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Recurrence Relations for Moment Generating Function Based on Progressive First Failure Censoring from Generalized Pareto Distribution and Characterization

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ABSTRACT

In this article, we establish recurrence relations (RR) for single and product moment generating function (MGF) based on based on progressive first failure censoring (PFFC) for generalized Pareto distribution (GPD). Characterization for GPD using RR of single and product MGF of PFFC are also obtained. Further, the results are specialized to the progressively type-II right censored (PTIIRCOS).

Keywords: Characterization; Generalized Pareto Distribution; Moment Generating Function; progressive first failure censoring.

1. Introduction

Suppose that n independent groups with k items within each group are put on a life test. R_1 groups and the group in which the first failure is observed are randomly removed from the test as soon as the first failure $X_{1:m:n,k}^{(R_1,R_2,\dots,R_m)}$ has occurred and finally when the m^{th} failure $X_{m:m:n,k}^{(R_1,R_2,\dots,R_m)}$ is observed, the remaining groups R_m are removed from the test. Then $X_{1:m:n,k}^{(R_1,R_2,\dots,R_m)} < \cdots < X_{m:m:n,k}^{(R_1,R_2,\dots,R_m)}$ are called progressively first-failure censored order statistics with the progressive censored scheme, where $n = m + \sum_{i=1}^{m} R_i$. If the failure times of the $n \times k$ items originally in the test are from a continuous population with cdf and pdf, the joint pdf for $X_{1:m:n,k}^{(R_1,R_2,\dots,R_m)}, \dots, X_{m:m:n,k}^{(R_1,R_2,\dots,R_m)}$ is defined as follows:

$$f_{X_{1:m:n},\dots,X_{m:m:n}}(x_1,\dots,x_{m-1},x_m) = K_{(n,m-1)} k^m \prod_{i=1}^{m} f(x_i)[\bar{F}(x_i)]^{kR_i+k-1},$$
$$0 < x_1 < x_2 < x_3 < \cdots < x_{m-1} < x_m < \infty, \qquad (1.1)$$

where,
$K_{(n,m-1)} = n(n-R_1-1)(n-R_1-R_2-2)\dots(n-R_1-R_2-R_3\dots-R_{m-1}-m+1)$.

Aggarwala and Balakrishnan [2] derived RR for single and product moments of PTIIRCOS from exponential distribution. Mohie El-Din et al. [7,8,9,10,11] derived RR of moments of the extended power Lindley, generalized Pareto, Gompertz, linear failure rate and Marshall-Olkin extended Burr XII distributions based on general PTIIRCOS and characterization. Sadek et al. [12] derived characterization for generalized power function distribution using RR based on general PTIIRCOS. Marwa Mohie El-Din and Sharawy [5,6] derived RR for the extension of exponential and generalized exponential distributions based on general PTIIRCOS. Kotb et al. [4] derived E-Bayesian estimation for Kumaraswamy distribution using PFFC. Abu-

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Moussa et al. [1] derived estimation of reliability functions for the extended Rayleigh distribution under PFFC.

Throughout this paper, we introduce RR of MGF based on PFFC. Also characterization for exponential distribution using RR of MGF based on PFFC, are obtained.

We introduce the RR of MGF for GPD based on PFFC.

The pdf is

$$f(x,\theta,\alpha,\sigma)=\frac{1}{\sigma}\left[1-\frac{\alpha}{\sigma}(x-\mu)\right]^{\left(\frac{1}{\alpha}-1\right)} \qquad \sigma>0\,, \qquad \mu,\alpha\in R. \quad (1.2)$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x,\theta,\alpha,\sigma)=1-\left[1-\frac{\alpha}{\sigma}(x-\mu)\right]^{\left(\frac{1}{\alpha}\right)} \quad (1.3)$$

For $\alpha\le 0$ we have $\mu\le x<\infty$ and for $\alpha>0$ we have $\mu\le x<\mu+\frac{\sigma}{\alpha}$.

It may be noticed that from (1.2) and (1.3) the relation between pdf and cdf is given by,

$$[\sigma+\alpha\mu-\alpha x]f(x)=1-F(x) \quad (1.4)$$

The single MGF of the PFFC in view of (1.1) as

$$M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)}=E\left[e^{tx_q X_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)}}\right]=$$
$$K_{(n,m-1)}\iint\cdots\int_{0<x_1<\cdots<x_m<\infty} e^{tx_q}k^m f(x_1)[\bar F(x_1)]^{kR_1+k-1}\times$$
$$f(x_2)[\bar F(x_2)]^{kR_2+k-1}\ldots f(x_m)[\bar F(x_m)]^{kR_m+k-1}dx_1\ldots dx_m, \quad (1.5)$$

and product MGF

$$M_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)}=K_{(n,m-1)}\iint\cdots\int_{0<x_1<\cdots<x_m<\infty} e^{(t_1x_q+t_2x_s)}k^m\times$$
$$f(x_1)[\bar F(x_1)]^{kR_1+k-1}\ldots f(x_m)[\bar F(x_m)]^{kR_m+k-1}dx_1\ldots dx_m. \quad (1.6)$$

2. Recurrence Relations

In this section, we introduce the RR for single and product MGF based on PFFC.

In the next theorem we introduce RR for single MGF based on PFFC.

Theorem 2.1

For $2\le q\le m-1,\ m\le n$, then

$$(\alpha t)M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(1)}}=\left[(\sigma+\alpha\mu)(t)-k\left(R_q+1\right)\right]M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)}$$
$$+\left(n-R_1-\cdots-R_{q-1}-q+1\right)M_{q-1:m-1:n,k}^{\left(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1\right)}$$
$$-\left(n-R_1-\cdots-R_q-q\right)M_{q:m-1:n,k}^{\left(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1\right)}. \quad (2.1)$$

Proof

From (1.4) and (1.5), we get

$$(\sigma+\alpha\mu)M_{q:m:n}^{(kR_1+k-1,\dots,kR_m+k-1)}-\alpha M_{q:m:n}^{(kR_1+k-1,\dots,kR_m+k-1)^{(1)}}=$$

$$K_{(n,m-1)}\iint\dots\int_{0<x_1<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty}A_1(x_{q-1},x_{q+1})k^m$$
$$f(x_1)[\bar F(x_1)]^{kR_1+k-1}\dots f(x_{q-1})[\bar F(x_{q-1})]^{kR_{q-1}+k-1}f(x_{q+1})\times$$
$$[\bar F(x_{q+1})]^{kR_{q+1}+k-1}\dots f(x_m)[\bar F(x_m)]^{kR_m+k-1}dx_1\dots dx_{q-1}dx_{q+1}\dots dx_m, \quad (2.2)$$

where

$$A_1(x_{q-1},x_{q+1})=\int_{x_{q-1}}^{x_{q+1}}e^{tx_q}\,[\bar F(x_q)]^{kR_q+k}\,dx_q. \quad (2.3)$$

Upon integrating the integral in (2.3) by parts, we get

$$A_1(x_{q-1},x_{q+1})=\frac{e^{tx_{q+1}}[\bar F(x_{q+1})]^{kR_q+k}-e^{tx_{q-1}}[\bar F(x_{q-1})]^{kR_q+k}}{t}$$
$$+\left(\frac{kR_q+k}{t}\right)\int_{x_{q-1}}^{x_{q+1}}e^{tx_q}f(x_q)[\bar F(x_q)]^{kR_q+k-1}\,dx_q. \quad (2.4)$$

Substituting by Eq. (2.4) in Eq. (2.2) and simplifying, yields Eq. (2.1).
This completes the proof.

Special cases

1- Theorem 2.1 will be valid for the PTIIRCOS when $k=1$,

$$\alpha tM_{q:m:n}^{(R_1,R_2,\dots,R_m)^{(1)}}=[(\sigma+\alpha\mu)(t)-(R_q+1)]M_{q:m:n}^{(R_1,R_2,\dots,R_m)}$$
$$+(n-R_1-\dots-R_{q-1}-q+1)M_{q-1:m-1:n}^{(R_1,R_2,\dots,R_{q-2},(R_{q-1}+R_q+1),R_{q+1},\dots,R_m)}$$
$$-(n-R_1-\dots-R_q-q)M_{q:m-1:n}^{(R_1,R_2,\dots,R_{q-1},(R_q+R_{q+1}+1),R_{q+2},\dots,R_m)}.$$

2- For $r=0$ *and* $q=m$

$$\alpha tM_{m:m:n}^{(R_1,R_2,\dots,R_m)^{(1)}}=[(\sigma+\alpha\mu)(t)-(R_m+1)]M_{m:m:n}^{(R_1,R_2,\dots,R_m)}$$
$$+(n-R_1-\dots-R_{m-1}-m+1)M_{m-1:m-1:n}^{(R_1,R_2,\dots,R_{m-2},(R_{m-1}+R_m+1))}.$$

3- For $r=0,\ m=1, n=1$ and $R_1=\dots=R_m=0$,

$$\alpha tM^{(1)}=[(\sigma+\alpha\mu)(t)-1]M+e^{t\mu},$$

and when $t=0$ we obtained the expected value $E[X]=\frac{\sigma+\alpha\mu+\mu}{1+\alpha}$.

4- From 3 we obtained the second moment,

$$E[X^2]=\frac{2\delta^2+4\delta\mu\alpha+2\delta\mu+2\mu^2\alpha^2+2\mu^2\alpha+\mu^2}{1+3\alpha+2\alpha^2}.$$

5- From 3 and 4, we obtained the variance

$$Var(x)=\frac{\delta^2}{(1+2\alpha)(1+\alpha)^2}.$$

In the next two theorems, we introduce RR for product continuous function based on PFFC.

Theorem 2.2

for $1\le q<s\le m-1,\ m\le n$, then

$$(\alpha t)M_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(1,0)}}=[(\sigma+\alpha\mu)(t)-k(R_q+1)]M_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)}$$
$$+(n-R_1-\dots-R_{q-1}-q+1)M_{q-1,s-1:m-1:n,k}^{((kR_1+k-1,\dots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\dots,kR_m+k-1)}$$

$$-\left(n-R_1-\cdots-R_q-q\right)M_{q,s-1:m-1:n,k}^{\left(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1\right)}. \quad (2.5)$$

Proof

Similarly as proved in Theorem 2.1.

Theorem 2.3

For $1 \le q < s \le m-1$, $m \le n$, then

$$[\alpha t]M_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(0,1)}} = [(\sigma+\alpha\mu)(t)-k(R_s+1)]M_{q,s:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)}$$
$$+(n-R_1-\cdots-R_{s-1}-s+1)M_{q,s-1:m-1:n,k}^{(kR_1+k-1,\ldots,kR_{s-1}+kR_s+2k-1,kR_{s+1}+k-1,\ldots,kR_m+k-1)}$$
$$-(n-R_1-\cdots-R_s-s)M_{q,s:m-1:n,k}^{\left(kR_1+k-1,\ldots,kR_q+kR_{s+1}+2k-1,kR_{s+2}+k-1,\ldots,kR_m+k-1\right)}. \quad (2.6)$$

Proof

Similarly as proved in Theorem 2.1.

Special case

For $k=1$, we obtain the recurrence relation of PTIIRCOS.

$$(\alpha t)M_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(0,1)}} = [(\sigma+\alpha\mu)(t)-(R_s+1)]M_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)}$$
$$+(n-R_1-\cdots-R_{s-1}-s+1)M_{q-1,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{s-2},(R_{s-1}+R_s+1),R_{s+1},\ldots,R_m)}$$
$$-(n-R_1-\cdots-R_s-s)M_{q,s:m-1:n}^{(R_1,R_2,\ldots,R_{s-1},(R_s+R_{s+1}+1),R_{s+2},\ldots,R_m)},$$

and for $s=m$

$$[\alpha t]M_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)^{(1,0)}} = [(\sigma+\alpha\mu)(t)-(R_m+1)]M_{q,s:m:n}^{(R_1,R_2,\ldots,R_m)}$$
$$+(n-R_1-\cdots-R_{m-1}-m+1)M_{q-1,s-1:m-1:n}^{(R_1,R_2,\ldots,R_{m-2},(R_{m-1}+R_m+1))}.$$

3. The Characterizations

In this section, we introduce the characterization of the GPD using the relation between pdf and cdf and using RR for single and product MGF based on PFFC.

3.1 Characterization via differential equation for the GPD

In the next theorem, we introduce the characterization of the GPD using relation between pdf and cdf.

Theorem 3.1

Let X be a continuous random variable with pdf $f(\cdot)$, cdf $F(\cdot)$ and survival function $[\bar{F}(\cdot)]$. Then X has GPD iff

$$[\bar{F}(x)] = [\sigma+\alpha\mu-\alpha x]f(x) \quad (3.1)$$

Proof

Necessity:

From Eq. (1.2) and Eq. (1.3) we can easily obtain Eq. (3.1).

Sufficiency:

Suppose that Eq. (3.1) is true. Then we have:

$$\frac{-d[\bar{F}(x)]}{\bar{F}(x)} = \frac{1}{\sigma+\alpha\mu-\alpha x}dx.$$

By integrating, we get

$$\ln|\bar{F}(x)| = \frac{1}{\alpha}\ln|\sigma+\alpha\mu-\alpha x|+C, \quad (3.2)$$

where C is an arbitrary constant.

Now, since $[\bar{F}(\mu)] = 1$, then putting $x=\mu$ in (3.2), we get $C = \frac{-1}{\alpha}\ln|\sigma|$.

Therefore,

$$\ln|[\bar{F}(x)]| = \frac{1}{\alpha}\ln\left|\frac{\sigma+\alpha\mu-\alpha x}{\sigma}\right|,$$

hence,

$$F(x) = 1 - \left[\frac{\sigma + \alpha\mu - \alpha x}{\sigma}\right]^{\frac{1}{\alpha}}.$$

That is the distribution function of GPD.

This completes the proof.

3.2 Characterization via RR for MGF

In the next theorem, we will introduce the characterization of the GPD using RR for single MGF based on RFFC.

Theorem 3.2

Let $X_{1:n} \leq \ldots \leq X_{n:n}$ be the order stasistics of a random sample of size n Then X has GPD iff, for $2 \leq q \leq m-1, m \leq n$ and $i \geq 0$,

$$(\alpha t) M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(1)}} = \left[(\sigma + \alpha\mu)(t) - k\left(R_q + 1\right)\right] M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)}$$
$$+\left(n - R_1 - \cdots - R_{q-1} - q + 1\right) M_{q-1:m-1:n,k}^{\left(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1\right)}$$
$$-\left(n - R_1 - \cdots - R_q - q\right) M_{q:m-1:n,k}^{\left(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1\right)}. \quad (3.3)$$

Proof

Necessity:

2.1 proved the necessary part of this theorem. Theorem

Sufficiency:

Assuming that (3.3) holds, then we have:

$$(\alpha t) M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)^{(1)}} = \left[(\sigma + \alpha\mu)(t) - k\left(R_q + 1\right)\right] M_{q:m:n,k}^{(kR_1+k-1,\ldots,kR_m+k-1)}$$
$$+\left(n - R_1 - \cdots - R_{q-1} - q + 1\right) M_{q-1:m-1:n,k}^{\left(kR_1+k-1,\ldots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\ldots,kR_m+k-1\right)}$$
$$-\left(n - R_1 - \cdots - R_q - q\right) M_{q:m-1:n,k}^{\left(kR_1+k-1,\ldots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\ldots,kR_m+k-1\right)}. \quad (3.4)$$

where,

$$\mu_{q:m:n}^{(kR_1+k-1,\ldots,kR_m+k-1)} =$$
$$K_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} A_2\left(x_{q-1}, x_{q+1}\right) k^m$$
$$f(x_1)[\bar{F}(x_1)]^{kR_1+k-1} \ldots f\left(x_{q-1}\right)\left[\bar{F}\left(x_{q-1}\right)\right]^{kR_{q-1}+k-1} f\left(x_{q+1}\right)\left[\bar{F}\left(x_{q+1}\right)\right]^{kR_{q+1}+k-1} \times$$
$$\ldots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1} dx_1 \ldots dx_{q-1} dx_{q+1} \ldots dx_m, \quad (3.5)$$

where

$$A_2\left(x_{q-1}, x_{q+1}\right) = \int_{x_{q-1}}^{x_{q+1}} e^{tx_q} \left[1 - F\left(x_q\right)\right]^{kR_q+k} dx_q. \quad (3.6)$$

Upon integrating the integral in (3.6) by parts, we get

$$A_2\left(x_{q-1}, x_{q+1}\right) = \frac{e^{tx_{q+1}}\left[\bar{F}\left(x_{q+1}\right)\right]^{kR_q+k} - e^{tx_{q-1}}\left[\bar{F}\left(x_{q-1}\right)\right]^{kR_q+k}}{t}$$
$$+\left(\frac{kR_q + k}{t}\right) \int_{x_{q-1}}^{x_{q+1}} e^{tx_q} f\left(x_q\right)\left[\bar{F}\left(x_q\right)\right]^{kR_q+k-1} dx_q. \quad (3.7)$$

Substituting in (3.5), we get

$$K_{(n,m-1)} \iint \cdots \int_{0<x_1<\cdots<x_{q-1}<x_{q+1}<\cdots<x_m<\infty} e^{tx_q} \left(\sigma + \alpha\mu - \alpha x_q\right) \times$$
$$k^m f\left(x_{q-1}\right)\left[\bar{F}\left(x_q\right)\right]^{kR_q+k-1} f(x_1)[\bar{F}(x_1)]^{kR_1+k-1} \ldots \times$$

$$f(x_{q-1})[\bar{F}(x_{q-1})]^{kR_{q-1}+k-1}f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\times$$
$$\dots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1}dx_1\dots dx_{q-1}dx_{q+1}\dots dx_m=$$
$$K_{(n,m-1)}\iint\dots\int_{0<x_1<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty}e^{tx_q}\times$$
$$k^m[\bar{F}(x_q)]^{kR_q+k}f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\dots f(x_{q-1})\times$$
$$[\bar{F}(x_{q-1})]^{kR_{q-1}+k-1}f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}\dots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1}$$
$$dx_1\dots dx_{q-1}dx_{q+1}\dots dx_m.$$

We get

$$K_{(n,m-1)}\iint\dots\int_{0<x_1<\dots<x_{q-1}<x_{q+1}<\dots<x_m<\infty}e^{tx_q}k^m\times$$
$$[\bar{F}(x_q)]^{R_q}\left[(\sigma+\alpha\theta-\alpha x_q)f(x_q)-[\bar{F}(x_q)]\right]f(x_1)[\bar{F}(x_1)]^{kR_1+k-1}\times$$
$$\dots f(x_{q-1})[\bar{F}(x_{q-1})]^{kR_{q-1}+k-1}f(x_{q+1})[\bar{F}(x_{q+1})]^{kR_{q+1}+k-1}$$
$$\dots f(x_m)[\bar{F}(x_m)]^{kR_m+k-1}dx_1\dots dx_{q-1}dx_{q+1}\dots dx_m=0. \qquad (3.8)$$

Using Müntz-Szasz theorem, [See, Hwang and Lin [3]], we get

$$(\sigma+\alpha\mu-\alpha x_q)f(x_q)=[\bar{F}(x_q)].$$

Using Theorem 3.1, we get

$$F(x)=1-\left[1-\frac{\alpha}{\sigma}(x-\mu)\right]^{\left(\frac{1}{\alpha}\right)}.$$

That is the distribution function of GPD.

This completes the proof.

In the next two theorems, we will introduce the characterization of the GPD using RR for product MGF based on PFFC.

Theorem 3.3

Let $X_{1:n}\le X_{r+2:n}\le\dots\le X_{n:n}$ be the order stasistics of a random sample of size n. Then X has GPD iff, for $1\le q<s\le m-1,\ \ m\le n$,

$$(\alpha t)M_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(1,0)}}=[(\sigma+\alpha\mu)(t)-k(R_q+1)]M_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)}$$
$$+(n-R_1-\dots-R_{q-1}-q+1)M_{q-1,s-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{q-1}+kR_q+2k-1,kR_{q+1}+k-1,\dots,kR_m+k-1)}$$
$$-(n-R_1-\dots-R_q-q)M_{q,s-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_q+kR_{q+1}+2k-1,kR_{q+2}+k-1,\dots,kR_m+k-1)}.$$

Proof

Similarly as proved in Theorem 3.2.

Theorem 3.4

Let $X_{1:n}\le\dots\le X_{n:n}$ be the order stasistics of a random sample of size n. Then X has GPD iff, for $1\le q<s\le m-1\,, m\le n$,

$$(\alpha t)M_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)^{(0,1)}}=[(\sigma+\alpha\mu)(t)-k(R_s+1)]M_{q,s:m:n,k}^{(kR_1+k-1,\dots,kR_m+k-1)}$$
$$+(n-R_1-\dots-R_{s-1}-s+1)M_{q,s-1:m-1:n,k}^{(kR_1+k-1,\dots,kR_{s-1}+kR_s+2k-1,kR_{s+1}+k-1,\dots,kR_m+k-1)}$$
$$-(n-R_1-\dots-R_s-s)M_{q,s:m-1:n,k}^{(kR_1+k-1,\dots,kR_s+kR_{s+1}+2k-1,kR_{s+2}+k-1,\dots,kR_m+k-1)}.$$

Proof

Similarly as proved in Theorem 3.2.

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