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Asymptotic Confidence Intervals for MRL and TVaR of the Inverse Weibull Distribution with Type II Censored Data

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ABSTRACT

We consider interval estimation for the parameters, the Mean Residual Life (MRL) and the Tail Value at Risk (TVaR) of the inverse Weibull distribution. We constructed the confidence intervals based on the asymptotic normality of the maximum likelihood estimator in addition to intervals based on the asymptotic Chi-Square distribution of the likelihood ratio (LR) statistic under Type II Censoring. The performance of intervals is investigated and compared using the lower, upper, and total error rates, in addition to expected lengths using simulation. We found that LR intervals showed better overall performance and are more accurate than confidence intervals based on the asymptotic normality of maximum likelihood estimator.

Keywords: Likelihood ratio interval; Mean Residual Life; Tail Value at Risk; Inverse Weibull distribution; Type II Censoring

1. Introduction

In most studies, researchers used confidence intervals based on the asymptotic normal distribution theory of the maximum likelihood estimators (MLE) to get the upper and lower confidence limits for the distribution parameters and functions of the parameters. However, these intervals are generally inaccurate for small sample sizes. For that reason, researchers prefer to use confidence intervals based on the likelihood ratio statistic. In this article we will derive and investigate the performance of both types of intervals for the parameters and some important functions of the parameters of the inverse Weibull distribution.

The Inverse Weibull (IW) distribution; named sometimes complementary Weibull distribution, or reverse Weibull distribution offers a versatile distribution useful to model lifetime data. It was introduced by Keller and Kamath (1982) as an appropriate model to explain deterioration manifestations of mechanical ingredients of diesel engines like pistons and crankshafts. This distribution is found to be useful in several other areas like medical, engineering, industrial and social sciences.

The Inverse Weibull probability density function (pdf) for random variable X and two parameters, shape (α) and scale (λ), is written as:

$$f(x;\alpha,\lambda)=\alpha\lambda\exp\{-\lambda x^{-\alpha}\}x^{-(\alpha+1)}, x>0\,,\alpha>0\,,\lambda>0. \tag{1}$$

The survival function of the Inverse Weibull distribution is defined as:

$$S(x;\alpha,\lambda)=1-\exp\{-\lambda x^{-\alpha}\}, x>0\,,\alpha>0\,,\lambda>0. \tag{2}$$

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Several authors have considered inference on the parameters of this distribution under various types of data including Sultan et al. (2014); Helu (2015); Kumar and Kumar (2019) and Kazemi and Azizpoor (2021). Intervals based on the likelihood ratio statistic has gained a considerable attention in the literature. Doganaksoy (2021) used a property of the likelihood ratio statistic to obtain a new simple method for finding likelihood ratio confidence interval and explained it using the Weibull distribution. Alsheikh and Baklizi (2022) considered maximum likelihood estimation of the parameters, the mean residual life (MRL) and the tail value-at-risk (TVaR) for this model with type II censored data. In this paper we derived and studied the performance of the confidence intervals for Mean Residual Life (MRL) and Tail Value at Risk (TVaR) of the IW distribution based on type II censoring.

The Mean Residual Life (MRL) is a function of time t. It represents the expected future lifetime given that a component has survived until time t. It plays an important role in reliability and life testing. This function is an attractive alternative to the survival function or the hazard function of a survival time in practice, see Gupta (1981); Tang et al. (1999) and Hall and Wellner (2017).

The MRL function of IW distribution is derived in Alsheikh and Baklizi (2022) and is given by

$$m(t) = \int_t^\infty \frac{xf(x)}{S(t)}dx = \frac{\lambda^{\alpha^{-1}}\left[\Gamma\left(1-\alpha^{-1}\right)-\Gamma\left(1-\alpha^{-1},\lambda t^{-\alpha}\right)\right]}{1-\exp\{-\lambda t^{-\alpha}\}} \tag{3}$$

The TVaR is a statistical measure of risk associated with the more general value at risk (VaR) approach, which measures the maximum amount of loss that is expected with an investment portfolio over a specified period, with a degree of confidence. It is a measure of risk important in actuarial studies. See Christoffersen et al. (2001).

The Tail Value-at-Risk with confidence level p is defined as:

$$TVaR_p(X) = \frac{\int_{VaR_p(X)}^\infty xf(x)dx}{1-p}$$

Alsheikh and Baklizi (2022) derived the following expression for the Tail Value-at-Risk:

$$TVaR_p(x) = \left(\frac{-\lambda}{\ln}\right)^{\alpha^{-1}} + \frac{\lambda^{\alpha^{-1}}\left[\Gamma\left(1-\alpha^{-1}\right)-\Gamma\left(1-\alpha^{-1},-\ln p\right)\right]}{1-p} \tag{4}$$

2. Confidence Intervals Based on the Asymptotic Normality of the MLE

To find the confidence intervals for MRL and TVaR based on the asymptotic distribution of a function of the maximum likelihood estimator with type II censored data (Bhtattachariya, 1985). Using the delta method (Lawaless, 2003), the variance of a function of the MLE, say $k\left(\hat{\alpha},\hat{\lambda}\right)$ can be found as:

$$var\left(k\left(\hat{\alpha},\hat{\lambda}\right)\right) = var(\hat{\alpha})\left(\frac{\partial k}{\partial \alpha}\right)^2\Big|_{\hat{\alpha},\hat{\lambda}} + 2\frac{\partial k}{\partial \alpha}\Big|_{\hat{\alpha},\hat{\lambda}}\frac{\partial k}{\partial \lambda}\Big|_{\hat{\alpha},\hat{\lambda}} cov\left(\hat{\alpha},\hat{\lambda}\right) + var\left(\hat{\lambda}\right)\left(\frac{\partial k}{\partial \lambda}\right)^2\Big|_{\hat{\alpha},\hat{\lambda}} \tag{5}$$

Where $\hat{\alpha}$ and $\hat{\lambda}$ are the maximum likelihood estimators (MLEs) of the parameters $\alpha\ and\ \lambda$. The variances of the parameters MLEs and their covariance are obtained from the inverse of the observed information matrix.

To find the first derivative of MRL or $TVaR$ with respect to the parameters, it is necessary to find the derivative of incomplete gamma function for these parameters as the following:

$$\frac{\partial \Gamma(\mathrm{s,x})}{\partial \mathrm{x}} = -\mathrm{x}^{\mathrm{s}-1}\exp[-\mathrm{x}],$$

$$\frac{\partial \Gamma(\mathrm{s,x})}{\partial s} = \ln \mathrm{x}\, \Gamma(\mathrm{s},\mathrm{x}) + \mathrm{x}\, \mathrm{T}(3,\mathrm{s},\mathrm{x}),$$

where the function $\mathrm{T}(3,\mathrm{s},\mathrm{x})$ is a special case of the Meijer G-function,

$$\mathrm{T}(\mathrm{m},\mathrm{s},\mathrm{x}) = \mathrm{G}_{\mathrm{m}-1,\mathrm{m}}^{\mathrm{m},\ 0}\left(\begin{matrix} 0,0,\dots,0 \\ \mathrm{s}-1,-1,\dots,-1 \end{matrix}\middle| \mathrm{x}\right).$$

To simplify the expressions for the derivatives of the MRL and TVaR, let

$t_1 = \exp[-\lambda t^{-\alpha}]$, $t_2 = \lambda^{\alpha^{-1}}$, $t_3 = \lambda\, \mathrm{t}^{-\alpha} \ln t$, $t_4 = \alpha^{-2}$, $t_5 = (\alpha\lambda)^{-1}$, $t_6 = \mathrm{t}^{-\alpha}$, $t_7 = \frac{-\lambda}{\ln p}$,
$G = \Gamma(1-\alpha^{-1})$, $GD = \Gamma'(1-\alpha^{-1})$, $IG = \Gamma(1-\alpha^{-1}, \lambda \mathrm{t}^{-\alpha})$, $IG1 = \Gamma(1-\alpha^{-1}, -\ln p)$,
$ShapeIGD = \frac{\partial \Gamma(\mathrm{s,x})}{\partial \mathrm{x}}$; $qIGD = \frac{\partial \Gamma(\mathrm{s,x})}{\partial s}$ where $s = 1-\alpha^{-1}$ *and* $x = \lambda \mathrm{t}^{-\alpha}$, $\alpha > 1, \lambda > 0, \mathrm{t} > 0$,
$qIGD1 = \frac{\partial \Gamma(\mathrm{s,y})}{\partial \mathrm{y}}$, *where* $y = -\ln p$.

Then, the first derivatives of MRL with respect to parameters are the following:

$$\frac{\partial m}{\partial \alpha} = \frac{t_2\{(\mathrm{G}-\mathrm{IG})[(t_1-1)t_4 \ln\lambda + t_3 t_1] + [t_4(GD - qIGD) + t_3 ShapeIGD](1-t_1)\}}{(1-t_1)^2},$$

$$\frac{\partial m}{\partial \lambda} = \frac{t_2\{(\mathrm{G}-\mathrm{IG})[(1-t_1)t_5 - t_6 t_1] - t_6 ShapeIGD(1-t_1)\}}{(1-t_1)^2},$$

and the first derivatives of TVaR with respect to parameters are as follows:

$$\frac{\partial TVaR}{\partial \alpha} = t_4\left\{\frac{t_2}{1-p}(GD - qIGD1 - \ln\lambda\,[G - IG1]) - t_7^{\alpha^{-1}} \ln t_7\right\},$$

$$\frac{\partial TVaR}{\partial \lambda} = t_5 TVaR(p),$$

Then $100(1-\gamma)\%$ confidence intervals for MRL and TVaR based on MLE can be written as:

$$\widehat{m}(t) \pm z_{\gamma/2}\sqrt{\widehat{Var}\big(\widehat{m}(t)\big)}, \tag{6}$$

$$\widehat{TVaR}(x) \pm z_{\gamma/2}\sqrt{\widehat{Var}\left(\widehat{TVaR}(x)\right)}, \tag{7}$$

where $z_{\gamma/2}$ is the $(\gamma/2)$ quantile of the standard normal distribution, $\widehat{Var}\big(\widehat{m}(t)\big)$ and $\widehat{Var}\left(\widehat{TVaR}(x)\right)$ can be obtained by substituting the appropriate quantities in the general expression for the asymptotic variance of a function of the MLE given in (5).

3. Likelihood Ratio Intervals

The likelihood ratio (LR) statistic for θ is given by:

$$W(\theta) = -2[l(\theta, \tilde{\lambda}_\theta) - l(\hat{\theta}, \hat{\lambda})] \tag{8}$$

where θ is the scalar parameter of interest, λ is vector of ρ nuisance parameters, $(\hat{\theta}, \hat{\lambda})$ are the MLEs of (θ, λ), $\tilde{\lambda}_\theta$ is constrained MLE of λ for a given value of θ, $l(\theta, \lambda)$ is the log-likelihood function, $l(\theta, \tilde{\lambda}_\theta)$ is the profile log-likelihood function for θ, and $l(\hat{\theta}, \hat{\lambda})$ is the maximized value of the log-likelihood function.

Assuming the the log-likelihood function is unimodal, the lower and upper $100(1-\gamma)\%$ LR confidence limits are the two values of θ that satisfy:

$$l(\theta, \tilde{\lambda}_\theta) = \kappa,$$

where

$$\kappa = l(\hat{\theta}, \hat{\lambda}) - \left(\frac{1}{2}\right) \chi^2_{1;1-\gamma}.$$

Here $\chi^2_{1;1-\gamma}$ is the $(1-\gamma)$ quantile of the chi-square distribution with one degree of freedom, so, κ is a constant when data are observed.

Doganaksoy (2021) proposed a new method to construct numerically the confidence limits of likelihood ratio intervals. This method is based on the observation that the lower (upper) LR confidence limit for θ is the smallest (Largest) value of θ that satisfy $l(\theta, \lambda) = \kappa$. Therefore, the lower (upper) limit for θ can be obtained by

$$\text{Min (Max) } \theta \text{ subject to } l(\theta, \lambda) = \kappa \tag{9}$$

with (θ, λ) handled as the optimization variables.

Equation (9) can be extended to present a formula of LR confidence limits on a function $\emptyset(\theta, \lambda)$ of model parameters. The lower (upper) LR confidence limit for $\emptyset(\theta, \lambda)$ can obtained by

$$\text{Min (Max) } \emptyset(\theta, \lambda) \text{ subject to } l(\theta, \lambda) = \kappa \tag{10}$$

This new method is easier to compute numerically than other traditional methods. Note that the new method proposed by Doganaksoy (2021) given by (9) and (10), does not include explicit calculation of $\tilde{\lambda}_\theta$. The final solution is achieved at $\lambda = \tilde{\lambda}_\theta$ even if this condition is not imposed at the beginning.

Confidence intervals for MRL and TVaR based on likelihood ratio intervals are constructed numerically by using the new method in (31) by using the "gosolnp" function of the R package "Rsolnp" to solve the nonlinear constrained optimization problems.

4. Simulation Study and Results

The simulation design is as follows. We used 2000 replications. We used all combinations of sample size ($n = 50, 80, 100$), the first r failure times ($r = 0.6n, 0.8n, n$), confidence level ($\gamma = 0.05$ and 0.1), $p = 0.95$ (for TVaR), and time t = 3 (for MRL). We investigated the performance of the intervals using the expected lengths and the error rates.

To judge the accuracy of an interval and to compare competing confidence intervals we will use the expected length of the interval (EL) and the lower (LER), upper (UER), total (TER) error rates.

Assuming that we have (N) simulation replications, each resulting in a confidence interval from each type, then the simulated value of the expected length is calculated as

$$EL = \frac{\sum_{i=1}^{N}(Upper\ Limit - Lowe\ \ Limit)}{N}$$

Intervals with smallest EL are considered the better.

The lower error rate (LER) is the proportion of times for which the parameter is less than the lower limits of its interval (LCI), while the upper error rate (UER) mean that the proportion of times in which the parameter is greater than the upper limit of its interval (UCI). The total error rate (TER) is the sum of lower and upper error of each interval.

Formulas used to estimate these errors rate for N intervals of a parameter θ are:

$$LER = \frac{\sum_{i=1}^{N}(\ \theta < LCI)}{N}$$

$$UER = \frac{\sum_{i=1}^{N}(\ \theta > UCI)}{N}$$

$$TER = LER + UER$$

The interval with closer error rates to the nominal ones is considered better. The results are given in Tables 1 and 2.

Table 1. Expected Average Length for LR and AN when $\alpha = 3,\ \lambda = 1, t = 3$ and $p = 0.95$

n	r		$\gamma = 0.05$				$\gamma = 0.1$			
		EL	α	λ	MRL	$TVaR$	α	λ	MRL	$TVaR$
50	30	LR	1.538	0.531	105.4	9.100	1.18	0.39	79.91	7.34
		AN	1.709	0.619	71.29	10.52	1.43	0.52	59.24	8.84
	40	LR	1.301	0.510	70.28	7.913	1.02	0.39	56.25	6.35
		AN	1.463	0.599	51.50	9.423	1.23	0.50	44.23	7.77
	50	LR	1.158	0.512	59.11	7.284	0.92	0.37	48.5	5.90
		AN	1.338	0.592	44.68	8.757	1.13	0.49	38.80	7.24
80	48	LR	1.120	0.417	58.48	6.891	0.83	0.34	46.27	5.60
		AN	1.320	0.488	45.14	8.385	1.11	0.41	37.48	6.99
	64	LR	0.937	0.390	43.56	6.033	0.65	0.30	34.89	4.98
		AN	1.142	0.472	34.86	7.436	0.96	0.40	28.96	6.26
	80	LR	0.847	0.380	38.85	5.550	0.60	0.29	30.83	4.62
		AN	1.049	0.466	31.62	6.861	0.88	0.39	25.86	5.81
100	60	LR	0.975	0.359	47.08	6.058	0.74	0.29	37.76	5.02
		AN	1.174	0.436	37.77	7.488	0.98	0.37	31.36	6.32
	80	LR	0.832	0.354	37.13	5.273	0.63	0.26	30.52	4.37
		AN	1.020	0.421	30.55	6.587	0.86	0.35	25.76	5.54
	100	LR	0.733	0.349	32.09	4.965	0.57	0.26	26.81	4.09
		AN	0.931	0.418	26.66	6.188	0.78	0.35	22.85	5.17

Table 2. Error Rates of likelihood intervals when $\alpha = 3,\ \lambda = 1,$ t = 3 and $p = 0.95$

n	r			$\gamma = 0.05$				$\gamma = 0.1$			
			ER	α	λ	MRL	$TVaR$	α	λ	MRL	$TVaR$
50	30	AN	LER	0.0345	0.0125	0	0	0.0635	0.0415	0	0
			UER	0.0165	0.045	0.0865	0.0425	0.0395	0.0780	0.1135	0.0705
			TER	0.051	0.0575	0.0865	0.0425	0.1030	0.1195	0.1135	0.0705
		LR	LER	0.135	0.1725	0.044	0.0195	0.1995	0.1940	0.0760	0.0385
			UER	0.0515	0.0695	0.0165	0.0345	0.1045	0.1835	0.0325	0.0715
			TER	0.1865	0.242	0.0605	0.054	0.3040	0.3775	0.1085	0.1100
	40	AN	LER	0.031	0.0165	0	0	0.0620	0.0305	0	0
			UER	0.023	0.041	0.088	0.031	0.0440	0.0715	0.0930	0.0570
			TER	0.054	0.0575	0.088	0.031	0.1060	0.1020	0.0930	0.0570
		LR	LER	0.1315	0.1615	0.043	0.0225	0.2005	0.1740	0.0720	0.0360
			UER	0.0555	0.0745	0.018	0.0305	0.1010	0.1565	0.0365	0.0640
			TER	0.187	0.236	0.061	0.053	0.3015	0.3305	0.1085	0.1000
	50	AN	LER	0.028	0.0145	0	0	0.0605	0.0385	0	0.0015
			UER	0.0225	0.041	0.089	0.0305	0.0385	0.0700	0.1085	0.0555
			TER	0.0505	0.0555	0.089	0.0305	0.0990	0.1085	0.1085	0.0570
		LR	LER	0.136	0.158	0.0355	0.0265	0.1925	0.1970	0.0740	0.0420
			UER	0.0605	0.0685	0.019	0.0315	0.1125	0.1645	0.0325	0.0645
			TER	0.1965	0.2265	0.0545	0.058	0.3050	0.3615	0.1065	0.1065
80	48	AN	LER	0.035	0.0195	0	0	0.0620	0.0430	0	0.0005
			UER	0.0195	0.0415	0.0915	0.039	0.0355	0.0635	0.1040	0.0470
			TER	0.0545	0.061	0.0915	0.039	0.0975	0.1065	0.1040	0.0475
		LR	LER	0.1495	0.1575	0.0445	0.028	0.2530	0.2175	0.0655	0.0430
			UER	0.07	0.089	0.0225	0.041	0.0875	0.1115	0.0365	0.0570
			TER	0.2195	0.2465	0.067	0.069	0.3405	0.3290	0.1020	0.1000
	64	AN	LER	0.031	0.017	0	0	0.0490	0.0450	0.0005	0.0025
			UER	0.0295	0.0295	0.082	0.021	0.0460	0.0540	0.1095	0.0320
			TER	0.0605	0.0465	0.082	0.021	0.0950	0.0990	0.1100	0.0345
		LR	LER	0.147	0.1515	0.033	0.0255	0.2400	0.2320	0.0540	0.0470
			UER	0.071	0.0855	0.0255	0.0265	0.1395	0.1165	0.0430	0.0450
			TER	0.245	0.237	0.0585	0.052	0.3795	0.3485	0.0970	0.0920
	80	AN	LER	0.029	0.0165	0	0	0.0490	0.0430	0.0045	0.0015
			UER	0.022	0.031	0.071	0.026	0.0435	0.0550	0.1065	0.0330
			TER	0.051	0.0475	0.071	0.026	0.0925	0.0980	0.1110	0.0345
		LR	LER	0.1765	0.1615	0.0325	0.0185	0.2190	0.2310	0.0505	0.0445
			UER	0.067	0.079	0.018	0.0355	0.1530	0.1270	0.0410	0.0490
			TER	0.2435	0.2405	0.0505	0.054	0.3720	0.3580	0.0915	0.0935

Table 1 presents results of expected length (EL) for likelihood ratio intervals (LR) and confidence intervals depending on the asymptotic normality of the MLE (AN). The results show that the expected length for LR tend to be less than AN in all cases except for MRL where the expected length results of AN are less than those of LR, especially with small sample size. When the sample size increases, the expected length values decreases for all cases.

The results in Table 2 show that error rates for confidence intervals depending on MLE for the parameters are symmetric and attain the nominal rates in almost all simulations. However, AN intervals for the MRL and TVaR are highly asymmetric. Moreover, the intervals for the TVaR are highly conservative in the sense that the actual error rates are considerably less than the nominal rates. On the other hand, LR intervals for the parameters are generally highly anticonservative, especially for small sample sizes. However, for the MRL and TVaR, the intervals tend to be symmetric and attain nominal error rates.

5. Conclusion and Suggestions for Further Research

In this paper, confidence intervals for the parameters, the Mean Residual Life (MRL) and the Tail Value at Risk (TVaR) of the Inverse Weibull distribution based on type II censored data are derived and their performance is studied. The intervals are compared using the lower, upper, total error rate, and the expected length using simulation.

The conclusion we get from this paper is that LR intervals give more accurate results than confidence intervals depending on MLE (AN) under for MRL and TVaR. However, for the parameters themselves the AN intervals appear to perform better.

A suggestion for further research is to explore the LR and AN intervals for different types of censoring data like hybrid censoring or progressive type II censoring and to see how they compare and give recommendations on their use in these situations.

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