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Transmuted Modified Weibull Distribution for Modeling Skewed Lifetime Dataset; Properties and Application

Awopeju Kabiru Abidemi
Nnamdi Azikiwe University, Nigeria

Alfred A. Abiodun
University of Ilorin, Nigeria

ABSTRACT

A new five parameter model called Generalized Modified Weibull distribution is proposed and studied. The new distribution generalizes the Modified Weibull Distribution introduced by Lai et al. (2003) using transformed – Transformer framework of Alzaatreh et al. (2014) which resulted to a distribution capable of modeling skewed data (positively or negatively skewed data), as well as, symmetric data. Property of a proper probability density function was used to ascertain that the resulting function is a proper probability density function. Statistical properties of the newly generated distribution were studied. Graph of probability density function of the distribution was used to show that it is capable of modeling skewed data. Graph of cumulative density functions of the distribution was plotted using varying parameter values. Monte Carlo simulation approach was used for the test of homogeneity of the distribution and it was observed that the parameters in the distribution approach the true values as sample size increases. Maximum likelihood Estimation method was used for estimation of the model parameters. Real life dataset was used for model comparison as well as demonstration of its application.

Keywords: Maximum Likelihood, Probability Density Function, Survival Function, Hazard Function, Moment, Characteristic Function, Simulation

1. Introduction

Weibull as probability distribution was introduced in the year 1951 by Swedish Scientist and Engineer, Waloddi Weibull. The distribution became most important parametric model in survival analysis (Weibull, 1951). The distribution gained wide applications in material science, reliability, forestry, quality control, geography, geology, astronomy (Rinne, 2008). Murthy *et al.* (2003) presented more than 40 applications of the Weibull distribution with references. Even, as an extreme value distribution, it has also been applied in the modelling of climate, weather data such as rainfall, wind speed and flood.

Gumbel (1959) proposed a modified Weibull distribution called Log Weinull distribution and studied its properties. This distribution later became known as the Gumbel or Extreme Value distribution. Cohen (1973) introduced the reflected Weibull distribution as the distribution of the negative of a Weibull random variable to produce a more robust version of the distribution.
Kies (1958) introduced a modification of the Weibull distribution by adding lower and upper limits to the Weibull random variable. This became known as the Kies modified Weibull distribution. In order to offer more flexibility distribution to the Kies modified Weibull distribution, Phani (1987) modified the Kies Weibull distribution by adding an extra parameter. This became known as the Phani modified Weibull distribution.

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- Awopeju Kabiru Abidemi (corresponding author), Department of Statistics, Faculty of Physical Sciences, Nnamdi Azikiwe University, Awka, Anambra State, Nigeria.. ak.awopeju@unizik.edu.ng and Alfred A. Abiodun , Department of Statistics, Faculty of Physical Sciences, University of Ilorin, Ilorin, Kwara State, Nigeria. alfredabiodun5@gmail.com

Mudholkar and Srivastava (1993) proposed the exponentiated Weibull distribution by raising the Weibull cumulative distribution to a positive real constant parameter. The distribution was found to exhibit both bathtub and unimodal hazard rates. Xie and Lai (1996) introduced the additive Weibull distribution with bathtub shape hazard. The distribution arose by adding the hazard functions of two Weibull random variables. Bebbington et al. (2006) studied a particular case of the additive Weibull distribution and applied same to real life data set.

Chen (2000) developed a certain variant of the Weibull distribution with bathtub hazard. This became known as the Chen distribution. Lai et al. (2003) proposed a modification of the Weibull distribution. This later became known as the modified Weibull distribution with three parameters. Xie et al. (2002) defined the modified Weibull extension distribution with bathtub shape hazard and presented its statistical properties. Famoye et al. (2005) used the beta distribution to generalize the Weibull distribution and arrived at a proper probability density function. This generalization became known as the beta-Weibull distribution which is both unimodal and bimodal. Cooray (2006) defined the odd Weibull distribution which has three parameters and Carrasco et al. (2008) generalized the modified Weibull distribution of Lai et al. (2003) to define the generalized modified Weibull distributions with four parameters.

Alizadeh *et al*. (2017) introduced a new family of continuous distributions called the transmuted Weibull-G family of distributions which class pioneered by Shaw and Buckley (2007). Mathematical properties of the new family were studied and some useful characterizations based on the ratio of two truncated moments as well as based on hazard function were presented.

Mashail (2022) proposed an extension of Weibull distribution to capture skewed data. The new distribution is called Log-logistic Lindley Weibull (OLLLW) distribution and the researcher claimed that the model has greater flexibility. In the report, it was stated that OLLLW distribution exhibits monotone, increasing, decreasing as well as non-monotone which makes it superior among lifetime distributions. The distribution was constructed using the odd log-logistic Lindley-G (OLLi-G) family proposed by Alizadeh et al., 2020. The Cumulative Density Function of Alizadeh et al., (2020) was given as;

$$F(x)=\frac{\left(1-\left(1+\frac{\theta}{\theta+1}\frac{G(x:\phi)}{\bar{G}(x:\phi)}\right)e^{-\frac{\vartheta G(x:\phi)}{\bar{G}(x:\phi)}}\right)^{\alpha}}{\left(1-\left(1+\frac{\theta}{\theta+1}\frac{G(x:\phi)}{\bar{G}(x:\phi)}\right)e^{-\frac{\vartheta G(x:\phi)}{\bar{G}(x:\phi)}}\right)^{\alpha}+\left(1-\left(1+\frac{\theta}{\theta+1}\frac{G(x:\phi)}{\bar{G}(x:\phi)}\right)e^{-\frac{\vartheta G(x:\phi)}{\bar{G}(x:\phi)}}\right)^{\alpha}}$$

where $G(x:\phi)$ denotes the baseline cdf with parameters vector ϕ:x > 0 and $\theta>0$. The corresponding pdf of the function is given as

$$f(x:x,\theta,\varphi)=\frac{\frac{\alpha\theta^2 g(x:\varphi)}{(1+\theta)\bar{G}(x:\varphi)^3}e^{-\frac{\theta G(x:\theta)}{\bar{G}(x:\theta)}}\left(1+\frac{\theta}{\theta+1}\frac{G(x:\theta)}{\bar{G}(x:\theta)}\right)^{\alpha-1}\left(1-\left(1+\frac{\theta}{\theta+1}\frac{G(x:\phi)}{\bar{G}(x:\phi)}\right)e^{-\frac{\vartheta G(x:\phi)}{\bar{G}(x:\phi)}}\right)^{\alpha-1}}{\left[\left(1-\left(1+\frac{\theta}{\theta+1}\frac{G(x:\phi)}{\bar{G}(x:\phi)}\right)e^{-\frac{\vartheta G(x:\phi)}{\bar{G}(x:\phi)}}\right)^2+\left(1-\left(1+\frac{\theta}{\theta+1}\frac{G(x:\phi)}{\bar{G}(x:\phi)}\right)e^{-\frac{\vartheta G(x:\phi)}{\bar{G}(x:\phi)}}\right)^2\right]}$$

where $g(x:\varphi)$ is the baseline density with a vector of parameters φ. The probability density function of the new distribution proposed by the researchers was;

$$f(x)=\frac{\left(\alpha\beta\theta^2(\theta+1)^{1-2\alpha}\lambda^\beta x^{\beta-1}\left(\left(\theta e^{(\lambda x)^\beta}+1\right)(\theta+1)e^{\theta e^{(\lambda x)^\beta}}-e^{\theta}\left(\theta e^{(\lambda x)^\beta}+1\right)\right)\right)^{\alpha-1}}{\left(\left(\frac{e^{\theta-\theta e^{(\lambda x)^\beta}}(\theta e^{(\lambda x)^\beta}+1)}{\theta+1}\right)^\alpha+\left(1-\frac{e^{\theta-\theta e^{(\lambda x)^\beta}}(\theta e^{(\lambda x)^\beta}+1)}{\theta+1}\right)^\alpha\right)^2 e^{-(\alpha\theta+(\theta-2\alpha\theta)e^{(\lambda x)^\beta}+2(\lambda x)^\beta}}$$

the survival and hazard functions were also derived.

Malik and Ahmad (2022) proposed a three parameter transmuted Weibull distribution using the new transmutation technique proposed by Bakouch et al. (2017). The distribution was named New Transmuted Weibull Distribution (NTWD). a new transmutation technique was introduced by Bakouch et al (2017). It was reported that random variable X is said to follow New Transmuted distribution with β as transmuted parameter if its cdf and pdf take respectively the forms given by;

$$G(x)=M(x)+\beta\frac{M(x)(1-M(x)}{(1+M(x))};x\in R,-1\le\beta\le1 \text{ and}$$

$$g(x)=(1-\beta+\left[\frac{2\beta}{(1+M(x))^2}\right]m(x)$$

where m(x) and M(x) are pdf and cdf of the baseline distribution. Using the expression, the researchers obtained a new distribution as

$$G(x)=1-e^{-\theta x^\lambda}+\frac{\beta e^{-\theta x^\lambda}\left(1-e^{-\theta x^\lambda}\right)}{2-e^{-\theta x^\lambda}};x,\lambda,\theta>0,-1\le\beta\le1 \text{ and}$$

$$g(x)=\left(1-\beta+\frac{2\beta}{\left(2-e^{-\theta x^\lambda}\right)^2}\right)\theta\lambda x^{\lambda-1}e^{-\theta x^\lambda};x,\lambda,\theta>0,-1\le\beta\le1$$

The pdf was tested for validation and simulation study was done to study the symptotic property of the new distribution. Mathematical properties of new model were derived. Entropy and parameter estimation was done using MLE approach. A real life data set was incorporated to illustrate the utility of the model.

Among the modifications done to the Weibull distribution, there is no specific modification done to address the possibility of capturing skewed data (positive or negative). This led to the idea of generating a new distribution using one of the modified Weibull distributions as baseline function. *Transformed – Transformed framework* introduced by *Alzaatreh et al. (2014)* method of generalization was adopted in the study. This led to increase in the parameters in the distribution and improves its performance among its class. The research was extended to derivation of Statistical properties of the distribution, graphs of its probability density function, cumulative density function, survival function and hazard function. Method of Maximum Likelihood Estimation was used to estimate the parameters in the newly generalized distribution.. Also, simulation study was carried out for the determination of asymptotic property of the distribution. The generalized distribution was compared with distributions in its class for the determination of its suitability in solving problems.

This research will serve as a guide for the users of Statistics in the field of human endevour such as weather prediction, rainfall prediction, determination of failure rate of an event, survival of patients in terms of a particular ailment.

2.1 Material and methods

Many methods abounds for generalizing a given probability distribution. These methods are usually based on the ideas of combining two or more distributions or by adding extra parameter(s) to an already existing distribution (Lee *et al*. 2013). The goal has been to make more flexible the distribution that is being generalized – usually called the baseline distribution. The process of generalization usually involve high mathematical thinking. However, the whole labour becomes productive when the generalized distribution begins to portray the characteristic(s) that informed the need to do such generalization. In this paper, modified Weibull distribution introduced by Lai *et al.* (2003) was used as baseline function. It was generalized using a method *the transformed – Transformed framework* introduced by *Alzaatreh et al. (2014)*.

2.2 Baseline Function; Weibull Distribution

Modified Weibull Distribution by Lai *et al*. (2003) with cdf and pdf as;

$$F(x) = 1 - \exp\left[-\beta x^{\gamma} e^{\lambda x}\right], \qquad (1)$$

$$f(x) = \beta(\gamma + \lambda x)x^{\gamma-1} e^{\lambda x} \exp\left[-\beta x^{\gamma} e^{\lambda x}\right], \quad (2)$$

$$x > 0, \beta > 0, \lambda > 0, \gamma > 0,$$

where $\beta \; and \; \lambda$ are scale parameters while γ is shape parameter.

2.3 Generalization of Baseline Function

Following Alzaatreh et al., (2014), the cdf of the random variable X following the Transformed – Transformed family of distribution is defined as

$$F_X(x) = \int_a^{Q_Y(F_R(x))} f_T(t)dt = P\left[T \le Q_Y(F_R(x))\right] = F_T\left(Q_Y(F_R(x))\right). \qquad (3)$$

The corresponding pdf associated with Equation (3) is

$$f_X(x) = f_R(x) \times Q_Y'\big(F_R(x)\big)^{-1} \times f_T\Big(Q_Y\big(F_R(x)\big)\Big)$$

$$= f_R(x) \times \frac{f_T\Big(Q_Y\big(F_R(x)\big)\Big)}{f_Y\Big(Q_Y\big(F_R(x)\big)\Big)}. \quad (4)$$

2.4 Generalization of Modified Weibull Distribution using the Transformed-transformer Framework

Suppose R is a random variable following the modified Weibull distribution with CDF, $F_R(x)$, and pdf, $f_R(x)$ such that the CDF is

$$F_R(x) = 1 - e^{-\beta x^r e^{\lambda x}}; \quad \beta, r, \lambda > 0, x > 0. \quad (5)$$

and the pdf is three parameter modified Weibull distribution given as

$$f_R(x) = \beta(r + \lambda x)x^{r-1}e^{\lambda x}e^{-\beta x^r e^{\lambda x}} \quad (6)$$

Allowing T and Y to be from Weibull and standard exponential distribution respectively, we have;

$$Q_Y(F_R(x)) = -log\left[1 - \left(1 - e^{-\beta x^r e^{\lambda x}}\right)\right] \quad (7)$$

$$= -log\left[e^{-\beta x^r e^{\lambda x}}\right] = \beta x^r e^{\lambda x}$$

Using the Transformed-transformer system, the cdf of the Weibull-Modified Weibull distribution is given as;

$$F_X(x) = 1 - exp\left[-\theta\left(\beta x^r e^{\lambda x}\right)^w\right] \quad (8)$$

and the pdf is given as;

$$f_X(x) = \beta(r + \lambda x)x^{r-1}e^{\lambda x}e^{-\beta x^r e^{\lambda x}} \times \frac{\theta w\left(\beta x^r e^{\lambda x}\right)^{w-1} exp\left[-\theta\left(\beta x^r e^{\lambda x}\right)^w\right]}{exp\left[-\left(\beta x^r e^{\lambda x}\right)\right]}$$

Then the pdf of the new distribution is

$$f_X(x) = \frac{\beta\theta w(r+\lambda x)x^{r-1}e^{\lambda x}e^{-\beta x^r e^{\lambda x}}\left(\beta x^r e^{\lambda x}\right)^{w-1} exp\left[- \left(\beta x^r e^{\lambda x}\right)^w\right]}{exp\left[-\left(\beta x^r e^{\lambda x}\right)\right]} \quad (9)$$

Survival and hazard functions of the generalized distributions can be determined using the relationship among cdf, pdf, s(t) and h(t).

3.1 Study of the newly derived distribution

It this section, the newly derived probability density function (pdf) is studied for completeness, structure in terms of graph, statistical properties as well as estimation of its parameters using MLE (Maximum Likelihood Estimate).

3.2 Area under Curve of the Newly Generated Distribution

One of the properties of a proper pdf is

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Using above property by substituting the probability dnesity function of Genealized Modified Weibull-Exponential (GMWE) distribution into the expression;

$$\int_{0}^{\infty} \frac{\beta\theta w(\lambda + rx)x^{\lambda-1}e^{rx}e^{-\beta x^{\lambda}e^{rx}}(\beta x^{\lambda}e^{rx})^{w-1}exp[-\theta(\beta x^{\lambda}e^{rx})^{w}]}{exp[-(\beta x^{\lambda}e^{rx})]}dx$$

Simplify the expression, we have;

$= \int_0^{\infty} \beta\theta w(\lambda + rx)x^{\lambda-1}e^{rx}(\beta x^{\lambda}e^{rx})^{w-1}exp[-\theta(\beta x^{\lambda}e^{rx})^{w}]\, dx$

Let U = $-\theta(\beta x^{\lambda}e^{rx})^{w}$

To get the derivative of U, let $V = \beta x^{\lambda}e^{rx}$

this implies U = $-\theta V^{w}$

$\frac{dU}{dV} = -\theta w V^{w-1}$

$V = \beta x^{\lambda}e^{rx}$

For the derivative of V, use product rule such that

$$\frac{dV}{dx} = P\frac{dq}{dx} + q\frac{dp}{dx}$$

Where $P = \beta x^{\lambda}$ and $q = e^{rx}$

$\frac{dp}{dx} = \beta\lambda x^{\lambda-1}$ and $\frac{dq}{dx} = re^{rx}$

By substitution, the derivative of V is

$e^{rx}(\beta\lambda x^{\lambda-1}) + \beta x^{\lambda}(re^{rx})$

$= (\beta\lambda x^{\lambda-1}\, e^{rx} + \beta r x^{\lambda}\, e^{rx})$

Then,

$\frac{dU}{dx} = \frac{dU}{dV}\left(\frac{dV}{dx}\right) = -\theta w V^{w-1}(\beta\lambda x^{\lambda-1}\, e^{rx} + \beta r x^{\lambda}\, e^{rx})$

Recall that

$$V = \beta x^{\lambda}e^{rx}$$

Then, the derivative of U is

$-\theta w(\beta x^{\lambda}e^{rx})^{w-1}(\beta\lambda x^{\lambda-1}\, e^{rx} + \beta r x^{\lambda}\, e^{rx})$

$= -\theta w(\beta x^{\lambda}e^{rx})^{w-1}\beta e^{rx}(\lambda x^{\lambda-1} + rx^{\lambda})$

$= -\theta w(\beta x^{\lambda}e^{rx})^{w-1}\beta e^{rx}(\lambda x^{\lambda}x^{-1} + rx^{\lambda})$

$= -\theta w(\beta x^{\lambda} e^{rx})^{w-1} \beta x^{\lambda} e^{rx} (\lambda x^{-1} + r)$
$= -\theta w(\beta x^{\lambda} e^{rx})^{w-1} \beta x^{\lambda} e^{rx} (\frac{\lambda}{x} + r)$
$= -\theta w(\beta x^{\lambda} e^{rx})^{w-1} \beta x^{\lambda} e^{rx} (\frac{\lambda + r}{x})$
$= -\theta w(\beta x^{\lambda} e^{rx})^{w-1} \beta x^{\lambda} e^{rx} (\lambda + rx) x^{-1}$
$= -\theta w(\beta x^{\lambda} e^{rx})^{w-1} \beta x^{\lambda} x^{-1} e^{rx} (\lambda + rx)$
$= -\theta w(\beta x^{\lambda} e^{rx})^{w-1} \beta x^{\lambda-1} e^{rx} (\lambda + rx)$
$= -\theta \beta w(\lambda + rx) x^{\lambda-1} e^{rx} (\beta x^{\lambda} e^{rx})^{w-1}$
Therefore,
$\frac{du}{dx} = -\theta \beta w(\lambda + rx) x^{\lambda-1} e^{rx} (\beta x^{\lambda} e^{rx})^{w-1}$
Then, $dx = \frac{du}{-\theta \beta w(\lambda + rx) x^{\lambda-1} e^{rx} (\beta x^{\lambda} e^{rx})^{w-1}}$

$$\int_0^{\infty} \frac{\beta \theta w(\lambda+rx) x^{\lambda-1} e^{rx} e^{-\beta x^{\lambda} e^{rx}} (\beta x^{\lambda} e^{rx})^{w-1} exp\left[-\theta(\beta x^{\lambda} e^{rx})^{w}\right]}{exp[-(\beta x^{\lambda} e^{rx})]} dx$$

$$= \int_0^{\infty} \frac{\beta \theta w(\lambda+rx) x^{\lambda-1} e^{rx} e^{-\beta x^{\lambda} e^{rx}} (\beta x^{\lambda} e^{rx})^{w-1} e^{U}}{-\theta \beta w(\lambda+rx) x^{\lambda-1} e^{rx} (\beta x^{\lambda} e^{rx})^{w-1}} du$$

$= - \int_0^{\infty} e^{U} \, du$
Since U = $-\theta(\beta x^{\lambda} e^{rx})^{w}$,
We have;
$- \int_0^{\infty} e^{u} \, du = - \lim_{N \to \infty} \int_o^{N} e^{u} \, du = - \lim_{N \to \infty} (e^{u}|_0^N$
$= - \lim_{N \to \infty} (e^{-\theta(\beta x^{\lambda} e^{rx})^{w}}|_0^N$
$= - \lim_{N \to \infty} e^{-\theta(\beta x^{\lambda} e^{rx})^{w}} + (e^{-\theta(0)^{w}})$
$= - \lim_{N \to \infty} e^{-\theta(\beta N^{\lambda} e^{rN})^{w}} + (e^{0})$
$= - \lim_{N \to \infty} e^{-\theta(\beta N^{\lambda} e^{rN})^{w}} + (e^{0})$
$= - e^{-\theta(\beta \infty^{\lambda} e^{r\infty})^{w}} + (e^{0}) = - e^{-\theta(\infty)} + 1 = -e^{-\infty} + 1 = 0 + 1 = 1$
This implies the newly generated function (f(x)) is a complete probability density function.

3.3 Estimation of the Parameters in the Generalized Modified Weibull-Exponential (GMWE) Distribution

The maximum likelihood method of parameter estimation was adopted in the study to estimate the parameters of the proposed distribution. The maximum Likelihood function is given by;

$$L(n, \theta, \beta) = \prod_{i=1}^{n} [F_X(x_i)] \tag{10}$$

Taking the partial derivative of the equation and equating it to zero yielded a non-linear system of equations. The solution to the non-linear system of equations yielded Maximum Likelihood Estimation of the parameters of the new distribution (GMWE).

Suppose $x_1, x_2, ..., x_n$ are independent random variables with sample size from Generalized Modified Weibull-Exponential (GMWE) distribution, its likelihood function is given by

$$\mathrm{L}(\mathrm{x};\beta,\theta,w,\lambda,r)=\left(\theta w\beta^{w}\right)^{n}\underset{i=1}{\overset{n}{\pi}}\left(\lambda+rx_i\right)x_i^{\lambda\omega-1}e^{rwx_i}e^{\left(-\theta\beta^{w\left(x_ie^{rx_i}\right)^{w}}\right)} \quad (11)$$

Taking the natural log of the equ.(11);

$$\mathrm{InL}(\mathrm{x};\beta,\theta,w,\lambda,r)=\mathrm{n\ In}\ \left(\theta w\beta^{w}\right)+$$

$$\sum_{i=1}^{n}\ln\left(\lambda+rx\right)+\left(\lambda w-1\right)\sum_{i=1}^{n}\ln\left(x_1\right)+rw\sum_{i=1}^{n}x_1-\theta\beta^{w}\sum_{i=0}^{n}\left(x_1^{\lambda}e^{rx_1}\right)^{w} \quad (12)$$

The estimates of the parameters are obtained by taking the derivatives of equations and set to zero;

$$\frac{\partial L}{\partial\beta}=\frac{nw}{\beta}-\theta w\beta^{w-1}\sum_{i=1}^{n}\ln\left(x^{\lambda}e^{rx}\right) \quad (13)$$

$$\frac{\partial L}{\partial\theta}=\frac{n}{\theta}-\beta^{w}\sum_{i=1}^{n}\ln\left(x^{\lambda}e^{rx_1}\right)^{w} \quad (14)$$

$$\frac{\partial L}{\partial\lambda}=\frac{1}{\sum_{i=1}^{n}\left(\lambda+rx_i\right)}+w\sum_{i=1}^{n}\ln\left(x_i\right)-\theta\beta^{w}\sum_{i=1}^{n}\ln\left(x_i^{\lambda}e^{rx_i}\right)^{w}w\ln\left(x_i\right) \quad (15)$$

$$\frac{\partial L}{\partial r}=\frac{\sum_{i=1}^{n}x_i}{\sum_{i=1}^{n}\left(\lambda+rx_i\right)}+w\sum_{i=1}^{n}\ln x_i-\theta\beta^{w}\sum_{i=1}^{n}\ln\left(x_i^{\lambda}e^{rx_i}\right)^{w}wx \quad (16)$$

$$\frac{\partial L}{\partial w}=\frac{n\left(\theta\beta^{w}+\theta w\beta^{w}\ln(\beta)\right)}{\theta w\beta^{w}}+\lambda\sum_{i=1}^{n}\left(x_i\right)+r\sum_{i=1}^{n}\ln\left(x_i\right)-\theta\beta^{w}\ln(\beta)\sum_{i=1}^{n}\left(x_i^{\lambda}e^{rx_i}\right)^{w}$$

$$-\theta\beta^{w}\sum_{i=1}^{n}\left(x_i^{\lambda}e^{rx_i}\right)^{w}\ln\sum_{i=1}^{n}\left(x_i^{\lambda}e^{rx_i}\right) \quad (17)$$

Above equations are non-linear in parameters and numerical optimization method can be used to obtain the parameter values.

3.4 Survival Characteristics of Newly Generated Distributions

Since it has been established that the newly generated probability density function is a proper pdf, then, the survival characteristics of the distribution can be obtained.

Let the probability distribution function of a distribution be f(x) and the corresponding Cummulative DensityFunction be F(x). Then, the Survival function S(x) is;

$$S(x)=(1-F(x))$$

Hazard function (h(x)) of the distribution becomes;

$h(x) = \frac{f(x)}{S(x)}$

since $S(x) = (1 - F(x))$,

therefore, hazard function can be expressed as

$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{(1-F(x))} = \frac{F^{|}(x)}{(1-F(x))}$

using the GMWE distribution as distribution of interest, its survival characteristics functions are;

$F(x) = 1 - exp[-\theta(\beta x^{\lambda} e^{rx})^{w}]$

and

$f(x) = \beta\theta w(\lambda + rx)x^{\lambda-1}e^{rx}(\beta x^{\lambda} e^{rx})^{w-1}exp[-\theta(\beta x^{\lambda} e^{rx})^{w}]$ respectively.

$$S(x) = 1 - [1 - exp[-\theta(\beta x^{\lambda} e^{rx})^{w}]] = e^{-\theta(\beta x^{\lambda} e^{rx})^{w}} \quad (18)$$

$$h(x) = \frac{\beta\theta w(\lambda+rx)x^{\lambda-1}e^{rx}(\beta x^{\lambda} e^{rx})^{w-1}e^{-\theta(\beta x^{\lambda} e^{rx})^{w}}}{e^{-\theta(\beta x^{\lambda} e^{rx})^{w}}}$$

$$h(x) = \beta\theta w(\lambda + rx)x^{\lambda-1}e^{rx}(\beta x^{\lambda} e^{rx})^{w-1} \quad (19)$$

3.5 Properties of the Newly Generated Probability (GMWE) Distribution

Some statistical properties of the newly generated distribution are discussed. The properties include moments, moment generating function, characteristic function, median, quantile function, mean, variance, mean deviation, incomplete moment, Lorenz and Bonferroni curves, conditional moments, order statistics and maximum likelihood of the estimates.

Given the probabilty densityfunction f(x),

$f(x) = \beta\theta w(\lambda + rx)x^{\lambda-1}e^{rx}(\beta x^{\lambda} e^{rx})^{w-1}exp[-\theta(\beta x^{\lambda} e^{rx})^{w}]$, x > 0

further simplication of the pdf;

$f(x) = \theta w\beta^{w}(\lambda + rx)x^{\lambda w-1}e^{rwx}\, e^{\left[-\theta\beta^{w}(x^{\lambda} e^{rx})^{w}\right]}$, x > 0

A. Moment of Generalized Modified Weibull Distribution

The p-th non-central moment of a continuous distribution is given by

$$E[X^{p}] = \int_{0}^{\infty} x^{p} f(x)dx$$

By substitution,

$$E[X^{p}] = \int_{0}^{\infty} x^{p}\theta w\beta^{w}(\lambda + rx)x^{\lambda w-1}e^{rwx}\, e^{\left[-\theta\beta^{w}(x^{\lambda} e^{rx})^{w}\right]}dx$$

$$= \theta w\beta^{w}\int_{0}^{\infty} x^{\lambda w+p-1}(\lambda + rx)e^{rwx}\, e^{\left[-\theta\beta^{w}(x^{\lambda} e^{rx})^{w}\right]}dx \quad (20)$$

but

$$e^{\left[-\theta\beta^{w}(x^{\lambda} e^{rx})^{w}\right]} = \sum_{m=0}^{\infty}\frac{(-\theta\beta^{w})^{m}x^{\lambda wm}e^{rmwx}}{m!} \quad (21)$$

Substituting Equation (21) into Equation (20),

$$E[X^{p}] = \theta w\beta^{w}\int_{0}^{\infty} x^{\lambda w+p-1}(\lambda + rx)e^{rwx}\sum_{m=0}^{\infty}\frac{(-\theta\beta^{w})^{m}x^{\lambda wm}e^{rmwx}}{m!}\, dx \quad (22)$$

$$E[X^{p}] = \theta w\beta^{w}\sum_{m=0}^{\infty}\frac{(-\theta\beta^{w})^{m}}{m!}\int_{0}^{\infty} x^{\lambda w+p-1}x^{\lambda wm}(\lambda + rx)e^{rwx}e^{rmwx}\, dx$$

$$E[X^{p}] = \theta w\beta^{w}\sum_{m=0}^{\infty}\frac{(-\theta\beta^{w})^{m}}{m!}\int_{0}^{\infty} x^{\lambda w(m+1)+p-1}e^{rw(m+1)x}(\lambda + rx)\, dx \quad (23)$$

Consider the integrand in Equation (23),

$$\int_0^\infty x^{\lambda w(m+1)+p-1} e^{rw(m+1)x} (\lambda + rx)\, dx$$

$$=\lambda \int_0^\infty x^{\lambda w(m+1)+p-1} e^{rw(m+1)x} dx + r \int_0^\infty x^{\lambda w(m+1)+p} e^{rw(m+1)x} dx \qquad (24)$$

From the first integrand in Equation (24),

$$\int_0^\infty x^{\lambda w(m+1)+p-1} e^{rw(m+1)x} dx$$

Let $-u = rw(m+1)x$

That implies $x = \frac{-u}{rw(m+1)}$

$dx = -\frac{du}{rw(M+1)}$

Then,

$$\int_0^\infty x^{\lambda w(m+1)+p-1} e^{rw(m+1)x} dx = \frac{(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \int_0^\infty u^{\lambda w(m+1)+p-1} e^{-u} du$$

$$= \frac{(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \Gamma(\lambda w(m+1)+p)$$

Similar, considering the second integrand in Equation (24),

$$\int_0^\infty x^{\lambda w(m+1)+p} e^{rw(m+1)x} dx = \frac{(-1)^{\lambda w(m+1)+p+1}}{[rw(m+1)]^{\lambda w(m+1)+p+1}} \Gamma(\lambda w(m+1)+p+1)$$

Therefore,

$$\int_0^\infty x^{\lambda w(m+1)+p-1} e^{rw(m+1)x} (\lambda + rx)\, dx$$

$$= \frac{\lambda(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \Gamma(\lambda w(m+1)+p) + \frac{r(-1)^{\lambda w(m+1)+p+1}}{[rw(m+1)]^{\lambda w(m+1)+p+1}} \Gamma(\lambda w(m+1)+p+1) \qquad (25)$$

recall that

$$\Gamma(n+1) = n\Gamma(n)$$

therefore,

$$\Gamma(\lambda w(m+1)+p+1) = (\lambda w(m+1)+p)\Gamma(\lambda w(m+1)+p)$$

This implies Equation (25) can be expressed as ;

$$\frac{(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \Gamma(\lambda w(m+1)+p) \left[\lambda - \frac{r(\lambda w(m+1)+p)}{rw(m+1)}\right] \qquad (26)$$

Substitute Equ. (25) into Equ. (23),

$$E[X^p] = \theta w \beta^w \sum_{m=0}^{\infty} \frac{(-\theta\beta^w)^m}{m!} \frac{(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \Gamma(\lambda w(m+1)+p) \left[\lambda - \frac{r(\lambda w(m+1)+p)}{rw(m+1)}\right] \qquad (27)$$

The mean of the distribution can be obtained by setting p = 1;

$$E[x] = \theta w \beta^w \sum_{m=0}^{\infty} \frac{(-\theta\beta^w)^m}{m!} \frac{(-1)^{\lambda w(m+1)+1}}{[rw(m+1)]^{\lambda w(m+1)+1}} \Gamma(\lambda w(m+1)+1) \left[\lambda - \frac{r(\lambda w(m+1)+1)}{rw(m+1)}\right] \qquad (28$$

B. The Q^th Central Moment

The q^th central moment is defined by

$$E[x-\mu]^q = \sum_{p=0}^{\infty} \binom{q}{p} (-\mu)^{q-p} E\left[X^p\right] \qquad (29)$$

Where $\mu = E(X)$ and $E\left[X^p\right]$ is as shown in Equation (27).

$$= \sum_{p=0}^{\infty} \binom{q}{p} (-\mu)^{q-p} \theta\omega\beta^w \sum_{m=0}^{\infty} \frac{\left(-\theta\beta^w\right)^m}{m!} \frac{(-1)^{w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \overline{)\lambda w(m+1)} \left[\lambda - \frac{r(\lambda w(m+1)+p)}{rw(m+1)}\right]$$

That implies

$$E[X-\mu]^q$$

$$= \theta w\beta^w \sum_{m=0}^{\infty} \frac{\left(-\theta\beta^w\right)^m}{m!} \sum_{p=0}^{q} \binom{q}{p} (-\mu)^{q-p} \frac{(-1)^{\lambda w(m+1)+p}}{(rw(m+1))^{\lambda w(m+1)+p}} \overline{)(\lambda w(m+1)+P)} \left[\lambda - \frac{r(\lambda w(m+1)+p)}{rw(m+1)}\right]$$

(30)

Equation (30) is the Q^th central moment. The variance of the distribution can be obtained by setting q=2 in Equation (30.

C. Moment Generating Function

The moment generating function of a random variable X is given by

$$M_x(t) = E\left(e^{tx}\right) = \sum_{p=0}^{\infty} \frac{t^p}{p!} E\left[X^P\right] \qquad (31)$$

Therefore,

$$M_x(t) = \theta w\beta^w \sum_{m=0}^{\infty} \frac{\left(-\beta^w\right)^m}{m!} \sum_{p=0}^{\infty} \frac{t^p}{p!} \frac{(-1)^{\lambda w(m+1)+p} \overline{)(\lambda w(m+1)+p)}}{[rw(m+1)]^{\lambda w(m+1)+p}} \left[\lambda - \frac{r(\lambda w(m+1)+p)}{rw(m+1)}\right] \qquad (32)$$

D. Characteristic Function

The characteristic function of a random variable X is given by

$$\varphi_x(it) = E\left(e^{itx}\right) = \sum_{p=0}^{\infty} \frac{(it)^p}{p!} E\left(X^p\right)$$

$$\varphi_x(it) = \theta w\beta^w \sum_{m=0}^{\infty} \frac{\left(-\theta\beta^w\right)^w}{m!} \sum_{p=0}^{\infty} \frac{(it)^p}{p!} \frac{(-1)^{\lambda w(m+1)+p} \overline{)(\lambda w(m+1)+p)}}{[rw(m+1)]^{\lambda w(m+1)+p}} \left[\lambda - \frac{r(\lambda w(m+1)+p)}{rw(m+1)}\right] \qquad (33)$$

E. The Q^th Quantile Function

It is the real solution of

$$F\left|X_q\right| = q$$

Where $F\left|X_q\right| = 1 - e^{\left|-\theta\left(\beta x^\lambda e^{rx}\right)^w\right|}$

That implies

$$1 - e^{\left[-\theta\left(\beta x^\lambda e^{rx}\right)^w\right]} = q$$

$$e^{\left[-\theta\left(\beta x^\lambda e^{rx}\right)^w\right]} = 1 - q$$

$$-\theta\left(\beta x^\lambda e^{rx}\right)^w = \ln(1-q)$$

$$\left(\beta x^{\lambda} e^{rx}\right)^{w} = -\tfrac{1}{\theta} In(1-q)$$

$$\beta x_{q}^{\lambda} e^{rxq} = \left(-\frac{1}{\theta} In(1-q)\right)^{\frac{1}{w}}$$

$$x^{\lambda} e^{rx} = \frac{1}{\beta}\left[-\frac{1}{\theta} In(1-q)\right]^{\frac{1}{w}}$$

$$W_{q} = x^{\lambda} e^{rm} = \frac{1}{\beta}\left(-\frac{1}{\theta} In(1-q)\right)^{\frac{1}{w}} \qquad (34)$$

F. The Median

The median is obtained by setting q= 0.5 in Equation (34)

$$\text{M}= x_{0.5}^{\lambda} e^{rx_{0.5}} = \frac{1}{\beta}\left(-\frac{1}{\theta} In(1-0.5)\right)^{\frac{1}{w}} \qquad (35)$$

G. The R'enyi Entropy

This is defined by $\text{I}_{\text{R}}[\sigma] = \frac{1}{1-\sigma}\log(I(\sigma))$

Where $\text{I}(\sigma) = \int_{R} f^{\sigma}(x)dx,\ \sigma > 0 \ \ and \ \ \sigma \neq 1$ (36)

$$I(\sigma) = \left(\theta w \beta^{w}\right)^{\sigma} \int_{0}^{\infty} x^{\sigma(\lambda w-1)} (\lambda + rx)^{\sigma} e^{rwx\sigma} e^{-\theta\left(\beta^{w}\left(x^{\lambda} e^{rx}\right)^{w}\right)^{\sigma}} dx \qquad (37)$$

But

$$e^{rwx\sigma} e^{-\theta\left(\beta^{w}\left(x^{\lambda} e^{rx}\right)^{w}\right)^{\sigma}} = \sum_{m=0}^{\infty} \frac{\left(-\theta\beta^{w\sigma}\right)^{m}}{m!} x^{\lambda\sigma r w} e^{rw\sigma x} \qquad (38)$$

Substitute Equation 38) into Equation (37);

$$I(\sigma) = \left(\theta w \beta^{w}\right)^{\sigma} \sum_{m=0}^{\infty} \frac{\left(-\theta\beta^{w\sigma}\right)^{m}}{m!} \int_{0}^{\infty} x^{\lambda\sigma w(m+1)-\sigma} e^{r\sigma w(m+1)x} (\lambda + rx)^{\sigma} dx \qquad (39)$$

where $\quad (\lambda + rx)^{\sigma} = \sum_{q=0}^{\infty} \binom{\sigma}{p} (r)^{p} x^{p} \lambda^{\sigma - p}$ (40)

substitute Equation (40) into (39)

$$\left(\theta w \beta^{w}\right)^{\sigma} \sum_{m=0}^{\infty} \frac{\left(-\theta\beta^{w\sigma}\right)^{m}}{m!} \sum_{p=0}^{\infty} \binom{\sigma}{p} r^{p} \lambda^{r-p} \int_{0}^{\infty} x^{\lambda\theta w(m+1)-\sigma+p} e^{r\sigma w(m+1)x} dx \qquad (41)$$

The integral in Equation (41) is

$$\int_{0}^{\infty} x^{\lambda\sigma w(m+1)-\sigma+p} e^{r\sigma w(m+1)x} dx = \frac{(-1)^{\lambda\sigma w(m+1)-\sigma+p+1}}{(r\sigma w(m+1))^{\lambda\sigma w(m+1)-\sigma+p+1}} \overline{)(\lambda\sigma w(m+1)-\sigma+p+1)} \qquad (42)$$

Substitute Equation (42) into Equation (41)

$$I(\sigma)=\left(\theta w\beta^{w}\right)^{\sigma}\sum_{m=0}^{\infty}\frac{\left(-\theta\beta^{w\sigma}\right)^{m}}{m!}\sum_{p=0}^{\infty}\binom{\sigma}{p}r^{p}\lambda^{\lambda-p}\frac{(-1)^{\lambda\sigma w(m+1)-\sigma+p+1}}{\left(r\sigma w(m+1)\right)^{\lambda\sigma w(m+1)-\sigma+p+1}}\overline{)\left(\lambda\sigma w(m+1)-\sigma+p+1\right)}$$
(43)

Hence, the entropy

$$I_{R}(\sigma)=\frac{1}{1-\sigma}\log\ ($$

$$\left(\theta w\beta^{w}\right)^{\sigma}\sum_{m=0}^{\infty}\frac{\left(-\theta\beta^{w\sigma}\right)^{m}}{m!}\sum_{p=0}^{\infty}\binom{\sigma}{p}r^{p}\lambda^{\lambda-p}\frac{(-1)^{\lambda\sigma w(m+1)-\sigma+p+1}}{\left(r\sigma w(m+1)\right)^{\lambda\sigma w(m+1)-\sigma+p+1}}\overline{)\left(\lambda\sigma w(m+1)-\sigma+p+1\right)})\qquad(44)$$

H. Incomplete Moment

This is defined as

$$Vp(z)=\int_{0}^{z}x^{p}f(x)dx\qquad(45)$$

$$Vp(z)=\theta w\beta^{w}\int_{0}^{z}x^{\lambda w+p-1}(\lambda+rx)e^{rwx}e^{-\theta\beta^{w}\left(x^{\lambda}e^{rx}\right)^{w}}dx\qquad(46)$$

But $e^{-\theta\beta^{w}\left(x^{\lambda}e^{rx}\right)^{w}}=\sum_{m=0}^{\infty}\frac{\left(-\theta\beta^{w}\right)^{m}}{m!}x^{\lambda wx}e^{rmwx}$ (47)

Substitute Equation (47) into (46)

$$Vp(z)=\theta w\beta^{w}\sum_{m=0}^{\infty}\frac{\left(-\theta\beta^{w}\right)^{m}}{m!}\int_{0}^{z}x^{\lambda w(m+1)+p-1}e^{rw(m+1)x}(\lambda+rx)\,dx\qquad(48)$$

Consider the integrand in Equation (48);

$$\int_{0}^{z}x^{\lambda w(m+1)+p-1}e^{rw(m+1)x}(\lambda+rx)\,dx$$

$$=\lambda\int_{0}^{z}x^{\lambda w(m+1)+p-1}e^{rw(m+1)x}dx+r\int_{0}^{z}x^{\lambda w(m+1)+p}e^{rw(m+1)x}dx\qquad(49)$$

The first integrand

$$\int_{0}^{z}x^{\lambda w(m+1)+p-1}e^{rw(m+1)x}dx=\frac{(-1)^{\lambda w(m+1)+p}}{\left[rw(m+1)\right]^{\lambda w(m+1)+p}}\int_{0}^{z}u^{\lambda w(m+1)+p-1}e^{-u}du\ \ *$$

The integrand in equation * is an incomplete integrand, hence

$$\int_{0}^{z}x^{\lambda w(m+1)+p-1}e^{rw(m+1)x}dx=\frac{(-1)^{\lambda w(m+1)+p}}{\left[rw(m+1)\right]^{\lambda w(m+1)+p}}\gamma(\lambda w(m+1)+p)\qquad(49A)$$

Similarly, the second integrand in Equation (49);

$$\int_0^z x^{\lambda w(m+1)+p} e^{rw(m+1)x} dx = \frac{(-1)^{\lambda w(m+1)+p+1}}{[rw(m+1)]^{\lambda w(m+1)+p+1}} \gamma(\lambda w(m+1)+p+1) \qquad (49B)$$

Hence, substitute Equation (49A) and (49B) into Equation (49);

$$\int_0^z x^{\lambda w(m+1)+p-1} e^{rw(m+1)x} (\lambda + rx)\, dx$$

$$= \frac{\lambda(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \gamma(\lambda w(m+1)+p) + \frac{r(-1)^{\lambda w(m+1)+p+1}}{[rw(m+1)]^{\lambda w(m+1)+p+1}} \gamma(\lambda w(m+1)+p+1) \qquad (50)$$

Substitute Equation (50) into Equation (48);

$$Vp(z) = \theta w \beta^w \sum_{m=0}^{\infty} \frac{\left(-\theta\beta^w\right)^m}{m!} [\frac{\lambda(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \gamma(\lambda w(m+1)+p)$$

$$+ \frac{r(-1)^{\lambda w(m+1)+p+1}}{[rw(m+1)]^{\lambda w(m+1)+p+1}} c+p+1)\gamma(\lambda w(m+1)+p=1)\,] \qquad (51)$$

Setting p=1 in Equation (51), gives the first incomplete moment;

$$V_p(z) = \theta w \beta \frac{\sum_{m=0}^{\infty}\left(-\theta\beta^w\right)^m}{m!} \left[\frac{\lambda(-1)^{\lambda w(m+1)+1}}{\left(rw(m+1)^{\lambda w(m+1)+1}\right)} \gamma(\lambda w(m+1)+1) + \frac{r(-1)^{\lambda w(m+1)+2}}{(rw(m+1))^{\lambda w(m+1)+2}} \gamma(\lambda w(m+1)+2) \right] \qquad (52)$$

I. Mean Deviation

The mean deviation about the mean of the distribution is given by

$$\sigma_1(m) = \int_0^\infty |x-\mu| f(x) dx = 2\mu_1 F(\mu_1) - 2V_1(\mu_1) \qquad (53)$$

The mean deviation about the median of the distribution is

$$\sigma_2(x) = \int_0^\infty |x-\mu| f(x) dx = \mu_1 - 2v_1(\mu_1) \qquad (54)$$

Where $\mu_1 = E(x)$ which is the first non- central moment given in Equation (52):

F(μ) can be obtained from the cdf of the distribution, M is the median of the distribution stated in Equation (35) and $V_1(\mu_1)$ is the first incomplete moment stated in Equation.(52)

J. Bonferroni and Lorenz Curves

The Bonferoni curve of the distribution is defined as;

$$B(\pi) = \frac{V_1(q)}{(\pi\mu_1)} \qquad (55)$$

Where q= w(x) can be defined from the quantile function

$V_1(q)$ can be obtained from the first incomplete moment and μ is the first central non-moment

$$B(\pi)=\frac{\theta w\beta\frac{\sum_{m=0}^{\infty}\left(-\theta\beta^{w}\right)^{w}}{m!}\left[\frac{\lambda(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}}\gamma(\lambda w(m+1)+p)+\frac{r(-1)^{\lambda w(m+1)+p+1}}{[rw(m+1)]^{\lambda w(m=1)+p+1}}\gamma(\lambda w(m+1)+p+1)\right]}{\pi\theta w\beta^{w}\sum_{m=0}^{\infty}\frac{\left(-\theta\beta^{w}\right)^{m}}{m!}\frac{(-1)^{\lambda w(m+1)+1}(\lambda w(m+1)+1)}{[rw(m+1)]^{\lambda w(m+1)+1}}\left(\lambda-\frac{r(\lambda w(m+1)+1)}{rw(m+1)}\right)}$$

$$B(\pi)=\frac{\frac{\sum_{m=0}^{\infty}\left(-\theta\beta^{w}\right)^{w}}{m!}\left[\frac{\lambda(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}}\gamma(\lambda w(m+1)+p)+\frac{r(-1)^{\lambda w(m+1)+p+1}}{[rw(m+1)]^{\lambda w(m=1)+p+1}}\gamma(\lambda w(m+1)+p+1)\right]}{\pi\sum_{m=0}^{\infty}\frac{\left(-\theta\beta^{w}\right)^{m}}{m!}\frac{(-1)^{\lambda w(m+1)+1}}{[rw(m+1)]^{\lambda w(m+1)+1}}\left(\lambda-\frac{r(\lambda w(m+1)+1)}{rw(m+1)}\right)}$$

(56)

Lorenz curve of the distribution is defined as

$$L(\pi)=\frac{V_1(q)}{\mu_1^{1}} \qquad (57)$$

K. Conditional Moment

The p^th^ conditional moment of the distribution is given as;

$$E[X^{p}/X>t]=\frac{1}{\overline{F}(t)}\int_{t}^{\infty}x^{p}f(x)dx.$$

where $\overline{F}(t)=1-F(t)$.

If *f(x)* is the pdf of GMW distribution, then,

$$E[X^{p}/X>t]=\frac{\theta w\beta^{w}}{\overline{F}(t)}\int_{0}^{\infty}x^{\lambda w+p-1}(\lambda+rx)e^{rwx}e^{-\theta\beta^{w}\left(x^{\lambda}e^{rx}\right)^{w}}dx \qquad (58)$$

But

$$e^{-\theta\beta^{w}\left(x^{\lambda}e^{rx}\right)^{w}}=\sum_{m=0}^{\infty}\frac{\left(-\theta\beta^{w}\right)^{m}}{m!}x^{\lambda wx}e^{rmwx} \qquad *$$

Then,

$$E[X^{p}/X>t]=\frac{\theta w\beta^{w}}{\overline{F}(t)}\sum_{m=0}^{\infty}\frac{\left(-\theta\beta^{w}\right)^{m}}{m!}\int_{0}^{\infty}x^{\lambda w(m+1)+p-1}e^{rw(m+1)x}(\lambda+rx)dx \qquad (59)$$

$$E[X^{p}/X>t]=\frac{\theta w\beta^{w}}{\overline{F}(t)}\sum_{m=0}^{\infty}\frac{\left(-\theta\beta^{w}\right)^{m}}{m!}\left[\lambda\int_{t}^{\infty}x^{\lambda w(m+1)+p-1}e^{rw(m+1)x}dx\right]+r\int_{t}^{\infty}x^{\lambda w(m+1)+p}e^{rw(m+1)x}dx$$

(60)

Consider the first integrand

$$\int_t^\infty x^{\lambda w(m+1)+p-1} e^{rw(m+1)x} dx = \frac{(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \int_t^\infty u^{\lambda w(m+1)+p-1} e^{-u} du \quad (6.18A)(60A)$$

The integrand in Equation (60A);

$$\int_{t^\beta}^\infty u^{\lambda w(m+1)+p-1} e^{-u} du = \int_0^\infty u^{\lambda w(m+1)+p-1} e^{-u} du - \int_0^{t^\beta} u^{\lambda w(m+1)+p-1} e^{-u} du \quad (60B)$$

The first integral in Equation (60B) is a complete gamma function while the second is incomplete gamma function;

$$\int_t^\infty u^{\lambda w(m+1)+p-1} e^{-u} du = \Gamma(\lambda w(m+1)+p) - \gamma(\lambda w(m+1)+p, t^\beta) \quad (60C)$$

Hence $\int_t^\infty x^{\lambda w(m+1)+p-1} e^{rw(m+1)x} dx = \frac{(-1)^{\lambda w(m+1)=p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \left[\Gamma(\lambda w(m+1)+p) - \gamma(\lambda w(m+1)+p, t^\beta)\right]$

(61)

Similarly the integral in Equation (60);

$$\int_t^\infty x^{\lambda w(m+1)+p} e^{rw(m+1)x} dx = \frac{(-1)^{\lambda w(m+1)=p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \left[\Gamma(\lambda w(m+1)+p) - \gamma(\lambda w(m+1)+p, t^\beta)\right]$$

(60D)

Substitute Equation (60C), and (60D) into Equation (60), gives the conditional moment.

Hence,

$$E(x/x>T] = \frac{\theta w \beta^w}{\overline{F}(t)} \sum_{m=0}^\infty \frac{(-\theta\beta^w)^w}{m!} \left[\begin{array}{l} \frac{\lambda(-1)^{\lambda w(m+1)+p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \left(\Gamma(\lambda w(m+1)+p) - \gamma(\lambda w(m+1)+p, t^\beta)\right) + \\ \frac{r(-1)^{\lambda w(m+1)=p}}{[rw(m+1)]^{\lambda w(m+1)+p}} \left[\Gamma(\lambda w(m+1)+p) - \gamma(\lambda w(m+1)+p, t^\beta)\right] \end{array}\right] \quad (62)$$

L. Order of Statistics

The pdf of order statistics of the distribution (GMWE) is given by

$$f_{xp:n^{(x)}} = \frac{n!}{(p-1)!(n-p)!} f_x(x)(f_x(x))^{p-1}(1-f_x(x))^{n-p} \quad (63)$$

Substitute the pdf and cdf of the distribution into Equation (63), gives

$$\frac{n!}{(p-1)!(n-p)!} \theta w \beta^w (\lambda trm) x^{\lambda w-1} e^{rx} e^{\left(-\theta\beta^w (x^\lambda e^{rx})^w\right)} \left(1 - e^{\left(-\theta(\beta x^\lambda e^{rx})^w\right)}\right)^{p-1} e^{\left(-\theta(\beta x^\lambda e^{rx})^w\right)^{n-p}} \quad (64)$$

The pdf is maximum when p=n and it is given as;

$$f_{m,n,n}(x) = n\theta w \beta^w (\lambda + rx) x^{\lambda w-1} e^{rx} e^{\left(-\theta(\beta x^\lambda e^{rx})^w\right)} \left(1 - e^{\left(-\theta(\beta x^\lambda e^{rx})^w\right)^{n-1}}\right) \quad (65)$$

The pdf of the minimum statistics is given when p=1 and it is written as;

$$f_{m,1,n}(x) = n\theta w\beta^{w}(\lambda + rx)x^{\lambda w-1}e^{rx}e^{\left(-\theta\left(\beta x^{\lambda}e^{rx}\right)^{w}\right)}\left(e^{\left(-\theta\left(\beta x^{\lambda}e^{rx}\right)^{w}\right)^{n-1}}\right) \quad (66)$$

4.0 Results and Discussion

This section has to do with the application of the generalized distribution and study of its stability using asymptotic property. Simulation study was done using the probability density function derived to ascertain its asymptotic property.

4.1 Graphs of Generalized Modified Weibull-Exponential Distribution

For better understanding of the behaviour of the distribution, there is need for graphical presentation with varying parameter values. The graphs include graph of pdf, cdf, survival and hazard functions

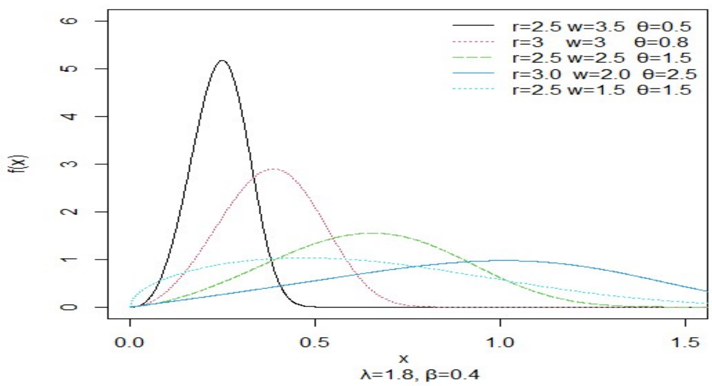

Fig. 1: Probability density function of GMWE distribution $(\lambda = 1.8, \beta = 0.4)$ and varying values of r, w and θ.

As shown in Figure 1, it can be observed that the distribution is capable of modeling positively skewed data as well as symmetric data. This shows its flexibility as parameter values change. In Figure 1, constant value was used for both lambda and beta with varying parameter values for r, w and ϑ. From the graph, the distribution is positively skewed when for the parameter values 2.5, 3.5 and 0.5 for r, w and ϑ respectively. Change in positively skewed position with reduced peak top was observed when parameter values changed to 3, 3, and 0.8 for r, w and ϑ respectively. It can be observed that the distribution is symmetric for parameter values 2.4, 2.5, 1.5, 1.8 and 0.4 for r, w, ϑ, λ and β respectively. This is an indication that the distribution is capable of modeling positively skewed and symmetric data which is an advantage over some of the existing Weibull class of distributions.

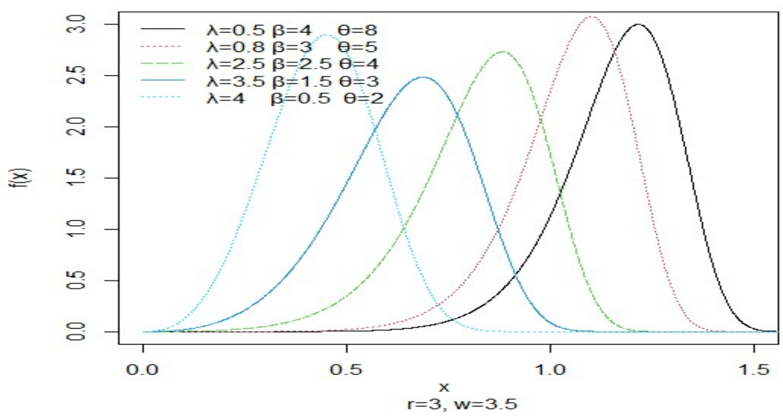

Fig. 2: Probability density function of GMWE distribution $(r=3, \beta=3.5)$ and varying values of λ, β and θ.

Figure 2 shows probability density function of GMWE distribution as parameters value changes. It shows the tendency of the distribution to model negatively skewed data. Varying parameter values were used to obtain negatively skewed nature of the distribution. As shown in figure 2, parameter values for r and w were kept constant at 3 and 3.5 respectively with varying values of $\lambda, \beta, and\ \theta$. The distribution became symmetry when lambda was 3.5, beta was 1.5 and theta 3 but negatively skewed at lambda of 0.5, beta was 4 and theta was 8. This shows the flexibility of the distribution in modeling different shapes of data. This property distinguished the newly derived distribution from the existing distributions in terms of modeling negatively skewed data.

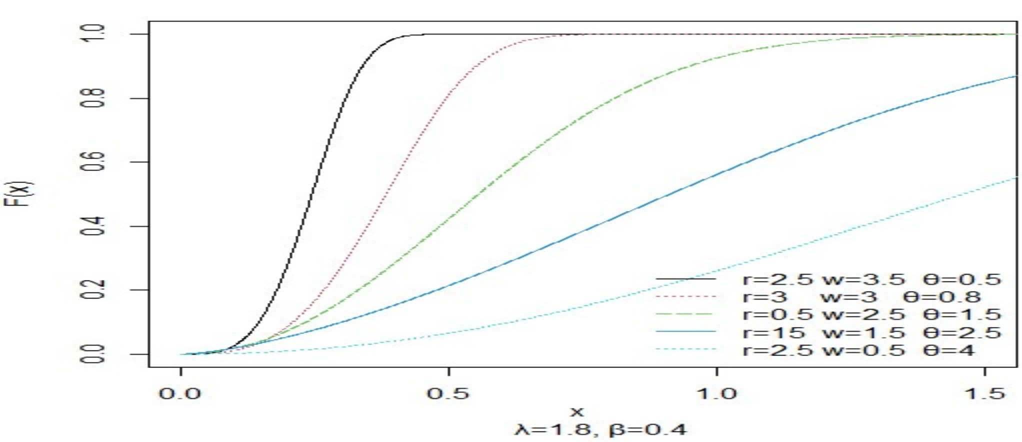

Fig. 3: Cumulative density function of GMWE distribution $(\lambda=1.8, \beta=0.4)$ and varying values of r, w and θ

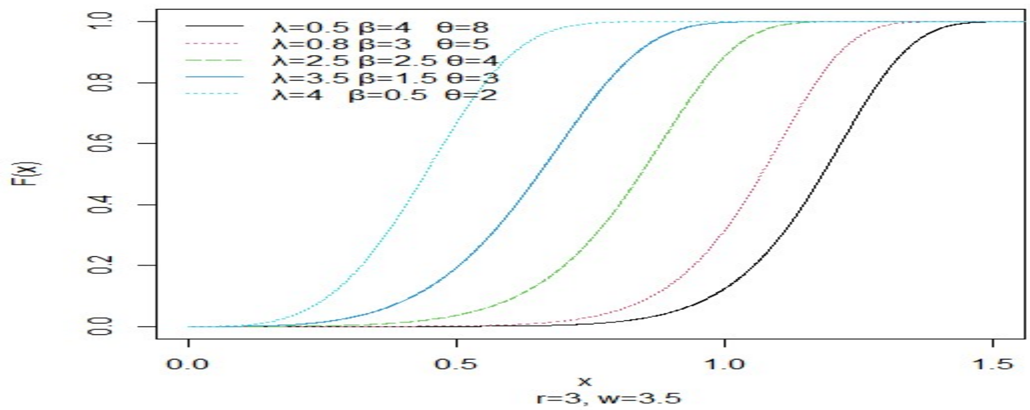

Fig. 4: Cumulative density function of GMWE distribution $(r=3, \beta=3.5)$ and varying values of λ, β and θ

Figures 3 and 4 are cumulative density function graphs of the newly derived distribution. Different pattern of graphs of cumulative density function were derived as a result of varying parameters value. It can be observed that the graphs show typical structure of a proper cumulative density function. This implies the resulting cumulative function is ideal and can be applied in the related studies.

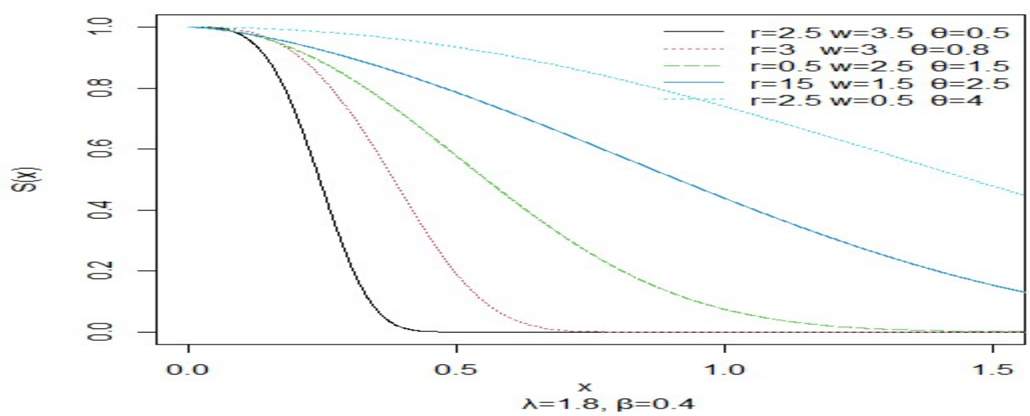

Fig. 5: Survival plot of GMWE distribution with $(\lambda=1.8, \beta=0.4)$ and varying values of r, w and θ

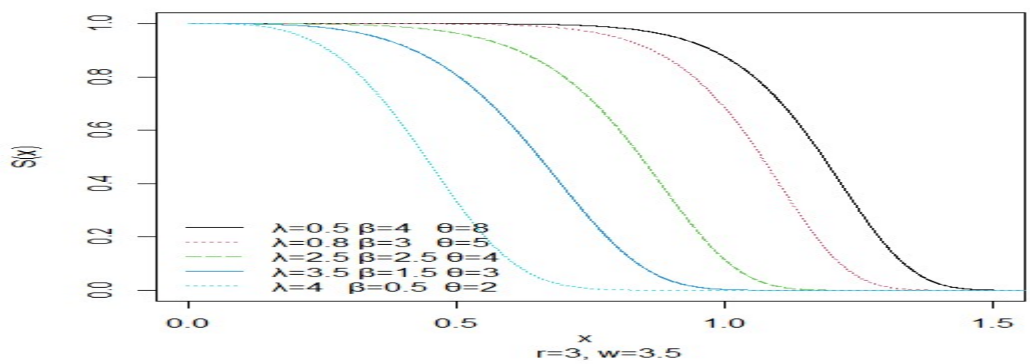

Fig. 6: Survival plot of GMWE distribution with $(r=3, \beta=3.5)$ and varying values of λ, $\beta$and θ

Figures 5 and 6 show the graph of survival function of the GMWE distribution. Parameters values were varied in order to get different shapes from the function. It can be observed that irrespective of the parameter values used on the function, the shape of a typical survival function is maintained.

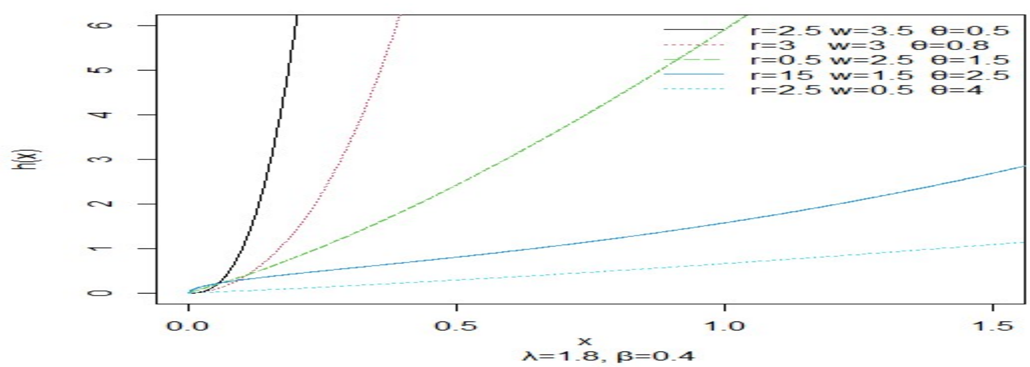

Fig. 7: Hazard plot of GMWE distribution with $(\lambda=1.8, \beta=0.4)$ and varying values of r, w and θ

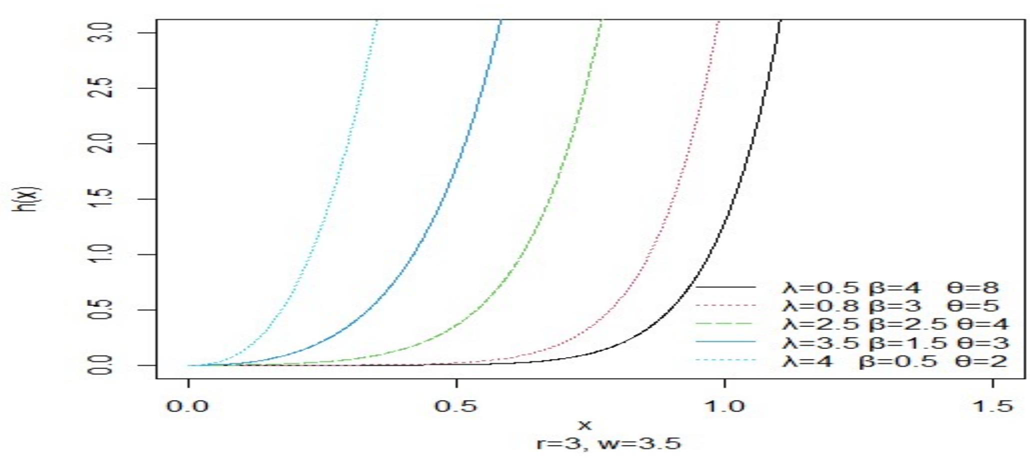

Fig. 8: Hazard plot of GMWE distribution with $(r=3, \beta=3.5)$ and varying values of λ, β and θ

As shown in figures 7 and 8, the pattern of the graphs of hazard function of the GMWE distribution show the function can be used to model different categories of hazard functions which show its superiority over some of the existing hazard functions.

4.2 Montė Carlo Simulation Study

The Montė Carlo simulation study was carried out for the study of homogenous properties of the distribution. Sample sizes considered are 50, 100, 200, and 500. The simulation was repeated with varying parameter values in order to get a clear picture of the homogenous property of the distribution. The modified distribution was also applied in modelling real life data set to show its application and for comparison.

Table 1: Simulation Study for GMWE Distribution

θ = 0.5, w = 2.0, r = 0.5		β = 0.5, λ = 0. 5			β = 1.0, λ = 0. 5			β = 0.5, λ = 1. 0			β = 1.0, λ = 1. 0		
N	**Par**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**
50	β	0.66996	0.16996	0.04326	1.05931	0.05931	0.03955	1.07604	0.57604	0.49288	2.47400	1.47400	4.11245
	θ	0.27408	0.22592	0.07908	0.36146	0.13854	0.02475	0.77901	0.22099	0.11052	0.71843	0.28157	0.09661
	λ	0.17980	0.32020	0.11247	0.55941	0.05941	0.11108	0.02598	0.47402	0.22507	0.04881	0.45119	0.20943
	W	2.04238	0.04238	0.01387	2.07391	0.07391	0.03821	2.30442	0.30442	0.60290	4.35585	2.35585	11.93941
	R	0.02085	0.47915	0.22986	0.02477	0.47523	0.22601	0.21348	0.28652	0.13788	0.01612	0.48388	0.23420
N	**Par**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**
100	β	0.63195	0.13195	0.02220	3.01853	2.01853	43.63015	0.79716	0.29716	0.09035	2.13987	1.13987	1.71791
	θ	0.25128	0.24872	0.07399	0.36783	0.13217	0.03884	1.03442	0.03442	0.02277	0.76391	0.23609	0.05970
	λ	0.26389	0.23611	0.06669	0.67682	0.17682	0.11356	0.05354	0.44646	0.19994	0.02444	0.47556	0.22758
	W	1.98371	0.01629	0.00147	6.34223	6.34223	193.86750	2.00434	0.00434	0.00084	3.71713	1.71713	3.95751
	R	0.03127	0.46873	0.22013	0.02285	0.47715	0.22791	0.47503	0.02497	0.02531	0.01710	0.48290	0.23326
N	**Par**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**	**Est**	**BIAS**	**MSE**
200	β	1.03219	0.53219	0.78174	0.90957	0.09043	0.01024	0.92569	0.42569	0.31348	1.92623	0.92623	1.28545
	θ	0.43110	0.06890	0.00731	0.31010	0.18990	0.03620	0.91748	0.08252	0.07501	0.72997	0.27003	0.09343
	λ	0.07298	0.42702	0.18862	0.83169	0.33169	0.11948	0.10139	0.39861	0.16140	0.04395	0.45605	0.21197
	W	2.64090	0.64090	2.92184	1.93788	0.06212	0.00475	2.26229	0.26229	0.72838	3.40372	1.40372	2.91138
	R	0.02377	0.47623	0.22706	0.03479	0.46521	0.21678	0.41331	0.08670	0.04354	0.02952	0.47048	0.22148
500	β	0.78162	0.28162	0.08731	0.91335	0.08665	0.00813	0.87085	0.37085	0.13760	2.28873	1.28873	2.04657
	θ	0.48702	0.01298	0.00432	0.31382	0.18618	0.03472	1.00802	0.00802	0.00007	0.77545	0.22455	0.05524
	λ	0.08781	0.41219	0.17212	0.81773	0.31773	0.10370	0.10018	0.39982	0.15993	0.03052	0.46948	0.22379
	W	2.15298	0.15298	0.03722	1.93973	0.06027	0.00389	2.00658	0.0000001	0.00004	3.97799	1.97799	4.98638
	R	0.02648	0.47352	0.22448	0.02716	0.47284	0.22409	0.50297	0.00297	0.00001	0.02496	0.47504	0.22602

As shown in Table 1, four different categories were considered by varying the parameter values in the model. Biasness and Mean Square Error (MSE) were used for the evaluation of the model. Estimates of the parameters were computed and can be compared with the actual values of the parameters as presented in Table 1. Considering the results in Table 1, increase in sample size led to closeness of the estimates to the true values of the parameters which signifies stability of the parameters as sample size increases. This implies the model is reliable.

4.3 Model Comparison

As part of perfomance test, there is need to compare the distribution with existing ones in the same category using secondary data. Using a data set previously used by Ahmed et al. (2015) on lenght of 10mm from Kandu and Raqab (2009), the data set consists of 63 observations. See Table 2

Table 2: Lenght of 10mm Extracted from work of Kanda and Raqab (2009)

1.901	2.132	2.203	2.228	2.257	2.35	2.361	2.396	2.397	2.445
2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614	2.616	2.618
2.624	2.659	2.675	2.738	2.74	2.856	2.917	2.928	2.937	2.937
2.977	2.996	3.03	3.125	3.139	3.145	3.22	3.223	3.235	3.243
3.264	3.272	3.294	3.332	3.346	3.377	3.408	3.435	3.493	3.501
3.537	3.554	3.562	3.628	3.852	3.871	3.886	3.971	4.024	4.027
4.225	5.395	5.02							

It was used to show the suitability and superiority of Transmitted Weibull-Pareto (TWPa) distribution over Weibull pareto (WPa), Transmuted Weibull Lonax (TWL), Transmutted Complimentary Weibull (TCW) and McDonald Lomax (McL) distributions. Using the data for modelling, the output is as shown below;

Table 3: Output of the Analysis of 10mm data modelling

Models	Estimates	AIC
GMWE	β = 1.2057 λ = 0.0471 θ = 1.7427 w = 1.5285 r = 0.2932	124.331
TWPa	a=0.1885 b=0.0909 c=14.4535 d=0.7280	127.282
Wpa	a=0.1834 b=0.0755 c=13.9522	127.790
TWL	a=0.3922 b=0.6603 c=0.5287 d=8.4451 e=0.7364	129.688
McL	a=45.9249 b=48.3024 c=18.1192 d=195.4633 e=353.1435	140.597
TCW	a=0.2022 b=3.3482 c=0.3076 d=-0.0001	134.895

Table 3 shows the superiority of Generalized Modified Weibull-Exponential distribution over five other existing distributions. It can be deduced that in modeling the data extracted from the work of Ahmed et al. (2016) on lenght of 10mm, the most appropriate model is Generalized Modified Weibull-Exponential distribution as it has the lowest AIC value among the AIC values for the distributions compared with.

5.0 Summary of Findings

In this paper, Transform-transformer approach was used as a method of generalization of distribution. Modified Weibull of three parameters was used as the baseline function which results to a five-parameter modified Weibull distribution called Generalized Modified Weibull-Exponential (GMWE) distribution. The newly generated distribution, GMWE, was tested for completeness using one of the properties of a proper probability density function. The test of completeness was done using the property called area under curve and it yielded 1 as a result of integral of the probability density function.

To buttress the point, statistical properties of the GMWE were studied which include moment, moment generating function, characteristic function, median, quantile function, mean, variance, mean deviation, incomplete moment, Lorenz and Bonferroni curves, conditional moments, order statistics. Parameters in the formulated model were estimated using the method of Maximum Likelihood.

Graph of probability density function, cumulative density function, survival function and hazard function of the distribution were plot using different parameter values. Also, Monte Carlo simulation approach was used for the study of stability (homogeneity) of the distribution. In the simulation, three replicates were used at varying parameter values and different sample sizes of 50, 100, 200 and 500. Biasness and Mean Square Error (MSE) were used for the appropriateness of the estimates. It was observed that the estimates approach true values of the parameters as sample size increases which lead to significant reduction in the biasness and MSE. Based on these facts, it was concluded that the resulting distribution is stable and can be used for modeling.

For more fact finding, the newly generalized distribution was showed to have the tendency to model skewed data (positive or negative) which distinguish it from the baseline function and some of the existing distributions in its category. It was also compared with existing distributions in its category using secondary data. Data on length of 10mm rod previously used by Kandu and Raqab (2009) and Ahmed et al. (2015) was used for the comparison. In the ranking of the distributions using AIC, it was observed that GMWE performed better than the distributions compared with. Therefore, the newly generated distribution is recommended for usage in studies involving probability density function of its category, especially in modelling skewed data; right or left skewed data.

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