defined as

$$(1+x)^{-n} = \sum_{j=0}^{\infty}\binom{-n}{j} x^j$$

Using the following lower incomplete gamma function is defined as

$$\gamma(s,x) = \int_0^x t^{s-1} e^{-t} dt$$

Substitute the limits of integration and simplify the expression.

$$B(p) = \frac{\left(\sum_{j=0}^{\infty}\binom{-1}{j}\gamma\left(j+1, \frac{\eta}{\beta}e^{\beta q} - 1\right) + \eta\beta\left[-\left(\frac{q^2}{2}(\eta q + 3) + \frac{3}{\eta}\left(q + \frac{1}{\eta}\right)\right)e^{-\eta q} + \frac{3}{\eta^2}\right]\right)}{p\left(\sum_{m=0}^{\infty} c_m \frac{\eta\Gamma(m+1)}{\beta^2\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{3\beta}{\eta^3}\right)}$$

$$L(p) = pB(p)$$

$$L(p) = \frac{\left(\sum_{j=0}^{\infty}\binom{-1}{j}\gamma\left(j+1, \frac{\eta}{\beta}e^{\beta q} - 1\right) + \eta\beta\left[-\left(\frac{q^2}{2}(\eta q + 3) + \frac{3}{\eta}\left(q + \frac{1}{\eta}\right)\right)e^{-\eta} + \frac{3}{\eta^2}\right]\right)}{\left(\sum_{m=0}^{\infty} c_m \frac{\eta\Gamma(m+1)}{\beta^2\left(\frac{\eta}{\beta}\right)^{m+1}} + \frac{3\beta}{\eta^3}\right)}$$

12. Estimations of Parameter

The MGG distribution parameter's maximum likelihood estimates provided in this section.

12.1 Maximum Likelihood estimation (MLE)

Consider $x_1, x_2, x_3, \dots, x_n$ be a random sample of size n from the mixture of Gompertz and gamma distribution with parameter η, β the likelihood function, which is defined as

$$L = (x; \eta, \beta) = \prod_{i=1}^{n} f(x_i; \eta, \beta) = \prod_{i=1}^{n} \frac{\eta}{1+\beta}\left(e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i} - 1\right)} + \frac{\eta^2 \beta}{2} {x_i}^2 e^{-\eta x_i}\right)$$

Then, the log-likelihood function is

$$\ell = \log L = n \log(\eta) - n\log(1+\beta) + \log \sum_{i=1}^{n}\left(e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i} - 1\right)} + \frac{\eta^2 \beta}{2} {x_i}^2 e^{-\eta x_i}\right)$$

Differentiating with respect to $\eta \ and \ \beta$

$$\frac{\partial \log L}{\partial \eta} = n\left(\frac{1}{\eta}\right) + \sum_{i=1}^{n} \frac{\left(-\frac{1}{\beta}\left(e^{\beta x_i} - 1\right) e^{-\frac{\eta}{\beta}\left(e^{\beta x_i} - 1\right)} + \beta {x_i}^2 e^{-\eta x_i}\left(\eta\left(1 - \frac{\eta x_i}{2}\right)\right)\right)}{\left(e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i} - 1\right)} + \frac{\eta^2 \beta}{2} {x_i}^2 e^{-\eta x_i}\right)} = 0 \quad (10)$$

$$\frac{\partial \log L}{\partial \beta} = -n\left(\frac{1}{\beta+1}\right) + \sum_{i=1}^{n} \frac{\left(x_i e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i} - 1\right)}\left(-\frac{\eta x_i e^{\beta x_i}}{\beta} + \frac{\eta\left(e^{\beta x_i} - 1\right)}{\beta^2}\right) + \frac{\eta^2 x_i^2 e^{-\eta x_i}}{2}\right)}{\left(e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i} - 1\right)} + \frac{\eta^2 \beta}{2} {x_i}^2 e^{-\eta x_i}\right)} = 0 \quad (11)$$

The maximum likelihood estimate of the parameters for the MGG distribution is provided by equations (10) and (11). The equation, however, cannot be solved analytically, so we used R programming and a data set to solve it numerically.

The asymptotic normality results are used to derive the confidence interval. Given that if $\hat{\lambda} = (\hat{\eta}, \hat{\beta})$ represents the MLE of $\lambda = (\eta, \beta)$, the results can be expressed as follows:

$\sqrt{n}\left(\hat{\lambda} - \lambda\right) \to N_2(0, I^{-1}(\lambda))$

In this case, $I(\lambda)$represents Fisher's Information Matrix.

$$I(\lambda) = -\frac{1}{n}\begin{pmatrix} E\left[\frac{\partial^2 \log L}{\partial \eta^2}\right] & E\left[\frac{\partial^2 \log L}{\partial \eta \partial \beta}\right] \\ E\left[\frac{\partial^2 \log L}{\partial \beta \partial \eta}\right] & E\left[\frac{\partial^2 \log L}{\partial \beta^2}\right] \end{pmatrix}$$

$$\left[\frac{\partial^2 \log L}{\partial \eta^2}\right] = -n\left(\frac{1}{\eta}\right) + \sum_{i=1}^{n} \frac{\left(\left(e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i - 1}\right)} + \frac{\eta^2 \beta}{2} {x_i}^2 e^{-\eta x_i}\right)\left(\frac{1}{\beta^2}\left(e^{\beta x_i - 1}\right)^2 e^{-\frac{\eta}{\beta}\left(e^{\beta x_i - 1}\right)} + \beta {x_i}^2 e^{-\eta x_i}\left(1 - 2\eta x_i + \frac{\eta^2 \beta}{2} {x_i}^2\right)\right) - \left(-\frac{1}{\beta}\left(e^{\beta x_i - 1}\right) e^{-\frac{\eta}{\beta}\left(e^{\beta x_i - 1}\right)} + \beta {x_i}^2 e^{-\eta x_i}\left(\eta\left(1 - \frac{\eta x_i}{2}\right)\right)\right)^2\right)}{\left(e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i - 1}\right)} + \frac{\eta^2 \beta}{2} {x_i}^2 e^{-\eta x_i}\right)^2}$$

$$\left[\frac{\partial^2 \log L}{\partial \beta^2}\right] = n\left(\frac{1}{(\beta+1)}\right) + \sum_{i=1}^{n} \frac{\left(\left(x_i e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i}-1\right)}\left(x_i - \frac{\eta x_i e^{\beta x_i}}{\beta} + \frac{\eta\left(e^{\beta x_i}-1\right)}{\beta^2}\right)\right)\left(-\frac{\eta x_i e^{\beta x_i}}{\beta} + \frac{2\eta x_i e^{\beta x_i}}{\beta^2} - \frac{2\eta e^{\beta x_i}}{\beta^3} + \frac{2\eta}{\beta^3}\right) - \left(x_i e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i}-1\right)}\left(-\frac{\eta x_i e^{\beta x_i}}{\beta} + \frac{\eta\left(e^{\beta x_i}-1\right)}{\beta^2}\right) + \frac{\eta^2 x_i^2 e^{-\eta x_i}}{2}\right)^2\right)}{\left(e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i}-1\right)} + \frac{\eta^2 \beta}{2} x_i^2 e^{-\eta x_i}\right)^2}$$

$$\left[\frac{\partial^2 \log}{\partial \eta \partial \beta}\right] = \sum_{i=1}^{n} \frac{\left(\begin{gathered}\left(e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i}-1\right)} + \frac{\eta^2 \beta}{2} x_i^2 e^{-\eta x_i}\right)\left(e^{-\frac{\eta}{\beta}\left(e^{\beta x_i}-1\right)}\left(-\frac{e^{\beta x_i}-1}{\beta^2} + \frac{x_i e^{\beta x_i}}{\beta} + \frac{\eta\left(e^{\beta x_i}-1\right)^2}{\beta^3} - \frac{\eta x_i e^{\beta x_i}\left(e^{\beta x_i}-1\right)}{\beta^2}\right) + x_i^2 e^{-\eta x_i}\left(\eta - \frac{\eta^2 x_i}{2}\right)\right) \\ -\left(-\frac{1}{\beta}\left(e^{\beta x_i}-1\right) e^{-\frac{\eta}{\beta}\left(e^{\beta x_i}-1\right)} + \beta x_i^2 e^{-\eta x_i}\left(\eta\left(1-\frac{\eta x_i}{2}\right)\right)\right)\left(x_i e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i}-1\right)}\left(-\frac{\eta x_i e^{\beta x_i}}{\beta} + \frac{\eta\left(e^{\beta x_i}-1\right)}{\beta^2}\right) + \frac{\eta^2 x_i^2 e^{-\eta x_i}}{2}\right)\end{gathered}\right)}{\left(e^{\beta x_i} e^{-\frac{\eta}{\beta}\left(e^{\beta x_i}-1\right)} + \frac{\eta^2 \beta}{2} x_i^2 e^{-\eta x_i}\right)^2}$$

13. Applications

Dat set 1: This data includes the life expectancy (in years) of forty patients with leukemia, a blood malignancy, from one of Saudi Arabia's Ministry of Health facilities, as published in [6]. 0.315,0.496,0.616,1.145,1.208,1.263,1.414,2.025,2.036,2.162,2.211,2.370,2.532,2.693,2.805,2.910,2.912,3.192,3.263,3.348,3.48,3.427,3.499,3.534,3.767,3.751,3.858,3.986,4.049,4.244,4.323, 4.381, 4.392,4.397,4.647,4.753,4.929,4.973,5.074,5.381.

Data set 2: The data under consideration are the life times of 19 leukemia patients who were treated by a certain drug [1]. The data are 1.013,1.034,1.109,1.169,1.226,1.509,1.533,1.563,1.716,1.929,1.965,2.061,2.344,2.546,2.626,2.778,2.951,3.413,4.118,5.136.

Data set 3: [20] The dataset included the survival rates of 121 breast cancer patients with were treated at a major hospital to 1929 to 1938 (Lee, 1992). (A1-kadim and Mahdi, 2018) has recently used this dataset. 0.3, 0.3, 4.0, 5.0, 5.6, 6.2, 6.3, 6.6, 6.8, 7.4, 7.5, 8.4, 8.4, 10.3,11.0, 11.8, 12.2, 12.3, 13.5, 14.4, 14.4, 14.8, 15.5, 15.7, 16.2, 16.3, 16.5, 16.8, 17.2, 17.3, 17.5,17.9, 19.8, 20.4, 20.9, 21.0, 21.0, 21.1, 23.0, 23.4, 23.6, 24.0, 24.0, 27.9, 28.2, 29.1, 30.0, 31.0,31.0, 32.0, 35.0, 35.0, 37.0, 37.0, 37.0, 38.0, 38.0, 38.0, 39.0, 39.0, 40.0, 40.0, 40.0, 41.0, 41.0,41.0, 42.0, 43.0, 43.0, 43.0, 44.0, 45.0, 45.0, 46.0, 46.0, 47.0, 48.0, 49.0, 51.0, 51.0, 51.0, 52.0,54.0, 55.0, 56.0, 57.0, 58.0, 59.0, 60.0, 60.0, 60.0, 61.0, 62.0, 65.0, 65.0, 67.0, 67.0, 68.0, 69.0,78.0, 80.0,83.0, 88.0, 89.0, 90.0, 93.0, 96.0, 103.0, 105.0, 109.0, 109.0, 111.0, 115.0, 117.0, 125.0,126.0, 127.0, 129.0, 129.0, 139.0, 154.0.

The Akaike Information Criteria (AIC), Bayesian Information Criteria (BIC), Akaike Information Criteria Corrected (AICC), and $-2\log L$.are used to compare the goodness of fit of the fitted distribution.

The following formula can be used to determine AIC, BIC, AICC, and $-2\log L$..

$$AIC = 2k - 2\log L, \qquad BIC = k\log n - 2\log L \ and \ AICC = AIC + \frac{2k(k+1)}{(n-k-1)}$$

Where, $k =$ number of parameters, n sample size and $-2\log L$ is the maximized value of loglikelihood function.

Table 1. The value of MLE's and goodness of fit criteria statistics for model selection based on cancer dataset.

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
Mixture of Gompertz and Gamma distribution	$\hat{\boldsymbol{\eta}} = \mathbf{2.445296e+07}$ $(\mathbf{4.094304e+03})$ $\hat{\boldsymbol{\beta}} = \mathbf{2.542801e-01}$ $(\mathbf{4.158017e-02})$	**-3894.226**	**-3890.266**	**-3886.896**	**-3980.5930**
Lindely	$\hat{\theta} = 0.2577071$ (0.06161721)	156.5028	158.5028	160.1664	158.6080
Shanker	$\hat{\theta} = 0.54972161$(0.05806214)	144.7945	155.9545	157.6181	156.0597
Rama	$\hat{\theta} = 1.10146523$(0.08055189)	143.3158	154.3158	147.1023	154.4210
Exponential	$\hat{\theta} = 0.31893857$(0.05107054)	167.1353	169.1353	170.7988	169.0405
Aradhana	$\hat{\theta} = 0.75060122$(0.07108124)	153.1793	155.1793	156.8682	155.2845
Akash	$\hat{\theta} = 0.79998356$ (0.0701655)	152.7582	154.7582	156.4471	154.8634
Ishita	$\hat{\theta} = 0.8047911$ (0.0642800)	151.6347	153.6347	155.3235	153.7399

Table 2. The value of MLE's and goodness of fit criteria statistics for model selection based on cancer dataset.

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
Mixture of Gompertz and Gamma distribution	$\hat{\boldsymbol{\eta}} = \mathbf{2.045989e+07}$ $(\mathbf{5.931642e+03})$ $\hat{\boldsymbol{\beta}} = \mathbf{3.344606e-01}$ $(\mathbf{7.925959e-02})$	**-1865.153**	**-1861.153**	**-1859.264**	**-1861.903**
Lindely	$\hat{\theta} = 0.7076860$ (0.1200725)	64.02158	66.02158	66.96602	66.2438
Shanker	$\hat{\theta} = 0.7124395$ (0.10777871)	63.08856	65.08856	66.033	65.3107
Rama	$\hat{\theta} = 1.3784229$ (0.1415338)	62.41991	64.41991	65.36435	64.6421
Exponential	$\hat{\theta} = 0.4463246$ (0.1023934)	68.65501	70.65501	71.59945	70.8772
Aradhana	$\hat{\theta} = 0.985545$ (0.135948)	60.60053	62.60053	63.54497	62.8227
Akash	$\hat{\theta} = 0.0297001$ (0.1317933)	62.69158	64.69158	65.63602	64.9138
Ishita	$\hat{\theta} = 0.9975990$ (0.1134076)	62.74297	64.74297	65.68741	64.9651

Table 3. The value of MLE's and goodness of fit criteria statistics for model selection based on cancer dataset.

Distribution	ML Estimates	-2log L	AIC	BIC	AICC
New Mixture of Gompertz and Gamma distribution	$\hat{\boldsymbol{\eta}} = \mathbf{2.042428e+07}$ $(\boldsymbol{NaN})$ $\hat{\boldsymbol{\beta}} = \mathbf{3.344612e-01}$ $(\mathbf{7.925974e-02})$	**-1864.954**	**-1860.954**	**-1856.109**	**-1038.8907**
Lindely	$\hat{\theta} = 0.042301604$ (0.002718848)	1160.863	1162.863	1165.659	1162.8966
Shanker	$\hat{\theta} = 0.043180645$ (0.002771516)	1165.784	1167.784	1170.586	1167.8176
Rama	$\hat{\theta} = 0.086335660$ (0.003923506)	1241.883	1243.883	1246.679	1243.9166
Exponential	$\hat{\theta} = 0.021597929$ (0.001959228)	1170.256	1172.256	1175.051	1172.2896
Aradhana	$\hat{\theta} = 0.042301604$ (0.002718848)	1187.828	1162.863	1165.659	1162.8966
Akash	$\hat{\theta} = 0.064664492$ (0.003390847)	1193.125	1195.125	1197.121	1195.1586
Ishita	$\hat{\theta} = 0.064904911$ (0.003403705)	1201.28	1203.28	1206.076	1203.3136

In comparison to the mixture of Gompertz and gamma distribution, Lindely, Shanker, Rama, Exponential, Aradhana, Akash, and Ishita distribution, it is evident from table 2,3 and 4 results that the (MGG) distribution has smaller AIC, BIC, AICC, and values. This suggests that the new mixture distribution fits the data better. Therefore, compared to the other distributions, the mixture of Gompertz and gamma distribution (MGGD) provides a better fit.

14. Conclusion

In this paper, a two-parameter distribution is called a MGGD, which is a mixture of two known distributions, the Gompertz and Gamma distributions. Some statistical properties of the moments, the moments-generating function, mean, variance, skewness, and kurtosis have been studied. A number of statistical characteristics of the proposed distribution have been derived, including order statistics, stochastic ordering, entropies, Bonferroni, and Lorenz curves, and the method of maximum likelihood estimation of the parameters has been estimated. The statistical approach of the cancer dataset was analyzed. Moreover, the derived distribution is applied to real data sets and compared with the other well-known distribution. Show that the result of the mixture of the Gompertz and Gamma distributions provides a better fit than other well-known distributions.

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