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Characteristics of the GLLE Family of Lifetime Distributions, and Applications

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ABSTRACT

A Generalized Log-Logistic (GLL) family of lifetime distributions is one in which any pair of distributions are related through a GLL transformation, for some (non-negative) value of the transformation parameter κ (the odds function of the second distribution is the κ^{th} power of the odds function of the first distribution). We consider GLL families generated from an exponential distribution. We derive some useful characteristics of these distributions; moreover, we discuss the hazard rates and Kullback-Leibler divergence, then illustrate the usefulness of this distribution family by fitting real data sets.

Keywords: Generalized Log-Logistic Exponential distribution; properties of MLE's; hazard rates; Kullback-Leibler divergence; data fitting.

1. Introduction

A Generalized Log-Logistic (GLL) family of lifetime distributions is one in which any pair of distributions are related through a GLL transformation, defined below, for some (non-negative) value of the transformation parameter κ. These families of distributions were first introduced by Gleaton and Lynch [9].

In this paper, we explore the Generalized Log-Logistic (GLL) family of lifetime distributions generated from an exponential distribution (GLLE). First, we will derive some useful characteristics of the GLLE family of lifetime distributions; moreover, we will discuss the hazard rates and then we will illustrate the usefulness of this family of distributions by fitting real data sets.

The foundational work on the properties of lifetime distributions belonging to families generated by a GLL transformation was introduced by Gleaton and Lynch in their seminal paper [9]. Their contribution has led to significant advancements in various fields where the GLL family of distributions is widely used [1, 2, 3, 4, 6, 7, 8, 11, 12, 13, 14, 15, 17].

In [9], Gleaton and Lynch discussed properties of lifetime distributions belonging to families generated by a Generalized Log-Logistic transformation:

$$G_\kappa(x) = \Lambda_\kappa \circ G(x) = \frac{\left(G(x)\right)^\kappa}{\left(G(x)\right)^\kappa + \left(\bar{G}(x)\right)^\kappa}, \quad \text{for } x > 0 \qquad \mathbf{(1.1)}$$

relating two lifetime distribution functions $G(x)$ and $G_\kappa(x)$. Here the distribution $G(x)$ may also be a function of an m-dimensional non-negative parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$. The transformation is defined for each κ > 0 by

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$$\Lambda_\kappa(u) = \left[1 + \left(\frac{\bar{u}}{u}\right)^\kappa\right]^{-1}, \text{ for } 0 < u < 1,$$

where $\bar{u} = 1 - u$.

In [9], Gleaton and Lynch established several fundamental results concerning the GLL families of lifetime distributions, which include:

1) The set of GLL transformations forms an Abelian group under the binary operation of composition.
2) The set of GLL transformations partitions the set of all lifetime distributions into equivalence classes, with any two distributions in an equivalence class being related to each other through a GLL transformation.
3) In each equivalence class, either every distribution has a moment generating function, or none does.
4) Every distribution in an equivalence class has the same number of moments.
5) The equivalence classes are linearly ordered according to the transformation parameter, with larger values of the parameter implying smaller dispersion of the distribution about the common class median.
6) Within each equivalence class, the Kullback-Leibler information for a pair of distributions is an increasing function of the ratio of the transformation parameters.

After their work on GLL distributions [9], Gleaton and Lynch further extended their work on GLL distributions by incorporating a proportional odds transformation, as discussed in their subsequent paper [10]. They established essential properties of such distribution families, in addition, they found that the GLLE, which is the primary focus of the current paper, provided a better fit compared to a 2-parameter Weibull distribution to a set of data consisting of the tensile breaking strengths of $n = 64$ ten-millimeter-long carbon fibers. These findings underscore the significance of the GLL family of distributions in accurately modeling data from a wide range of fields, including material science. Gleaton and Lynch's contribution has been instrumental in promoting the use of GLL distributions in practical applications and inspiring further research in the field.

The terms "generalized log logistic" and "odd log logistic" are used interchangeably in the literature to refer to the same distribution transformation. Although the former term was introduced earlier in the literature, both terms have been used to describe the same distribution families [1, 2, 3, 4, 6, 7, 8, 11, 12, 13, 14, 15, 17].

In this paper, we will derive some useful characteristics of the GLLE family of lifetime distributions where $g(x;\lambda) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$, is the p.d.f. of an exponential distribution, for $\lambda > 0$, and

$$g_\kappa(x;\lambda) = \Lambda'_\kappa\big(G(x;\lambda)\big)g(x;\lambda) = l_\kappa\big(g(x;\lambda)\big), \qquad \mathbf{(1.2)}$$

for $\kappa > 0$ is the p.d.f. of the corresponding GLLE distribution. Moreover, we will illustrate the usefulness of this family by using two real data sets in which we compare fits of GLLE distributions to fits of Weibull distributions analytically.

2. Characteristics of the GLLE Distribution.

In [9], Gleaton and Lynch proved the following theorem which is useful in examining the shape of the p.d.f. of a Generalized Log-Logistic distribution, including as a special case, the GLLE distribution family:

THEOREM 2.1: Let $G(x; \boldsymbol{\varphi})$ be the c.d.f. and $g(x; \boldsymbol{\varphi})$ the p.d.f. of a baseline lifetime distribution characterized by a vector parameter $\boldsymbol{\varphi}$. Let $\kappa > 0$, and let $G_\kappa(x; \boldsymbol{\varphi})$ be the c.d.f. and $g_\kappa(x; \boldsymbol{\varphi})$ of a lifetime distribution generated from the baseline distribution by a generalized log-logistic transformation with transformation parameter κ. We define $G^{-1}(x) = inf\{y: G(y) \geq x\}$. Let the distribution G have a unique median, $G^{-1}(0.5)$. Then

(i) The medians of G and G_κ are equal, i.e., $G^{-1}(0.5) = G_\kappa^{-1}(0.5)$ for all $\kappa > 0$.

(ii) G_κ is a decreasing function of κ on the interval $\left(0, G^{-1}(0.5)\right)$, with $\lim_{\kappa \to \infty} G_\kappa(x) = 0$, and is an increasing function of κ on the interval $(G^{-1}(0.5), \infty)$, with $\lim_{\kappa \to \infty} G_\kappa(x) = 1$.

(iii) For all $x > 0$, $\lim_{\kappa \downarrow 0} G_\kappa(x) = 0.5$.

(iv) The density of the GLLE distribution at the median is proportional to κ.

For the Generalized Log-Logistic Exponential distribution (GLLE) family, we wish to find the value of the argument at which the maximum of the density function occurs, as a function of the transformation parameter.

THEOREM 2.2: Let $g(x; \lambda) = \lambda e^{-\lambda x} I_{(0,\infty)}(x)$, the p.d.f. of an exponential distribution, for $\lambda > 0$, and let $g_\kappa(x; \lambda) = l_\kappa\left(g(x; \lambda)\right)$, for $\kappa > 0$, the p.d.f. of the corresponding GLLE distribution. Then

a) the point, x_m, at which the unique maximum of the GLLE p.d.f. occurs:
   i) is less than the common median, $x_{0.5} = \frac{ln(2)}{\lambda}$, of the GLLE equivalence class, and
   ii) approaches $x_{0.5}$ for larger values of κ;

b) quantiles below the median approach the median as $\kappa \to \infty$;

c) quantiles above the median approach the median as $\kappa \to \infty$; and

d) the median is the same for all possible values of $\kappa > 0$.

PROOF: a) We rewrite the p.d.f. in terms of the exponential odds function, $\Omega(x) = \frac{G(x)}{\bar{G}(x)} = e^{\lambda x} - 1$. Then

$$g_\kappa(x) = \kappa\lambda\left(1 + \Omega(x)\right)\frac{\left(\Omega(x)\right)^{\kappa-1}}{\left[1 + \left(\Omega(x)\right)^\kappa\right]^2} I_{(0,\infty)}(x).$$

We will find the value of x_m by differentiating the natural logarithm of the p.d.f.:

$$ln\left(g_\kappa(x)\right) = ln(\kappa\lambda) + ln\left(1 + \Omega(x)\right) + (\kappa - 1)ln\left(\Omega(x)\right) - 2ln\left(1 + \left(\Omega(x)\right)^\kappa\right).$$

Using the derivative of the exponential odds function:

$$\frac{d\Omega}{dx} = \lambda e^{\lambda x},$$

we find that the derivative of the log of the p.d.f. with respect to x is:

$$\frac{dln\left(g_\kappa(x)\right)}{dx} = \lambda e^{\lambda x}\left[\frac{1}{1 + \Omega(x)} + \frac{\kappa - 1}{\Omega(x)} - 2\kappa\frac{\left(\Omega(x)\right)^{\kappa-1}}{1 + \left(\Omega(x)\right)^\kappa}\right].$$

If we set this equal to zero and rearrange terms, we find that x_m must satisfy the equation:

$$\frac{1 - \left(\Omega(x_m)\right)^\kappa}{1 + \left(\Omega(x_m)\right)^\kappa} = \frac{1}{\kappa}\frac{1}{1 + \Omega(x_m)}. \qquad \mathbf{(2.1)}$$

Both sides of this equation must be positive. In particular, we have $\Omega(x_m) < 1$, i.e., the point at which the maximum of the p.d.f. occurs must be less than the common median of the distribution family, $x_{0.5}$. In addition, for larger values of the transformation parameter, the RHS of the equation decreases, in the limit going to 0. This means that the LHS must also approach 0. The implication of this is that $\Omega(x_m)$ must get closer to 1, so that x_m approaches the median as κ increases. It was already proved in [9] that the distribution approached a degenerate distribution at the median as κ goes to infinity.

b) Let $0 < q < 1$, and denote the q^{th} quantile of the distribution by $x_{\kappa,q}$. We have:

$$x_{\kappa,q} = \frac{1}{\lambda} ln\left(1 + \left(\frac{q}{\bar{q}}\right)^{\kappa^{-1}}\right).$$

Differentiating with respect to κ, we find

$$\frac{dx_{\kappa,q}}{d\kappa} = -\frac{1}{\kappa^2\lambda}\frac{1}{1+\left(\frac{q}{\bar{q}}\right)^{\kappa}} ln\left(\frac{q}{\bar{q}}\right).$$

For $0 < q < 0.5$, $ln\left(\frac{q}{\bar{q}}\right) < 0$, and the derivative is positive for all κ.

c) For $0.5 < q < 1$, $ln\left(\frac{q}{\bar{q}}\right) > 0$, and the derivative is negative for all κ.

d) For $q = 0.5$, $ln\left(\frac{q}{\bar{q}}\right) = 0$, and the derivative is 0 for all κ, implying that the medians are the same for all of the GLLE distributions with a given value of λ. ■

The value of x_m for a particular value of the transformation parameter may be found by solving (numerically) the equation:

$$\kappa\big(\Omega(x_m)\big)^{\kappa+1} + (\kappa+1)\big(\Omega(x_m)\big)^{\kappa} - \kappa\Omega(x_m) - \kappa + 1 = 0 \qquad \mathbf{(2.2)}$$

for $\Omega(x_m)$ and inverting to find $x_m = \frac{1}{\lambda} ln\big(1 + \Omega(x_m)\big)$.

We see that GLLE distributions are positively skewed, with tails that decay more rapidly than the baseline exponential distribution so long as the transformation parameter exceeds 1. A GLLE distribution with $\kappa > 1$ is an appropriate distribution to attempt to fit to reliability data sets having certain characteristics:

i) failure is unlikely to occur in a short time (or under relatively small levels of stress),
ii) failure is more likely to occur before very long times (or under less extreme conditions of stress),
iii) failure occurs sooner (or under less stress) than for the baseline exponential distribution,
iv) failure is highly likely to occur during some finite time interval (or interval of stress) bounded away from 0.

3. Hazard Rate for GLLE Distributions.

If the generating distribution is unit exponential, then its hazard rate function is constant, with $h(x) = 1$, for $x < 0$. The hazard rate function of the GLLE distribution with $\lambda = 1$ is given by

$$h_\kappa(x) = \frac{\kappa(1-e^{-x})^{\kappa-1}e^{-x}}{(1-e^{-x})^{\kappa} + e^{-\kappa}} = \frac{\kappa(e^x-1)^{\kappa}}{1+(e^x-1)^{\kappa}}\left(\frac{e^x}{e^x-1}\right). \qquad \mathbf{(3.1)}$$

It is clear from the second form of h_κ that the function asymptotically approaches the constant κ for large x; it is clear from the first form of h_κ that $h_\kappa(0) = 0$, for $\kappa > 1$, and that h_κ is unbounded at 0 for $\kappa < 1$. Graphs of the hazard rate functions are shown in Figures 3.1 and 3.2 for $\kappa = 2$ and $\kappa = 4$, respectively. In each of these two cases, there is a particular value x_κ of x such that $h_\kappa(x) > \kappa$ for all $x > x_\kappa$ and $h'_\kappa(x) > 0$ for all $x < x_\kappa$. We will show that this property holds in general when $\kappa > 1$.

Graphs of the hazard rate functions are shown in Figures 3.3 and 3.4 for $\kappa = 0.5$ and $\kappa = 0.2$, respectively. In each of these two cases, there is a particular value x_κ of x such that $h_\kappa(x) < \kappa$ for all $x > x_\kappa$ and $h'_\kappa(x) < 0$ for all $x < x_\kappa$. We will show that this property holds in general when κ < 1.

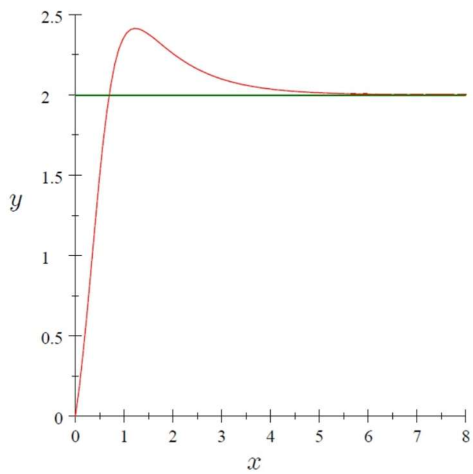

Figure 3.1: Hazard Rate, κ = 2

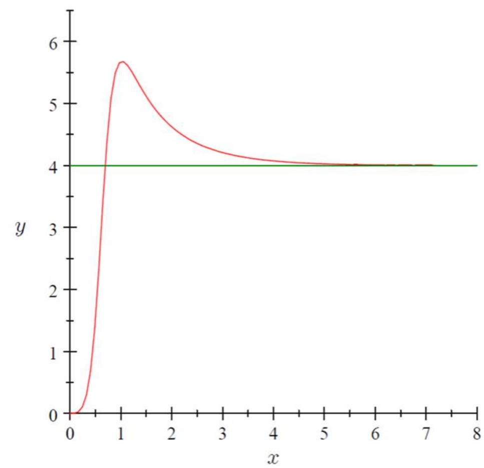

Figure 3.2: Hazard Rate, κ = 4

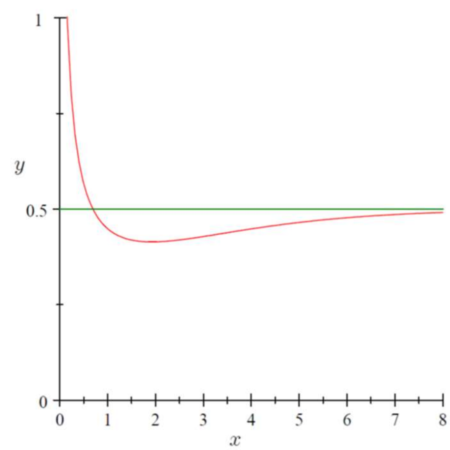

Figure 3.3: Hazard Rate, $\kappa = 0.5$

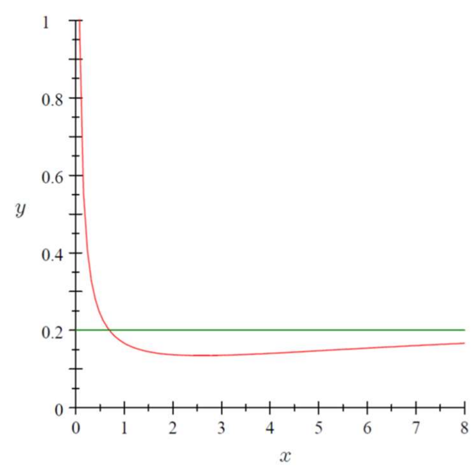

Figure 3.4: Hazard Rate, $\kappa = 0.2$

THEOREM 3.1: Let X be a non-negative continuous random variable with a GLLE ($\kappa,\ \lambda = 1$) distribution. Then the hazard rate function $h_\kappa(x)$ has the following properties:

For $\kappa > 1$,

i) $\lim\limits_{x \downarrow 0} h_\kappa(x) = 0$;

ii) $h_\kappa(x) > \kappa$ for $x > ln(2)$; $h_\kappa(x) < \kappa$ for $x < ln(2)$;

iii) There is a unique $x_{max} > 0$ such that $h'_\kappa(x_{max}) = 0$; $h'_\kappa(x) > 0$ for $0 < x < x_{max}$; $h'_\kappa(x) < 0$ for $x > x_{max}$; and

iv) $\lim_{x\to+\infty} h_\kappa(x) = \kappa$.

For $\kappa < 1$,

v) $\lim_{x\downarrow 0} h_\kappa(x) = +\infty$;

vi) $h_\kappa(x) < \kappa$ for $x > ln(2)$; $h_\kappa(x) > \kappa$ for $x < ln(2)$;

vii) There is a unique $x_{min} > 0$ such that $h'_\kappa(x_{min}) = 0$; $h'_\kappa(x) < 0$ for $0 < x < x_{min}$; $h'_\kappa(x) > 0$ for $x > x_{min}$; and

viii) $\lim_{x\to+\infty} h_\kappa(x) = \kappa$.

PROOF:

Let $\kappa > 1$. Properties (i) and (iv) are obvious from the functional form of h_κ in **(3.1)**.

ii) Let $y = e^x - 1$. If $x > ln(2)$, then $y > 1$, so that $y^\kappa > y$, which implies that $\frac{y^\kappa}{1+y^\kappa}\left(\frac{y+1}{y}\right) > 1$, or $h_\kappa(x) > \kappa$. It is also clear that for $0 < x < ln(2)$, we have $0 < y < 1$. This implies that, for $\kappa > 1$, $y^\kappa < y$, implying $\frac{y^\kappa}{1+y^\kappa}\left(\frac{y+1}{y}\right) < 1$, or $h_\kappa(x) < \kappa$.

iii) The derivative of h_κ is

$$h'_\kappa(x) = \frac{\kappa e^x (e^x - 1)^{\kappa-2}[\kappa(e^x - 1) + \kappa - 1 - (e^x - 1)^\kappa]}{[1 + (e^x - 1)^\kappa]^2}. \qquad (\mathbf{3.2})$$

It is clear that the sign of the derivative is determined by the sign of the quantity in brackets in the numerator. Letting $y = e^x - 1$, as before, we may write the quantity in brackets as

$$g(y) = -y^\kappa + \kappa y + \kappa - 1. \qquad (\mathbf{3.3})$$

This function is guaranteed to have exactly one positive real root for $\kappa > 1$. Let x_1 be the value of x corresponding to this solution. Then x_1 is a candidate for the maximum of the hazard rate. Further examination of **(3.3)** shows that the derivative of the hazard rate is positive for $x < x_1$ and negative for $x > x_1$.

Now, let $\kappa < 1$. Properties (v) and (viii) are obvious from the functional form of h_κ in **(3.1)**.

vi) As before, we let $y = e^x - 1$. For $x < \sqrt{2}$, we have $0 < y < 1$, which implies $y^\kappa > y$, implying $\frac{y^\kappa}{1+y^\kappa}\left(\frac{1+y}{y}\right) > 1$, or $h_\kappa(x) > \kappa$.

vii) As in the proof of (iii), the sign of the derivative is determined by the sign of the quantity in brackets in the numerator of **(3.2)**. Letting $y = e^x - 1$, we may write this quantity in the form of **(3.3).** For $\kappa < 1$, $g(y)$ has exactly one positive real root. Let x_2 be the value of x corresponding to this root. Then x_2 is a candidate for the minimum of the hazard rate. Further examination of **(3.3)** shows that $h'_\kappa(x) < 0$ for $0 < x < x_2$, and that $h'_\kappa(x) > 0$ for $x_2 < x < +\infty$.

Thus, the behavior exhibited by h_2 and h_4 in Figures 3.1 and 3.2 generalizes to all $\kappa > 1$; while the behavior exhibited by $h_{0.5}$ and $h_{0.2}$ in Figures 3.3 and 3.4 generalizes to all $\kappa < 1$.

4. Kullback-Leibler Divergence

Let $\kappa_2 > \kappa_1 > 0$, and let $r = \frac{\kappa_2}{\kappa_1} > 1$. Let $G_{\kappa_1}(x)$ and $G_{\kappa_2}(x)$ be c.d.f.'s related to each other by a GLL transformation. The Kullback-Leibler divergence of $G_{\kappa_2}(x)$, relative to $G_{\kappa_1}(x)$, was shown by Gleaton and Lynch [9] to be

$$K\left(G_{\kappa_2}(x);\ G_{\kappa_1}(x)\right) = 2(r-1) - ln(r) + 2\int_0^1 ln(u^r + \bar{u}^r)du, \qquad \mathbf{(4.1)}$$

a strictly increasing function of the ratio, r, of the GLL transformation parameters, and independent of the form of the baseline c.d.f., $G_{\kappa_1}(x)$. For $r = 1$, the divergence is 0. If we integrate by parts and use the fact that the GLL Unit Uniform distribution is symmetric about 0.5, we obtain

$$K\left(G_{\kappa_2}(x);\ G_{\kappa_1}(x)\right) = r - 2 - ln(r) + 2r\int_0^1 \frac{1-u}{u}\frac{u^r}{u^r + (1-u)^r}du.$$

The integral may be evaluated numerically for specific values of $r > 1$. The table below gives values of the K-L divergence for $r = 1$, 1.5, 2, 2.5, and 5.

r	1	1.5	2	2.5	5
$K(r)$	0	0.13734	0.44845	0.85463	3.483

5. Applications

We attempted to fit two reliability data sets to both a GLLE distribution, with p.d.f. given by Equation **(1.2)**, and to a Weibull distribution, with p.d.f.

$$f(x;\lambda,\rho) = \rho\lambda^{-\rho}x^{\rho-1}e^{-(x/\lambda)^{\rho}}I_{(0,\infty)}(x).$$

Maximum likelihood estimates of the parameters were calculated, and assessment of fit of each distribution to the data set was done.

a) The first data set, listed in the Appendix, consists of measurements of the tensile strengths of carbon fibres of length 10 mm (where $n = 64$). Strength measurements are in GPa. This data set is from Bader and Priest [5]. This data set was analyzed in [10]. The results are recapitulated here.

For the GLLE distribution, the MLE's are $\hat{\lambda} = 4.316169$, $\hat{\kappa} = 6.188709$. The Cramer-von Mises statistic was 0.086319. For the Weibull distribution, the MLE's are $\hat{\lambda} = 3.505193$, $\hat{\rho} = 5.02631$. The value of the Cramer-von Mises statistic was 0.135782. The Cramer-von Mises value was smaller for the GLLE than for the Weibull, indicating a somewhat better fit for the GLLE distribution.

b) The second data set include $n = 63$ observations of the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England which presented by Smith and Naylor [16].

In [15], the Zografos-Balakrishnan odd log-logistic generalized half-normal (ZOLL-GHN) family with four parameters is introduced and studied, and the usefulness of this family were illustrated using this data set.

We used the Kolmogorov-Smirnov test to assess the fit of the data to the two distributions.

For the GLLE distribution, the MLE's are $\hat{\lambda} = 0.453359$, $\hat{\kappa} = 5.913577$. The value of the K-S statistic is 0.148358, with p-value = 0.112646. For the Weibull distribution, the MLE's are $\hat{\lambda} = 1.628113$, $\hat{\rho} = 5.780701$. The value of the K-S statistic is 0.152236, with p-value = 0.096914.

We cannot say that either distribution does not fit the data; however, the GLLE fits the data slightly better than the Weibull.

6. Conclusion.

In this paper we considered family of lifetime distributions generated from an exponential distribution by a GLL transformation. We obtained some useful characteristics of these distributions; we found an equation for numerically calculating the point x_m at which the maximum of the density function occurs, derived the limiting behavior of the quantiles of the distribution. Moreover, we found the behavior of the hazard rate function for values of $\kappa > 1$ and $\kappa < 1$. We extended the Kullback-Leibler divergence studied by Gleaton and Lynch [9] for general GLL families, calculating the Kullback-Leibler divergence for pairs of GLLE distributions for various values of the ratio of the transformation parameters. Finally, we attempted to fit two reliability data sets to both a GLLE distribution and a Weibull distribution. For both data sets, the GLLE provided a somewhat better fit than the Weibull distribution.

Appendix: Data Sets

a) The first data set consists of measurements of the tensile strengths of carbon fibres of length 10 mm (where $n = 64$). Strength measurements are in GPa. This data set is from Bader and Priest [5]:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.454,

2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675,

2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139,

3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408,

3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024,

4.027, 4.225, 4.395, 5.020

b) The second data set include $n = 63$ observations of the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England which presented by Smith and Naylor [16]:

0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2,0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89

References

[1]. M. Alizadeh, L. Benkhelifa, M. Rasekhi & B. Hosseini. (2020). The odd log-logistic generalized Gompertz distribution: properties, applications and different methods of estimation, *Communications in Mathematics and Statistics,* 8(3), 295–317.

[2]. M. Alizadeh, M. Ç. Korkmaz, J. A. Almamy & A. A. E. Ahmed. (2018). Another odd log-logistic logarithmic class of continuous distributions, *Journal of Statisticians: Statistics and Actuarial Sciences* IDIA, 11(2), 55-72.

[3]. M. Alizadeh, S. M. T. K. Mirmostafee, E. M. M. Ortega, T. G. Ramires & G. M. Cordeiro. (2017). The odd log-logistic logarithmic generated family of distributions with applications in different areas, *Journal of Statistical Distributions and Applications,* 4(6).

[4]. M. Alizadeh, S. Tahmasebi & H. Haghbin.(2020). The Exponentiated Odd Log-Logistic Family of Distributions: Properties and Application, *Journal of Statistical Modelling: Theory and Applications*, 1(1), 29-52.

[5]. M. G. Bader & A. M. Priest. (1982). Statistical aspects of fiber and bundle strengths in hybrid composites, *Progress in Science and Engineering of Composites*, T. Hayashi, K. Kawata, & S. Umekawa, Eds., ICCM-IV, Tokyo.

[6]. G. M. Cordeiro, M. Alizadeh, G. Ozel, B. Hosseini, E. Moises, M. Ortega & E. Altun. (2017). The generalized odd log-logistic family of distributions: properties, regression models and applications, *Journal of Statistical Computation and Simulation,* 87(5), 908–932.

[7]. G. M. Cordeiro, M. Alizadeh, M. H. Tahir, M. Mansoor, M. Bourguignon & G. G. Hamedani. (2016). The beta odd log-logistic generalized family of distributions, *Haceteppe Journal of Mathematics and Statistics,* 45(4), 1175–1202.

[8]. F. Domma, A. Eftekharian, A. Z. Afify, M. Alizadeh, I. Ghosh. (2018). The odd log-logistic Dagum distribution: properties and applications, *Review de Colombiana Estadísticas,* 41(1), 109–135.

[9]. J. U. Gleaton & J. D. Lynch, Properties of generalized log-logistic families of lifetime distributions, *Journal of Probability and Statistical Science*, Vol. 4, No. 1, February, 2006.

[10]. J. U. Gleaton & J. D. Lynch. (2010). Extended generalized log-logistic families of lifetime distributions, with an application, *Journal of Probability and Statistical Science*, 8(1), 2010.

[11]. M. Goldoust, S. Rezaei, M. Alizadeh & S. Nadarajah. (2019). The odd log-logistic power series family of distributions: properties and applications, *STATISTICA, anno LXXIX*(1).

[12]. H. Karamikabir, M. Afshari, M. Alizadeh & H. M. Yousof. (2021). The Odd Log-Logistic Burr-X Family of Distributions: Properties and Applications, *Journal of Statistical Theory and Applications*, 20(2), 228–241.

[13]. M. Ç. Korkmaz, H. M. Yousof, M. Alizadeh & G. Hamedani. (2019). The Topp-Leone generalized odd log-logistic family of distributions: properties, characterizations and applications, *Communications Faculty of Science, University of Ankara, Series A1, Mathematics and Statistics*, 68(2), 1506–1527.

[14]. S. Marganpoor, V. Ranjbar, M. Alizadeh & K. Abdollahnezhad. (2020). The odd log-logistic logarithmic Lindley distribution: properties and applications, *Applied Mathematics E-Notes,* 20, 545–563.

[15]. M. Mozafari, M. Afshari, M. Alizadeh & H. Karamikabir. (2019). The Zografos-Balakrishnan odd log-logistic generalized half-normal distribution with mathematical properties and simulations, *Statistics, Optimization & Information Computing*, 7(1), 211–234.

[16]. R. L. Smith & J. C. Naylor. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, *Journal of Applied Statistics*, 36, 358–369.

[17]. M. H. Tahir, M. Mansoor, M. Zubair, G. G. Hamedani. (2014). McDonald log-logistic distribution with an application to breast cancer data, *Journal of Statistical Theory and Applications,* 13(1), 65–82.