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Bayesian Transmuted Normal Distribution With β and σ Parameters Where X' s Are Correlated

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ABSTRACT

Transmuted distribution emerged as new form of distribution in the literature recently, this is due to influx and changing nature of data from the conventional structured to semi and unstructured data. The study developed new distribution in practical term by incorporating regression variables into normal distribution and direct Bayesian gradient Monte Carlo simulation (DBGMS). The data were subjected to multicollinearity in a low dimension with specified ρ and transmuted parameter λ were specified as 0.3, 0.6 and 0.9. The outcome of the study pointed to the fact that Bayes estimate and posterior mean of DBGMS is superior and more efficient to classical maximum likelihood estimates. The study therefore recommended DBGMS when data are multicollinear and transmuted distribution is in use.

Keywords: Bayesian, transmuted, gradient, multicollinearity, Gibbs

1. Introduction

Ayman and Kisten (2013) claimed that most of the distribution nowadays that are used to model and fit the data do not provide enough evidence for the precision of estimates and its goodness of fit, this may be due to Balaswamy (2018) who pointed out that data are generated from various sources as a result of advancement in technology. Normal distribution is a general distribution belonging to the family of exponential distribution which is often used in econometrics to model various forms of regressions. Recently, transmuted probability distribution features in many probability and Statistics literature due to advancement and complexity of data collection, generation and used. Many fields of studies have begun to adopt transmuted probability distribution to solve their complexity problems, Muhammad *etal,* (2020a).

Muhammad *etal,* (2020b) examined the performance of Bayes estimator using different loss function with reference to the posterior risk under the mixture of two components of transmuted Frechet distribution. In an attempt to derive skewed distribution, quadratic rank transmuted Shaw, *etal* (2009) had been adopted by many literature in many families of distributions. This is done by inducing a parameter to the baseline continuous distribution, Muhammad *etal,* (2020b). It was keenly observed that transmuted distribution is flexible in modelling and analysis of contemporary data such as the area of engineering, reliability, survival analysis and many more. The probability density function of a transmuted random variables (X, y) can be expressed as follow, Muhammad *etal,* (2020b) .

$$g(X, y) = f(X, y)[(1 + \lambda) - 2\lambda F(X, y)] \quad (1)$$

$$G(X, y) = (1 + \lambda)F(X, y) - \lambda F(X, y)^2 \quad (2)$$

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where λ is the transmuted parameter which is set as $0 < \lambda \leq 1$, if $\lambda = 0$, the distribution turns out to baseline distribution, X and y are greater than 0, $f(X, y)$ and $f(X, y)$ are probability density function(pdf) and cumulative distribution function (cdf) of baseline distribution respectively, $g(X, y)$ and $G(X, y)$ are the transmuted pdf and cdf respectively. Muhammad *etal* (2020b) adopted Bayesian paradigm of the transmuted Pareto distribution using different performance metrics and applied it to censored and uncensored data. They made use of Marcov chain Monte Carlo simulation to obtain the Bayes estimate due to non-closed form of the model. Shaw and Buckley (2009) adopted quadratic rank transmutation approach to formulate new distribution, this is due to the fact that data are generated from various sources such as sensor, network, close circuit television, weblog and many more, the data may be structure, semi and unstructured with which the existing distribution may not be able to fit well Rahila, *etal* (2021).

Yousaf *etal* (2019) and Rahila *etal* (2021) examined transmuted Weibull distribution in a Bayesian paradigm by adopting uniform and informative gamma priors using three different loss functions(square error , quadratic and precautionary loss functions) and the relative importance of Bayes estimators using different loss function was examined as in Ali, (2015), problems of selecting appropriate priors and loss function for various sample sizes and concluded that Bayes estimates converged to assumed parameter values and observed property of consistency, Yousaf, *etal* (2018).

Amal *etal* (2021) observed that Bayes estimates in case of both minimum expected loss function when informative prior is used outperformed other estimates in most of the times. From the recent development in the transmutation of the distribution, none of the studies have applied it to regression modeling. The study aims at adapting Bayesian paradigm in transmuting normal distribution with normal-inverse gamma priors.

2.1 Transmuted Normal Distribution

The probability density function(pdf) and cumulative distribution function(cdf) are given as equation 3 and 4 respectively below:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-X\beta)^2}{2\sigma^2}} \tag{3}$$

$$F(x) = erf\left(\frac{y-X}{\sqrt{2\sigma^2}}\right) \tag{4}$$

where σ is the standard deviation with erf as error function(erf). Thus the pdf and cdf of transmuted distribution can be given in equations 5 and 6 respectively.

$$g(x) = f(X)[(1+\lambda) - 2\lambda F(X)] \tag{5}$$

$$G(x) = (1+\lambda)F(X) - \lambda F(X)^2 \tag{6}$$

The random variable X and y are said to have the transmuted normal distribution (TND) with parameters β, σ, λ where λ parameter ranges $-1 \leq \lambda \leq 1$. Thus the regression transmuted normal distribution can be expressed as:

$$g(x, y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-X\beta)^2}{2\sigma^2}} \left[(1+\lambda) - 2\lambda erf\left(\frac{y-X\beta}{\sqrt{2\sigma^2}}\right)\right] \tag{7}$$

While cumulative distribution function is expressed as :

$$G(x, y) = (1+\lambda) erf\left(\frac{y-X\beta}{\sqrt{2\sigma^2}}\right) - \lambda\left[erf\left(\frac{y-X\beta}{\sqrt{2\sigma^2}}\right)\right]^2 \tag{8}$$

2.2 Likelihood function

Let X, y be set of observations of size n for transmuted normal distribution of a regression equation modeling.

$$L = (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{(y-X\beta)^2}{2\sigma^2}} \left[(1+\lambda) - 2\lambda erf\left(\frac{y-X\beta}{\sqrt{2\sigma^2}}\right)\right] \quad (9)$$

Then the log-likelihood is expressed as:

$$logL = -\frac{n}{2}log(2\pi) - nlog(\sigma) - \sum_{i=1}^{n}\frac{(y-X\beta)^2}{2\sigma^2} + \sum_{i=1}^{n} log\left[(1+\lambda) - 2\lambda erf\left(\frac{y-X\beta}{\sqrt{2\sigma^2}}\right)\right] \quad (10)$$

Then the derivative of logL with respect to β is expressed as:

$$\frac{dlogL}{d\beta} = -\sum_{i=1}^{n}\frac{-2X(y-X\beta)}{2\sigma^2} + \frac{\sum_{i=1}^{n}[-2\lambda f(x)]}{\sum_{i=1}^{n} log\left[(1+\lambda)-2\lambda erf\left(\frac{y-X\beta}{\sqrt{2\sigma^2}}\right)\right]} \quad (11)$$

$$\frac{dlogL}{d\beta} = \sum_{i=1}^{n}\frac{X(y-X\beta)}{\sigma^2} + \frac{\sum_{i=1}^{n}[-2\lambda f(x)]}{\sum_{i=1}^{n} log\left[(1+\lambda)-2\lambda er\ \left(\frac{y-X\beta}{\sqrt{2\sigma^2}}\right)\right]} \quad (12)$$

$$\frac{dlogL}{d\beta} = \frac{(y-X\beta)X}{\sigma^2} + \frac{2\lambda exp\left(-\frac{(y-X\beta)^2}{2\sigma^2}\right)X\sqrt{2}}{\sqrt{\pi}\sigma^2\left(1+\lambda - 2\lambda erf\left(\frac{(y-X\beta)\sqrt{2})}{2\sqrt{\sigma^2}}\right)\right)}$$

(13)

$$\frac{dlogL}{d\sigma} = -\sum_{i=1}^{n}\frac{(y-X\beta)^2}{2\sigma^4} + \frac{\sum_{i=1}^{n}[-2\lambda f(x)]}{\sum_{i=1}^{n} log\left[(1+\lambda)-2\lambda er\ \left(\frac{y-X\beta}{\sqrt{2\sigma^2}}\right)\right]} \quad (14)$$

$$\frac{dlogL}{d\sigma} = -\frac{n}{\sigma} + \frac{(y-X\beta)^2}{\sigma^3} + \frac{2\lambda exp(-\frac{(y-X\beta)^2}{2\sigma^2}(y-X\beta)\sqrt{2}\sigma}{\sqrt{\pi}(\sigma^2)^{(3/2)}\left(1+\lambda - 2\lambda erf\left(\frac{(y-X\beta)\sqrt{2}}{2\sqrt{\sigma^2}}\right)\right)}$$

(15)

Due to complexity of the model and its non-closed form, the study adopted maximum likelihood estimator using Newton Raphson's algorithm to obtain maximum likelihood estimates of β and σ.

3.0 Bayesian Inference

The study examines the Bayesian analysis of transmuted normal distribution in this section, the posterior density is derived under conjugate priors using multivariate normal and inverse-gamma distributions.

The priors are expressed as expressed as follow:

$$p(\sigma^2) = \frac{b^a}{\Gamma(a)}(\sigma^2)^{-(a+1)}(-\frac{b}{\sigma^2}) \quad (16)$$

$$p(\beta) = d\ (2\pi|\Sigma|)^{-\frac{1}{2}}\exp\left(-\frac{1}{2}(\beta-\mu)'(\Sigma^{-1})(\beta-\mu)\right) \quad (17)$$

$$\mu = \exp(X\beta) \quad (18)$$

$\lambda \sim uniform(0,1)$

The joint posterior distribution of the parameters β and σ given data X and y can be expressed as :

$$p(\beta,\sigma|X,y) \propto L(X,y)p(\beta)p(\sigma^2)$$

$$p(\beta,\sigma|X,y) \propto (2\pi\sigma^2)^{-\frac{n}{2}} e^{-\frac{(y-X\ \ )^2}{2\sigma^2}} \left[(1+\lambda) - 2\lambda erf\left(\frac{y-X\beta}{\sqrt{2\sigma^2}}\right)\right] \times d\ (2\pi|\Sigma|)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\beta - \mu)'(\Sigma^{-1})(\beta-\mu)\right) \times \frac{b^a}{\Gamma(a)}(\sigma^2)^{-(a+1)}(-\frac{b}{\sigma^2}) \quad (19)$$

$$p(\beta,\sigma|X,y) \propto (2\pi\sigma^2)^{-\frac{n}{2}} \times d\ (2\pi|\Sigma|)^{-\frac{1}{2}} e^{-\frac{(y-X\beta)^2}{2\sigma^2}} \left(-\frac{1}{2}(\beta-\mu)'(\Sigma^{-1})(\beta-\mu)\right)\left[(1+\lambda) - 2\lambda erf\left(\frac{y-X\beta}{\sqrt{2\sigma^2}}\right)\right] \times \frac{b^a}{\Gamma(a)}(\sigma^2)^{-(a+1)}(-\frac{b}{\sigma^2}) \quad (20)$$

The marginal posterior density of parameters β and σ can be obtained as$\int_\beta \int_\sigma p(\beta,\sigma|X,y)d\beta d\sigma$, it is observed that posterior distribution has no closed form thereby the study adopted Direct Gradient Bayesian Monte Carlo simulation(DBGMS) which has superior advantage of Markov Chain Monte Carlo Simulation(MCMCS), since there are no need of burn-in and thinning.

3. Simulation Study and Data Generation Processes.

This section presents the outcome of the simulation study that examined the performances of Bayesian transmuted normal distribution as compared to classical Maximum Likelihood Estimation(MLE) using the slice balance loss function, (Oloyede, 2022) and Quadractic loss function, sample size was set as $n = 30$ to capture the problem of small sample size. Random variables $X's$ were generated by setting ρ as $0.1 \leq \rho \leq 0.9$, the covariance matrix p with specified value of ρ was decomposed with single value decomposition (SVD) and thereafter added to randomly generated variables. This ensured the same covariance matrix before and after the data had been generated. The study employed this strategy to stabilize the presence of multicollinearity in the dataset which serves as data uncertainty.

4. Data analysis and interpretation

Table 4.1 showing slice balance loss function

		Classical	**Direct Bayes gradient**	
λ	ρ	Monte	Bayes	Post. mean
	0.1	0.6328	0.0064	0.0076
	0.2	533.65	0.0245	0.0229
	0.3	17.10	0.0076	0.0087
	0.4	5.41	0.0437	0.0407
	0.5	22.672	0.0653	0.0618
0.3	0.6	41.36	0.0309	0.0313
	0.7	68.46	0.0318	0.0329
	0.8	234.78	0.0231	0.0272
	0.9	940.28	0.0258	0.0308
	0.1	24.55	0.0039	0.0044
	0.2	6.92	0.0224	0.018
	0.3	15.63	0.015	0.0155
	0.4	40.24	0.1404	0.1281
0.6	0.5	1784.24	0.1997	0.183
	0.6	21.81	0.0385	0.0381
	0.7	67.22	0.0597	0.0583
	0.8	35.76	0.0335	0.0371

	0.9	27.20	0.041	0.0448
0.9	0.1	8.92	0.0213	0.0204
	0.2	24.17	0.0782	0.0675
	0.3	25.53	0.032	0.0311
	0.4	9.83	0.2033	0.1827
	0.5	44.795	0.2022	0.1845
	0.6	17.72	0.0807	0.076
	0.7	24.17	0.0971	0.0907
	0.8	631.32	0.0416	0.0426
	0.9	218.63	0.0362	0.0397

In table 4.1 above, it was observed that DBGMS outperformed classical maximum likelihood estimation Monte Carlo simulation. This is observed based on the choice of transmuted parameter and coefficient infused in the data generated. Both Bayes estimate and posterior mean were compared with mle estimate under slice balance loss function, Oloyede(2022) which examined the precision of the estimate. The outcome of the study is in line with the findings of Amal *etal* (2021). The efficiency of Bayes estimate and posterior mean are both superior to classical maximum likelihood estimates.

Table 4.2 showing comparison of classical and Bayesian estimators using quadratic loss function

		Classical	Direct Bayes gradient	
λ	ρ	Monte	Bayes	Post. Mean
0.3	0.1	0.6328	0.0149	0.0177
	0.2	533.65	0.0574	0.0537
	0.3	17.10	0.0178	0.0203
	0.4	5.41	0.1022	0.0953
	0.5	22.672	0.1528	0.1446
	0.6	41.36	0.0725	0.0732
	0.7	68.46	0.0745	0.0770
	0.8	234.78	0.0541	0.0638
	0.9	940.28	0.0605	0.0722
0.6	0.1	24.55	0.0093	0.0103
	0.2	6.92	0.0524	0.042
	0.3	15.63	0.0351	0.0362
	0.4	40.24	0.3287	0.3
	0.5	1784.24	0.4675	0.4284
	0.6	21.81	0.0902	0.0893
	0.7	67.22	0.1398	0.1364
	0.8	35.76	0.0784	0.0869
	0.9	27.20	0.0959	0.1048
0.9	0.1	8.92	0.0498	0.0477
	0.2	24.17	0.1831	0.158
	0.3	25.53	0.075	0.0728
	0.4	9.83	0.4759	0.4278
	0.5	44.795	0.4733	0.432
	0.6	17.72	0.1888	0.1778
	0.7	24.17	0.2273	0.2124
	0.8	631.32	0.0975	0.0997
	0.9	218.63	0.0848	0.0929

Table 4.2 showing the comparison of classical and Bayesian gradient estimation using quadratic loss function. Quadratic loss function was adopted to compare DBGMS with respect to Bayes

estimate, posterior mean and classical mle, the study found out that DBGMS is more superior and efficient compare to classical Monte Carlo mle across the ρ and transmuted parameters.

5. Conclusion

The study has modeled transmuted normal distribution in a Bayesian framework, the study was able to develop transmuted normal distribution and direct Bayesian gradient Monte Carlo simulation using both slice balanced loss function and quadratic loss function to evaluate the performance of Bayesian framework and classical mle. The study concluded that Bayes estimate and posterior mean of DBGMCS is superior and efficient to classical maximum likelihood estate in a transmuted normal distribution.

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