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Theoretical Bayes Approach to the Parameter Estimation of Himanshu Distribution

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ABSTRACT

In this paper, Himanshu distribution is considered for Bayesian analysis. The expressions for Bayes estimators of the parameter have been derived under squared error, precautionary, entropy, K-loss, Al-Bayyati's loss, DeGroot and minimum expected loss functions by using beta prior.

Keywords: Bayesian method, Himanshu distribution, beta prior, squared error, precautionary, entropy, K-loss, Al-Bayyati's loss, DeGroot and minimum expected loss functions.

1. Introduction

The Himanshu distribution was first proposed by Abhishek Agarwal & Himanshu Pandey [1]. The probability density function of this distribution is given by

$$f\left(x;\theta\right)=\theta^{r}\left(1-\theta^{r}\right)^{x}\quad ;\begin{cases}x=0,1,2,.....\\ r\in I\end{cases}\tag{1}$$

The joint density function or likelihood function of (1) is given by

$$f\left(\underline{x};\theta\right)=\left(\theta^{r}\right)^{n}\left(1-\theta^{r}\right)^{\sum_{i=1}^{n}x_i}\tag{2}$$

The log likelihood function is given by

$$log\, f\left(\underline{x};\theta\right)=n\,log\left(\theta^{r}\right)+\sum_{i=1}^{n}x_i\,log\left(1-\theta^{r}\right)\tag{3}$$

Differentiating (3) with respect to θ and equating to zero, we get the maximum likelihood estimator of θ which is given by

$$\hat{\theta}=\left(n\Big/\left(n+\sum_{i=1}^{n}x_i\right)\right)^{1/r}.\tag{4}$$

2. Bayesian method of estimation

The Bayesian inference procedures have been developed generally under squared error loss function

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$$L\left(\hat{\theta},\theta\right)=\left(\hat{\theta}-\theta\right)^{2}. \tag{5}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_S = E(\theta). \tag{6}$$

Zellner [2], Basu and Ebrahimi [3] have recognized that the inappropriateness of using symmetric loss function. Norstrom [4] introduced precautionary loss function is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^{2}}{\hat{\theta}}. \tag{7}$$

The Bayes estimator under this loss function is denoted by $\hat{\theta}_P$ and is obtained as

$$\hat{\theta}_P = \left[E\left(\theta^2\right)\right]^{1/2}. \tag{8}$$

Calabria and Pulcini [5] points out that a useful asymmetric loss function is the entropy loss

$$L(\Delta)\propto\left[\Delta^{p}-p\ \log_e(\Delta)-1\right]$$

where $\Delta=\frac{\hat{\theta}}{\theta}$, and whose minimum occurs at $\hat{\theta}=\theta.$ Also, the loss function $L(\Delta)$ has been used in Dey et al. [6] and Dey and Liu [7], in the original form having $p=1.$ Thus $L(\Delta)$ can written be as

$$L(\Delta)=b\left[\Delta-\log_e(\Delta)-1\right];\ b>0. \tag{9}$$

The Bayes estimator under entropy loss function is denoted by $\hat{\theta}_E$ and is obtained by solving the following equation

$$\hat{\theta}_E=\left[E\left(\frac{1}{\theta}\right)\right]^{-1}. \tag{10}$$

Wasan [8] proposed the K-loss function which is given as

$$L\left(\hat{\theta},\theta\right)=\frac{\left(\hat{\theta}-\theta\right)^{2}}{\hat{\theta}\theta}. \tag{11}$$

Under K-loss function the Bayes estimator of θ is denoted by $\hat{\theta}_K$ and is obtained as

$$\hat{\theta}_K=\left[\frac{E(\theta)}{E(1/\theta)}\right]^{\frac{1}{2}}. \tag{12}$$

Al-Bayyati [9] introduced a new loss function which is given as

$$L\left(\hat{\theta},\theta\right)=\theta^{c}\left(\hat{\theta}-\theta\right)^{2}. \tag{13}$$

Under Al-Bayyati's loss function the Bayes estimator of θ is denoted by $\hat{\theta}_{Al}$ and is obtained as

$$\hat{\theta}_{Al} = \frac{E\left(\theta^{c+1}\right)}{E\left(\theta^{c}\right)}. \tag{14}$$

DeGroot [10] introduced several types of loss functions and he obtains Bayes estimators under this loss function. An example of a symmetric loss function is the DeGroot loss function defined by

$$L\left(\hat{\theta},\theta\right) = \left(\frac{\theta-\hat{\theta}}{\hat{\theta}}\right)^2 \tag{15}$$

Bayes estimator of θ under DeGroot loss function is denoted by $\hat{\theta}_{DG}$ and is obtained as

$$\hat{\theta}_{DG} = \frac{E\left(\theta^{2}\right)}{E\left(\theta\right)} \tag{16}$$

Consider the minimum expected loss function

$$L\left(\hat{\theta},\theta\right) = \left(\frac{\theta-\hat{\theta}}{\theta}\right)^2 \tag{17}$$

This loss function was used by Zellner [11] for estimating functions of parameters in econometric models and by Singh [12] for estimating the unknown parameter and reliability of the exponential distribution. Bayes estimator of θ under minimum expected loss function is denoted by $\hat{\theta}_{MEl}$ and is obtained as

$$\hat{\theta}_{MEl} = \frac{E\left(\theta^{-1}\right)}{E\left(\theta^{-2}\right)} \tag{18}$$

Let us consider beta prior distribution of θ to obtain the Bayes estimators.

$$g\left(\theta\right) = \frac{\left(\theta^{r}\right)^{a-1}\left(1-\theta^{r}\right)^{b-1}}{B\left(a,b\right)} \quad ; \; 0<\theta<1. \tag{19}$$

3. Posterior density

The posterior density of θ under beta prior, on using (2), is given by

$$f\left(\theta/\underline{x}\right) = \frac{\frac{1}{B\left(a,b\right)}\left(\theta^{r}\right)^{n}\left(1-\theta^{r}\right)^{\sum_{i=1}^{n}x_i}\left(\theta^{r}\right)^{a-1}\left(1-\theta^{r}\right)^{b-1}}{\int_0^1 \frac{1}{B\left(a,b\right)}\left(\theta^{r}\right)^{n}\left(1-\theta^{r}\right)^{\sum_{i=1}^{n}x_i}\left(\theta^{r}\right)^{a-1}\left(1-\theta^{r}\right)^{b-1}d\theta^{r}}$$

$$= \frac{1}{B\left(n+a,b+\sum_{i=1}^{n}x_i\right)}\left(\theta^{r}\right)^{n+a-1}\left(1-\theta^{r}\right)^{\left(b+\sum_{i=1}^{n}x_i\right)-1} \tag{20}$$

Theorem: On using (20), we have

$$E\left(\theta^{r}\right)^{c}=\frac{B\left(n+a+c,b+\sum_{i=1}^{n}x_{i}\right)}{B\left(n+a,b+\sum_{i=1}^{n}x_{i}\right)} \tag{21}$$

Proof. By definition,

$$E\left(\theta^{r}\right)^{c}=\int\theta^{rc}f\left(\theta/\underline{x}\right)d\theta$$

$$=\int_{0}^{\infty}\theta^{rc}\ \frac{1}{B\left(n+a,b+\sum_{i=1}^{n}x_{i}\right)}\ \left(\theta^{r}\right)^{n+a-1}\left(1-\theta^{r}\right)^{\left(b+\sum_{i=1}^{n}x_{i}\right)-1}d\theta^{r}$$

$$=\frac{B\left(n+a+c,b+\sum_{i=1}^{n}x_{i}\right)}{B\left(n+a,b+\sum_{i=1}^{n}x_{i}\right)}\ .$$

From equation (21), for $c=1$, we have

$$E\left(\theta^{r}\right)=\frac{B\left(n+a+1,b+\sum_{i=1}^{n}x_{i}\right)}{B\left(n+a,b+\sum_{i=1}^{n}x_{i}\right)} \tag{22}$$

From equation (21), for $c=2$, we have

$$E\left(\theta^{r}\right)^{2}=\frac{B\left(n+a+2,b+\sum_{i=1}^{n}x_{i}\right)}{B\left(n+a,b+\sum_{i=1}^{n}x_{i}\right)} \tag{23}$$

From equation (21), for $c=-1$, we have

$$E\left(\theta^{r}\right)^{-1}=\frac{B\left(n+a-1,b+\sum_{i=1}^{n}x_{i}\right)}{B\left(n+a,b+\sum_{i=1}^{n}x_{i}\right)} \tag{24}$$

From equation (21), for $c=-2$, we have

$$E\left(\theta^{r}\right)^{-2}=\frac{B\left(n+a-2,b+\sum_{i=1}^{n}x_{i}\right)}{B\left(n+a,b+\sum_{i=1}^{n}x_{i}\right)} \tag{25}$$

From equation (21), for $c = c+1$, we have

$$E\left(\theta^r\right)^{c+1} = \frac{B\left(n+a+c+1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)} \tag{26}$$

4. Bayes estimators

From equation (6), on using (22), the Bayes estimator of θ^r under squared error loss function is given by

$$\hat{\theta}_S^r = E\left(\theta^r\right)$$

$$= \frac{B\left(n+a+1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)}$$

$$\Rightarrow \quad \hat{\theta}_S = \left[\frac{B\left(n+a+1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)}\right]^{\frac{1}{r}} \tag{27}$$

From equation (8), on using (23), the Bayes estimator of θ^r under precautionary loss function is given by

$$\hat{\theta}_P^r = \left[E\left(\theta^r\right)^2\right]^{\frac{1}{2}}$$

$$= \left[\frac{B\left(n+a+2, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)}\right]^{\frac{1}{2}}$$

$$\Rightarrow \quad \hat{\theta}_P = \left[\frac{B\left(n+a+2, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)}\right]^{\frac{1}{2r}} \tag{28}$$

From equation (10), on using (24), the Bayes estimator of θ^r under entropy loss function is given by

$$\hat{\theta}^r_E = \left[E\left(\theta^r\right)^{-1} \right]^{-1}$$

$$= \left[\frac{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a-1, b+\sum_{i=1}^{n} x_i\right)} \right]$$

$$\Rightarrow \quad \hat{\theta}_E = \left[\frac{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a-1, b+\sum_{i=1}^{n} x_i\right)} \right]^{\frac{1}{r}} \tag{29}$$

From equation (12), on using (22) and (24), the Bayes estimator of θ^r under K-loss function is given by

$$\hat{\theta}^r_K = \left[\frac{E\left(\theta^r\right)}{E\left(\theta^r\right)^{-1}} \right]^{\frac{1}{2}}$$

$$= \left[\frac{B\left(n+a+1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)} \Bigg/ \frac{B\left(n+a-1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)} \right]^{\frac{1}{2}}$$

$$\Rightarrow \quad \hat{\theta}_K = \left[\frac{B\left(n+a+1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a-1, b+\sum_{i=1}^{n} x_i\right)} \right]^{\frac{1}{2r}} \tag{30}$$

From equation (14), on using (21) and (26), the Bayes estimator of θ^r under Al-Bayyati's loss function is given by

$$\hat{\theta}^r_{Al} = \frac{E\left(\theta^r\right)^{c+1}}{E\left(\theta^r\right)^{c}}.$$

$$= \frac{B\left(n+a+c+1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)} \Bigg/ \frac{B\left(n+a+c, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)}$$

$$\Rightarrow \quad \hat{\theta}_{Al} = \left[\frac{B\left(n+a+c+1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a+c, b+\sum_{i=1}^{n} x_i\right)} \right]^{\frac{1}{r}} \tag{31}$$

From equation (16), on using (22) and (23), the Bayes estimator of θ^r under DeGroot loss function is given by

$$\hat{\theta}^r_{DG} = \frac{E\left(\theta^r\right)^2}{E\left(\theta^r\right)}$$

$$= \frac{B\left(n+a+2, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)} \Bigg/ \frac{B\left(n+a+1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)}.$$

$$\Rightarrow \quad \hat{\theta}_{DG} = \left[\frac{B\left(n+a+2, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a+1, b+\sum_{i=1}^{n} x_i\right)} \right]^{\frac{1}{r}} \tag{32}$$

From equation (18), on using (24) and (25), the Bayes estimator of θ^r under minimum expected loss function is given by

$$\hat{\theta}^r_{MEL} = \frac{E\left(\theta^r\right)^{-1}}{E\left(\theta^r\right)^{-2}}$$

$$= \frac{B\left(n+a-1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)} \Bigg/ \frac{B\left(n+a-2, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a, b+\sum_{i=1}^{n} x_i\right)}.$$

$$\Rightarrow \quad \hat{\theta}_{MEL} = \left[\frac{B\left(n+a-1, b+\sum_{i=1}^{n} x_i\right)}{B\left(n+a-2, b+\sum_{i=1}^{n} x_i\right)} \right]^{\frac{1}{r}} \tag{33}$$

5. Conclusion:

In this paper, we have obtained a number of estimators for parameter of Himanshu distribution. In equation (4) we have obtained the maximum likelihood estimator of the parameter. In equation (27) (28), (29), (30), (31), (32), and (33) we have obtained the Bayes estimators under different loss functions using beta prior. In the above equation, it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

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