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Statistical Modeling for Flood Frequency in Upper Chaophraya River Basin

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ABSTRACT

The research deals with the probabilistic estimates of annual maximum flood peaks in the upper Chaophraya basin (Thailand) used the extreme value theory, the Block Maxima. The Generalized Extreme Value (GEV) distribution model, the Frèchet distribution (EV2), the extension of the Frèchet distribution such as the Kumaraswamy Frèchet distribution and the new distribution as called the Exponentiated Kumaraswamy Frèchet distribution satisfied the Goodness of fit test (Kolmokorov-smirnov test). The return levels are estimated for 3, 5, 10, 30, 50, 100, 500 and 1000 years which are consistently increasing for designs of flood protection in future. The return period of flood for each stations are estimated.

The investigation of the new distribution and appropriated estimation technique for the flood frequency in upper Chaophraya river basin as we called the Exponentiated Kumaraswamy Frèchet distribution and the differential evolution maximum likelihood estimation were done. We derived the properties of the Frèchet family; such as the Frèchet distribution (EV2), the Kumaraswamy Frèchet distribution (KF) and the Exponentiated Kumaraswamy Frèchet distribution (EKF). We also compared Bias, Variance, Mean Square Error and Mean Absolute Percentage Error for all parameters in each distribution by generating the Frèchet family random number. For the effectiveness of analytical solutions of the parameters we provided the numerical solutions (differential evolution method) to obtain estimates for all parameters by using Scilab program.

Accuracy of flood assessment of extreme event is of fundamental importance for many safety, engineering and financial application. In part of application we provided the probabilistic estimates of annual maximum flood peaks or momentary peak data in the upper Chaophraya river basin (Thailand). The Generalized Extreme Value (GEV) distribution model were used to be gain to compare with as the Frèchet distribution (EV2), the Kumaraswamy Frèchet distribution (KF) and the Exponentiated Kumaraswamy Frèchet distribution (EKF). The Goodness of fit test, the return level and return period were done. The return periods of flood were classified by hazard class using GEV found that in upper Chaophraya river basin flood occurred highly. The result from the Frèchet family also occurred highly, but the return period and return level from the Kumaraswamy Frèchet distribution (KF) quite closed to GEV more than another distribution.

Keywords: Generalized Extreme Value distribution, Frèchet distribution, Kumaraswamy Frèchet distribution, Exponentiated Kumaraswamy Frèchet distribution, differential evolution maximization estimation, Flood Frequency, Return Level, Retrun Period

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1 Introduction

Frequency analysis is an information problem: if one had a sufficiently long record of flood flows , rainfall, low flows, or pollutant loadings, then a frequency distribution for site could be precisely determined , so long as change over time due to urbanization or natural processes did not alter the relationships of concern. In most situations, available data are insufficient to precisely define the risk of large floods, rainfall, pollutant loadings, or low flows. This forces hydrologists to use practical knowledge of the processes involved, and efficient and robust statistical techniques, to develop the best estimate of risk that they can. These techniques are generally restricted, with 10 to 100 sample observations to estimate events exceeded with a chance of at least 1 in 100, corresponding to exceeded probabilities of 1 percent or more. In some cases, they are used to estimate the rainfall exceeded with a chance of 1 in 1000, and even the flood flows for spillway design exceeded with a chance of 1 in 10,000.

The Chao Phraya basin is mountainous with agriculturally productive valleys found in the upper region. The lower region contains alluvial plains that are highly productive for agriculture. The Chao Phraya River drains from north to south. Monsoon weather dominates, with a rainy season lasting from May to October and supplementary rain from occasional westward storm depressions originating in the Pacific. Temperatures range from 15°C in December to 40°C in April except in high altitude locations. The whole basin can be classified as a tropical rainforest with high biodiversity. The lower part has extensive irrigation networks and hence intensive rice paddy cultivation. In recent years, however, encroachment of people into forest area in the upper part of the basin and its conversion to agricultural use has become problematic.

The objective of the study of statistical modeling for flood frequency in upper Chaophraya river basin is to model new model extended from the Frèchet distribution and applied the Frèchet distribution and extended Frèchet distribution for flood frequency in upper Chaophraya river basin compared with the generalized extreme value distribution.

In order to achieve the objectives of the study, the scope of work can be summarized as follows:

Theoretical Part

- Derive the properties of the Fréchet family distribution.
- Develope the new model modified from the Fréchet distribution.
- Develope the parameter estimation for the Fréchet family distribution.

Computation Part

- Estimate parameters by generating random variable of the Fréchet family distribution; the Fréchet distribution, the Kumaraswamy Fréchet distribution and the Exponentiated Kumaraswamy Fréchet distribution using the differential evolution maximum likelihood estimation method for different sample sizes and parameter sets in order to calibrate Bias, Variance, Mean Square Error and Mean Absolute Percentage Error.

Application Part

- Applied the Fréchet distribution, the Kumaraswamy Fréchet distribution and the Exponentiated Kumaraswamy Fréchet distribution for flood frequency in upper Chaophraya river basin to find return period and return level in 29 stream gauging stations.

Study Area

Royal irrigation department of Thailand is the oldest organization of Thailand that collect data about runoff. Annual maximum flood peaks calculate from the maximum hourly discharge each days in a year (maximum of daily of yearly). The upper Chaophraya river basin conclude of Ping, Wang, Yom and Nan River basins. Ping, Wang, Yom and Nan rivers is confluence at Pagnumpho of NakornSawan province, there exist first runoff station of Chaophraya River at C.2 station. The selected stream gauging stations each river basin in this study are:

- The 6 selected stream gauging stations in Ping River are P.67, P.1, P.5, P.2A, P.7A and P.17.
- The 3 selected stream gauging stations in Wang River are W.1C, W.10A and W.4A.
- The 9 selected stream gauging stations in Yom River are Y.20, Y.1C, Y.14, Y.6, Y.3A, Y.33, Y.4, Y.16 and Y.17.
- The 10 selected stream gauging stations in Nan River are N.64, N.1, N.12A, N.13A, N.60, N.27A, N.5A, N.7A, N.8A and N.67.
- The selected runoff station in Chaophraya River is C.2.

The data employed in this study are 58 observations of annual maximum flood peaks in year 1956-2013 of 29 stream gauging stations in upper Chaophraya basin in Thailand from hydrology division, office of water management and hydrology, royal irrigation department, Thailand as shown in figure 1.1

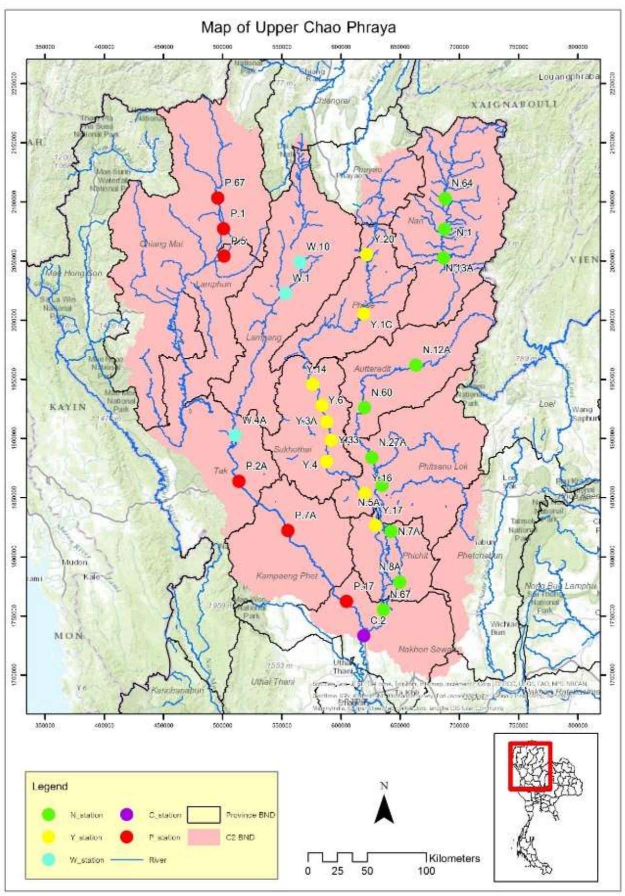

Figure 1.1 Selected stream gauging stations in upper Chaophraya river basin.

The structure of this paper is the following: the introduction in section 1, the methodology in section 2, the simulation study in section 3, the application for flood frequency in upper Chaophraya river basin in section 4 and the conclusion and outlook of the future work in section 5.

2 Methodology

2.1 The Fréchet distribution

During 1878-1973, Maurice Fréchet, the French mathematician collected empirical examples of heavy-tailed distribution. The Fréchet distribution has a heavy tail. He wrote the related paper in 1927, then in 1928 and in 1958 further work were done by Fisher and Tippett and by Gumbel respectively. The Fréchet distribution is a special case of the generalized extreme value distribution used to model the extreme natural event such as earthquake, floods, rainfall, sea currents, wind speeds etc. In this study we interested in floods as called flood frequency for planning, design, management and mitigation of water resources system. The cumulative distribution function (c.d.f.) of the Fréchet distribution is

$$F(x) = \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) \tag{2.1}$$

where $\mu < x < \infty$, μ - location parameter $(-\infty < \mu < \infty)$, σ-scale parameter $(\sigma > 0)$ and ξ-shape parameter $(\xi > 0)$

The probability density function (p.d.f.) is

$$f(x) = \xi\sigma^{\xi}(x-\mu)^{-(\xi+1)}\exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) \tag{2.2}$$

2.1.1 Shape of the Fréchet distribution

The first derivative of $\log f(x)$ for the Fréchet Distribution is,

$$\frac{\partial\log\big(f(x)\big)}{\partial x} = -\frac{(\xi+1)}{(x-\mu)} + \xi\sigma^{\xi}(x-\mu)^{-(\xi+1)} \tag{2.3}$$

The Standard calculation based on the first derivative show that $f(x)$ exhibits a singular mode at $x = x_0$ with $f(\mu) = f(\infty) = 0\,, (\mu < x < \infty)$. We can find x_0 by solving from $\frac{\partial\log(f(x))}{\partial x} = 0$, we get $x_0 = \mu + \left(\frac{\xi+1}{\xi\sigma^{\xi}}\right)^{\frac{-1}{\xi}}$.

The second derivative of $\log f(x)$ for the Fréchet Distribution is,

$$\frac{\partial^2\log\big(f(x)\big)}{\partial x^2} = \frac{(\xi+1)}{(x-\mu)^2} + \xi(\xi+1)\sigma^{\xi}(x-\mu)^{-(\xi+2)} \tag{2.4}$$

2.1.2 Survival Function of the Fréchet distribution

The survival function can be describes the relationship between the probability and events, as shown on the following form,

$$S(x) = P(X > x) = \int_x^{\infty} f(x)\,dx = 1 - F(x)$$

Therefore the survival function of the Fréchet Distribution is,

$$S(x) = 1 - \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) \tag{2.4}$$

2.1.3 Hazard Rate Function of the Fréchet distribution

The hazard rate function can be defined as the ratio of the density and the survival function (one minus the c.d.f.) as shown on the following form,

$$h(x) = \frac{f(x)}{1 - F(x)}$$

Then the hazard rate function of the Fréchet Distribution is,

$$h(x) = \frac{\xi\sigma^{\xi}(x-\mu)^{-(\xi+1)}\exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right)}{1 - \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right)} = \frac{\xi\sigma^{\xi}(x-\mu)^{-(\xi+1)}}{\exp\left(\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) - 1} \tag{2.5}$$

The first derivative of log of $h(x)$of the Fréchet Distribution is,

$$\frac{\partial \log h(x)}{\partial x} = -\frac{(\xi+1)}{(x-\mu)} + \frac{\xi\sigma^{\xi}(x-\mu)^{-(\xi+1)}}{1 - \exp(-\sigma^{\xi}(x-\mu)^{-\xi})} \tag{2.6}$$

The first derivative of log of $h(x)$ shows a single a singular mode at $x = x_0$ with$h(\mu) = f(\infty) = 0$, $(\mu < x < \infty)$, where $y_0 = (x_0 - \mu)^{\xi}$ is the solution of, $y_0\left(1 - \exp\left(-\frac{\sigma^{\xi}}{y_0}\right)\right) = \frac{\xi\sigma^{\xi}}{\xi+1}$.

The graph of some possible parameter sets of p.d.f., c.d.f., survival function, hazard rate function and lambda function were shown in figure 2.1-2.5

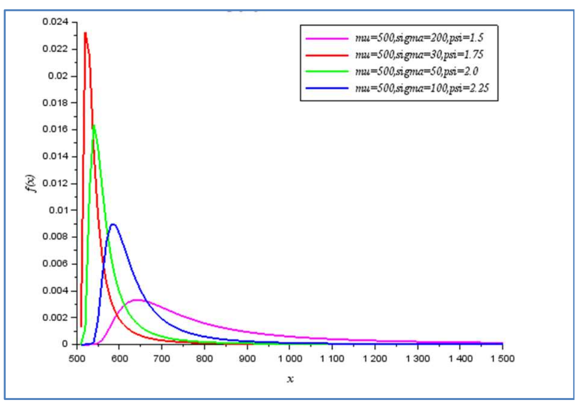

Figure 2.1 the graph of the p.d.f. of the Fréchet Distribution for some parameter values.

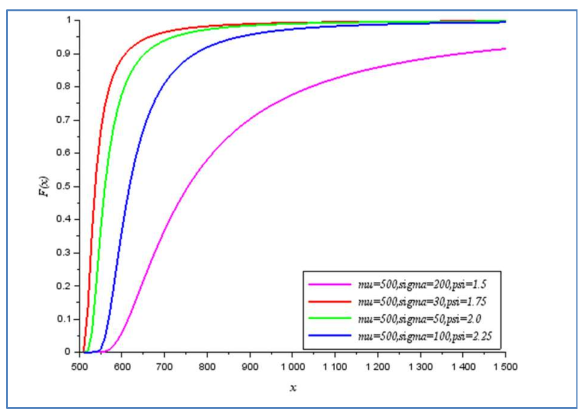

Figure 2.2 the graph of the c.d.f. of the Fréchet Distribution for some parameter values.

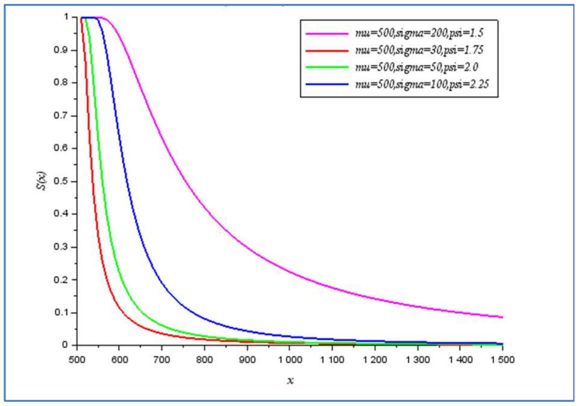

Figure 2.3 the graph of the survival functions of the Fréchet Distribution.

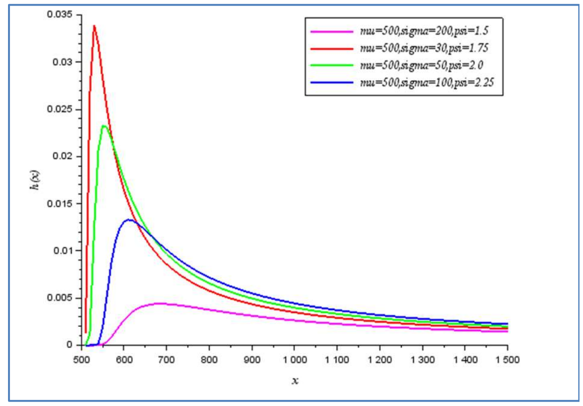

Figure 2.4 the graph of the hazard rate functions of the Fréchet Distribution.

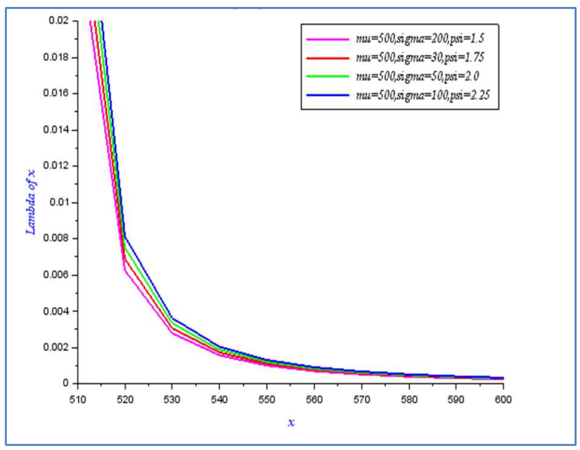

Figure 2.5 the graph of the lambda function of the Fréchet Distribution.

2.1.4 Quantile Function of the Fréchet distribution

The quantile function of the Fréchet Distribution is,

$$x_p = F^{-1}(p) = \mu + \sigma(-\log(p))^{-\frac{1}{\xi}} \tag{2.7}$$

The median can be derived from p=0.5 then,

$$Median(X) = \mu + \sigma(-\log(0.5))^{-\frac{1}{\xi}} = \mu + \sigma(\log(2))^{-\frac{1}{\xi}}$$

2.1.5 The Moment of the Fréchet distribution

The once of interesting characteristics of the distribution can be studies through the moment of distribution. We provided the *k*ith moment of the Fréchet distribution. We determined the *k*th moment

$$E(X^k) = \int_{\mu}^{\infty} x^k \xi \sigma^{\xi} (x-\mu)^{-(\xi+1)} \exp\left(-\sigma^{\xi}(x-\mu)^{-\xi}\right) dx$$

Let $y = \sigma^{\xi}(x-\mu)^{-\xi}$ then $x = \mu + \sigma y^{-\frac{1}{\xi}}$ and $dy = \xi \sigma^{\xi}(x-\mu)^{-(\xi+1)} dx$

$$x = \mu + \sigma y^{-\frac{1}{\xi}}$$

$$x^2 = \mu^2 + 2\mu\left(\sigma y^{-\frac{1}{\xi}}\right) + \left(\sigma y^{-\frac{1}{\xi}}\right)^2$$

$$x^3 = \mu^3 + 3\mu^2\left(\sigma y^{-\frac{1}{\xi}}\right) + 3\mu\left(\sigma y^{-\frac{1}{\xi}}\right)^2 + \left(\sigma y^{-\frac{1}{\xi}}\right)^3$$

$$\vdots$$

$$x^k = \binom{k}{0}\mu^k + \binom{k}{1}\mu^{k-1}\left(\sigma y^{-\frac{1}{\xi}}\right)^1 + \binom{k}{2}\mu^{k-2}\left(\sigma y^{-\frac{1}{\xi}}\right)^2 + \cdots + \binom{k}{k-1}\mu^1\left(\sigma y^{-\frac{1}{\xi}}\right)^{k-1} + \binom{k}{k}\left(\sigma y^{-\frac{1}{\xi}}\right)^k$$

Therefore,

$$E(X^1) = \int_0^{\infty} \left(\mu + \sigma y^{-\frac{1}{\xi}}\right) \exp(-y)\, dy$$
$$= \mu \int_0^{\infty} \exp(-y)\, dy + \sigma \int_0^{\infty} y^{\left(1-\frac{1}{\xi}\right)-1} \exp(-y)\, dy$$
$$= \mu + \sigma \int_0^{\infty} y^{-\frac{1}{\xi}} \exp(-y)\, dy$$
$$= \mu + \sigma\Gamma\left(1-\frac{1}{\xi}\right)$$

$$E(X^2) = \int_0^{\infty} \left(\mu + \sigma y^{-\frac{1}{\xi}}\right)^2 \exp(-y)\, dy$$
$$= \int_0^{\infty} \left(\mu^2 + 2\mu\left(\sigma y^{-\frac{1}{\xi}}\right) + \left(\sigma y^{-\frac{1}{\xi}}\right)^2\right) \exp(-y)\, dy$$
$$= \mu^2 + 2\mu\sigma\Gamma\left(1-\frac{1}{\xi}\right) + \sigma^2\Gamma\left(1-\frac{2}{\xi}\right)$$

$$E(X^3) = \int_0^{\infty} \left(\mu + \sigma y^{-\frac{1}{\xi}}\right)^3 \exp(-y)\, dy$$
$$= \int_0^{\infty} \left(\mu^3 + 3\mu^2\left(\sigma y^{-\frac{1}{\xi}}\right) + 3\mu\left(\sigma y^{-\frac{1}{\xi}}\right)^2 + \left(\sigma y^{-\frac{1}{\xi}}\right)^3\right) \exp(-y)\, dy$$
$$= \mu^3 + 3\mu^2\sigma\Gamma\left(1-\frac{1}{\xi}\right) + 3\mu\sigma^2\Gamma\left(1-\frac{2}{\xi}\right) + \sigma^3\Gamma\left(1-\frac{2}{\xi}\right)$$

$$E(X^4) = \int_0^\infty \left(\mu + \sigma y^{-\frac{1}{\xi}}\right)^4 \exp(-y)\, dy$$
$$= \int_0^\infty \left(\mu^4 + 4\mu^2\left(\sigma y^{-\frac{1}{\xi}}\right) + 6\mu^2\left(\sigma y^{-\frac{1}{\xi}}\right)^2 + 4\mu\left(\sigma y^{-\frac{1}{\xi}}\right)^3 + \left(\sigma y^{-\frac{1}{\xi}}\right)^4\right)\exp(-y)\, dy$$
$$= \mu^4 + 4\mu^3\sigma\Gamma\left(1-\tfrac{1}{\xi}\right) + 6\mu^2\sigma^2\Gamma\left(1-\tfrac{2}{\xi}\right) + 4\mu\sigma^3\Gamma\left(1-\tfrac{3}{\xi}\right) + \sigma^4\Gamma\left(1-\tfrac{4}{\xi}\right)$$
$$\vdots$$
$$E(X^k) = \int_0^\infty \left(\mu + \sigma y^{-\frac{1}{\xi}}\right)^k \exp(-y)\, dy$$
$$= \binom{k}{0}\mu^k + \binom{k}{1}\mu^{k-1}\sigma\Gamma\left(1-\tfrac{1}{\xi}\right) + \binom{k}{2}\mu^{k-2}\sigma^2\Gamma\left(1-\tfrac{2}{\xi}\right) + \cdots + \binom{k}{k-1}\mu^1\sigma^{k-1}\Gamma\left(1-\tfrac{k-1}{\xi}\right) + \binom{k}{k}\sigma^k\Gamma\left(1-\tfrac{k}{\xi}\right) \tag{2.8}$$

By the property of gamma function $\Gamma(x), x > 0$, then $\left(1-\frac{k}{\xi}\right) > 0\,, k < \xi$

Mean of the Fréchet distribution is,

$$\bar{X} = E(X) = \mu + \sigma\Gamma\left(1-\tfrac{1}{\xi}\right) \tag{2.9}$$

Variance of the Fréchet distribution is,

$$Var(X) = E(X^2) - \bar{X}^2$$
$$= \mu^2 + 2\mu\sigma\Gamma\left(1-\tfrac{1}{\xi}\right) + \sigma^2\Gamma\left(1-\tfrac{2}{\xi}\right) - \left(\mu + \sigma\Gamma\left(1-\tfrac{1}{\xi}\right)\right)^2$$
$$= \mu^2 + 2\mu\sigma\Gamma\left(1-\tfrac{1}{\xi}\right) + \sigma^2\Gamma\left(1-\tfrac{2}{\xi}\right) - \mu^2 - 2\mu\sigma\Gamma\left(1-\tfrac{1}{\xi}\right) - \sigma^2\Gamma\left(1-\tfrac{1}{\xi}\right)^2$$
$$= \sigma^2\left(\Gamma\left(1-\tfrac{2}{\xi}\right) - \Gamma\left(1-\tfrac{1}{\xi}\right)^2\right) \tag{2.10}$$

2.1.6 The Moment of the Fréchet distribution

We consider the maximum likelihood estimator (MLE) of the Fréchet distribution. Let $x_1, x_2, \ldots, x_n$ are the random sample of size n from the Fréchet distribution with 3 parameters (μ, σ, ξ) . , where $\mu < x < \infty$, μ -location parameter $(-\infty < \mu < \infty)$, σ -scale parameter $(\sigma > 0)$ and ξ -shape parameter $(\xi > 0)$. The likelihood function for the vector of parameter $\boldsymbol{\theta} = (\mu, \sigma, \xi)^T$ can be expressed as

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i) = \left(\xi\sigma^{\xi}\right)^n \prod_{i=1}^{n} (x_i - \mu)^{-(\xi+1)} \exp\left(-\left(\frac{x_i - \mu}{\sigma}\right)^{-\xi}\right) \tag{2.11}$$

Then, the log-likelihood function for the vector of parameter $\boldsymbol{\theta} = (\mu, \sigma, \xi)^T$ can be expressed as

$$\ell(\boldsymbol{\theta}) = \log(L(\boldsymbol{\theta}))$$
$$= n(\log(\xi) + \xi\log(\sigma)) - (\xi + 1)\sum_{i=1}^{n}\log(x_i - \mu)$$
$$- (\xi + 1)\sum_{i=1}^{n}\left(\frac{x_i - \mu}{\sigma}\right)^{-\xi}$$
(2.12)

The maximum likelihood estimators (MLE) $\widehat{\boldsymbol{\theta}} = \left(\hat{\mu}, \hat{\sigma}, \hat{\xi}\right)^T$ of $\boldsymbol{\theta} = (\mu, \sigma, \xi)^T$ can be obtained by differential evolution method.

2.2 The Kumaraswamy Fréchet distribution

Kumaraswamy P. (1980) proposed the Kumaraswamy distribution distribution (or Kum distribution) denoted by $Kum(\alpha, \beta)$. The cumulative distribution function of the Kumaraswamy Distribution is,

$$G_{Kum}(x; \alpha, \beta) = 1 - (1 - x^{\alpha})^{\beta} \quad , x \in (0,1)$$

where $\beta > 0$, shape parameter

The probability density function (p.d.f) of it is,

$$g_{Kum}(x; \alpha, \beta) = \alpha\beta x^{\alpha-1}(1 - x^{\alpha})^{\beta-1} \qquad , x \in (0,1), \alpha, \beta > 0$$

If we started with a parent continuous a parent continuous c.d.f $(G(x))$ and $g(x) = \frac{\partial G(x)}{\partial x}$be the associated c.d.f by combining the work of Kamaraswamy 1980 and Jones (2009) defined the c.d.f of the Kum-G distribution by

$$F(x) = 1 - (1 - G(x)^a)^b \qquad \text{where } a, b > 0$$
(2.13)

The p.d.f corresponding to (2.18) is,

$$f(x) = abg(x)G(x)^{a-1}(1 - G(x)^a)^{b-1}$$
(2.14)

Proof:

$$f(x) = \frac{\partial F(x)}{\partial x} = \frac{\partial[1 - (1 - G(x)^a)^b]}{\partial x}$$
$$= -b(1 - G(x)^a)^{b-1}\frac{\partial(1 - G(x)^a)}{\partial x}$$
$$= -b(1 - G(x)^a)^{b-1}(-1)\frac{\partial G(x)^a}{\partial x}$$
$$= b(1 - G(x)^a)^{b-1}aG(x)^{a-1}\frac{\partial G(x)}{\partial x}$$
$$= abg(x)G(x)^{a-1}(1 - G(x)^a)^{b-1}$$

The study about the Kum-G distribution are the Kumaraswamy Weibull distribution by Cordeiro et al. (2010), Kumaraswamy generalized gamma distribution by de Pascoa et al. (2011) , the Kumaraswamy Gumbel distribution by Cordeiro et al. (2012) , the Kumaraswamy log-logistic distribution by de Santana et al. (2012), the Kumaraswamy-geometric distribution by Akinsete et al. (2014). We concern with flood peak in upper Chaophraya river basin that followed the generalized extreme value distribution with mostly Fréchet distribution (EV2), therefore we provide the Kumaraswamy Fréchet distribution (KF). The cumulative distribution function (c.d.f) of the Fréchet distribution,

$$G_{EV2}(x;\mu,\sigma,\xi) = \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) \tag{2.15}$$

where $\mu < x < \infty, \sigma, \xi > 0, -\infty < \mu < \infty$
The probability density function (c.d.f) of the Fréchet distribution,

$$g_{EV2}(x;\mu,\sigma,\xi) = \left(\frac{\xi}{\sigma}\right)\left(\frac{x-\mu}{\sigma}\right)^{-(\xi+1)} \cdot \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) \tag{2.16}$$

Therefore, the c.d.f of the Kumaraswamy Fréchet distribution is,

$$F(x) = 1 - \left(1 - \exp\left(-a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right)\right)^{b} \tag{2.17}$$

and the p.d.f corresponding to (2.17) is

$$f(x) = ab\left(\frac{\xi}{\sigma}\right)\left(\frac{x-\mu}{\sigma}\right)^{-(\xi+1)} \exp\left(-a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) \cdot \left(1 - \exp\left(-a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right)\right)^{b-1} \tag{2.18}$$

2.2.1 Shape of the Kumaraswamy Fréchet distribution

The log of p.d.f of the Kumaraswamy Fréchet Distribution is

$$\log\big(f(x)\big) = \log\left(\frac{ab\xi}{\sigma}\right) - (\xi+1)\log\left(\frac{x-\mu}{\sigma}\right) - a\left(\frac{x-\mu}{\sigma}\right)^{-\xi} + (b-1)\log\left(1 - \exp\left(-a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right)\right) \tag{2.19}$$

The first derivative of log of f(x) we get,

$$\frac{\partial \log\big(f(x)\big)}{\partial x} = -\left(\frac{\xi+1}{x-\mu}\right) + \frac{a\xi}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{-(\xi+1)} + \frac{(1-b)\left(\frac{a\xi}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{-(\xi+1)}\right)}{\left(\exp\left(a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) - 1\right)} \tag{2.20}$$

The second derivative of log of f(x) is,

$$\frac{\partial^2 \log\big(f(x)\big)}{\partial x^2} = \frac{(\xi+1)}{(x-\mu)^2} - (\xi+1)a\xi\sigma^{\xi}(x-\mu)^{-(\xi+2)} + (b-1)a\xi\sigma^{\xi} \cdot \frac{(x-\mu)^{-(\xi+2)}(\xi+1)}{\exp\left(a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) - 1} + (b-1)\left(a\xi\sigma^{\xi}\right)^2 \cdot \frac{\left[(x-\mu)^{-2(\xi+1)}\exp\left(a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right)\right]}{\left(\exp\left(a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) - 1\right)^2} \tag{2.21}$$

2.2.2 Shape of the Kumaraswamy Fréchet distribution

The survival function can be describes the relationship between the probability and events, as shown on the following form,

$$S(x) = P(X > x) = \int_{x}^{\infty} f(x)\, dx = 1 - F(x)$$

Therefore the survival function of the Kumaraswamy Fréchet Distribution is,

$$S(x) = \left(1 - \exp\left(-a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right)\right)^{b} \tag{2.22}$$

2.2.3 The Hazard Rate Function of the Kumaraswamy Fréchet distribution

$$h(x) = \frac{f(x)}{1-F(x)} = \frac{f(x)}{S(x)} = \frac{ab\left(\frac{\xi}{\sigma}\right)\left(\frac{x-\mu}{\sigma}\right)^{-(\xi+1)}}{\left(\exp\left(a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) - 1\right)} \tag{2.23}$$

The graph of some possible parameter sets of p.d.f, c.d.f, survival function, hazard rate function and lambda function were shown in figure 2.6-2.10

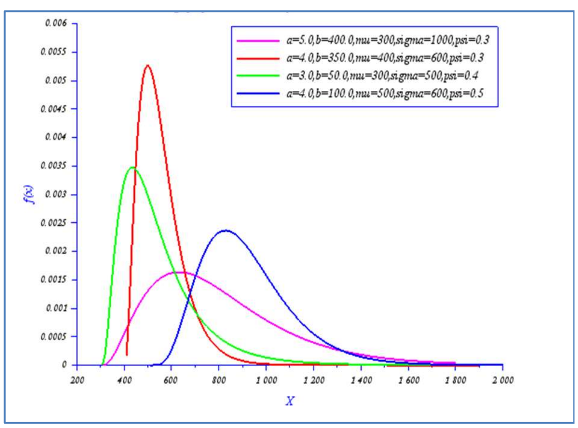

Figure 2.6 The graph of the p.d.f of the Kumaraswamy Fréchet Distribution for some parameter values.

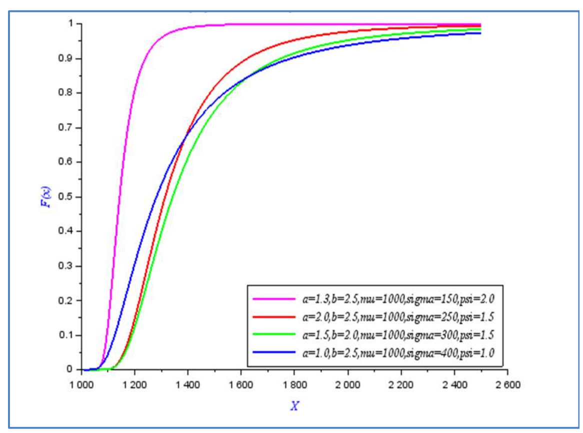

Figure 2.7 The graph of the c.d.f of the Kumaraswamy Fréchet Distribution for some parameter values.

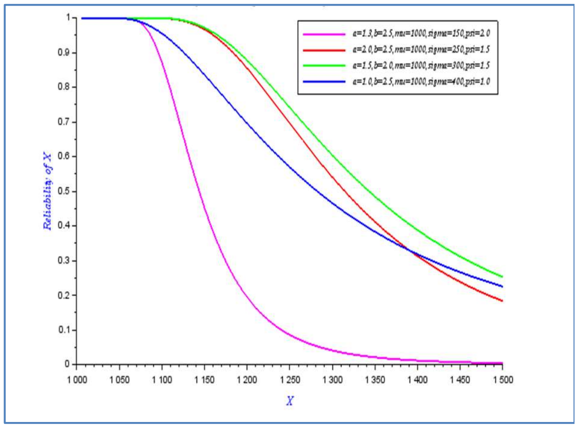

Figure 2.8 The graph of the survival functions of the Kumaraswamy Fréchet Distribution.

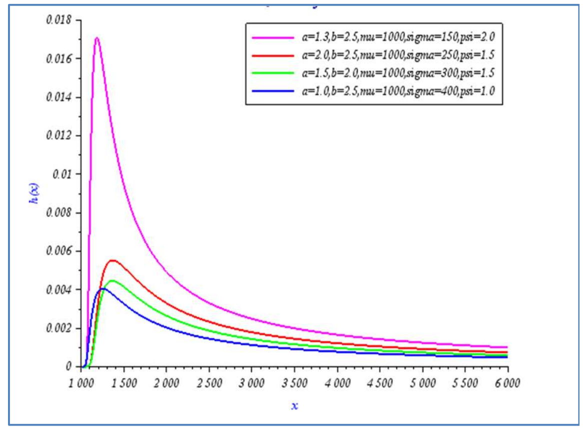

Figure 2.9 The graph of the hazard rate functions of the Kumaraswamy Fréchet Distribution.

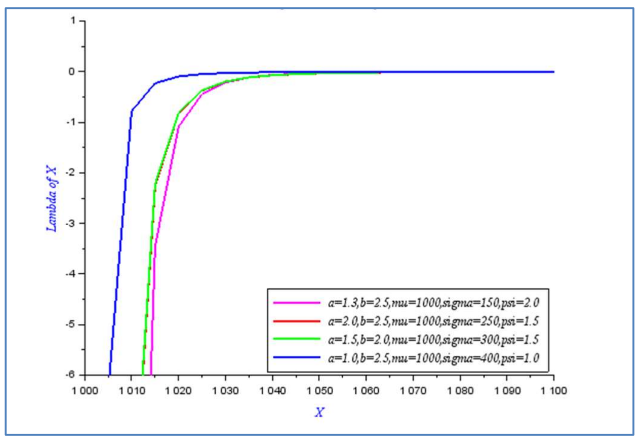

Figure 2.10 The graph of the lambda function of the Kumaraswamy Fréchet Distribution.

2.2.4 The Quantile of the Kumaraswamy Fréchet distribution

The quantile function of the Kumaraswamy Fréchet Distribution is,

$$x_p = \mu + \sigma\left\{-\ln\left\{\left[1-(1-p)^{1/b}\right]^{1/a}\right\}\right\}^{-1/\xi} \tag{2.24}$$

The median can be derived from p=1/2 then,

$$Median(X) = \mu + \sigma\left\{-\ln\left\{\left[1-0.5^{1/b}\right]^{1/a}\right\}\right\}^{-1/\xi} \tag{2.25}$$

2.2.5 The Expansion of p.d.f. and c.d.f. of the Kumaraswamy Fréchet distribution

The p.d.f of the Kumaraswamy Fréchet distribution is

$$f(x) = ab\left(\frac{\xi}{\sigma}\right)\left(\frac{x-\mu}{\sigma}\right)^{-(\xi+1)} \exp\left(-a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) \cdot \left(1-\exp\left(-a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right)\right)^{b-1}$$

The exponential function,

$$e^s = \sum_{k=0}^{\infty} \frac{s^k}{k!}$$

The binomial series,

$$(1+x)^s = \sum_{k=0}^{\infty} \frac{\Gamma(s+1)x^k}{k!\,\Gamma(s+1-k)}$$

By the exponential function, we have

$$\exp\left(-a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) = \sum_{j=0}^{\infty} \frac{\left(-a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right)^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j a^j \left(\frac{x-\mu}{\sigma}\right)^{-j}}{j!}$$

By the binomial series, we have

$$\begin{aligned}\left(1-\exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^{b-1} &= \sum_{k=0}^{\infty} \frac{\Gamma(b-1+1)\left(-\exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^k}{k!\,\Gamma(b-1+1-k)} \\ &= \sum_{k=0}^{\infty} \frac{\Gamma(b)\left(-\exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^k}{k!\,\Gamma(b-k)} \\ &= \Gamma(b)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(b-k)} \exp\left(-ak\sigma^{\xi}(x-\mu)^{-\xi}\right) \\ &= \Gamma(b)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(b-k)} \sum_{j=0}^{\infty} \frac{(-1)^j a^j \left(\frac{x-\mu}{\sigma}\right)^{-j}}{j!} \\ &= \Gamma(b)\sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(b-k)} \sum_{j=0}^{\infty} \frac{(-1)^j a^j \left(\frac{x-\mu}{\sigma}\right)^{-j\xi}}{j!} \\ &= \Gamma(b)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k+j}(ak)^j \left(\frac{x-\mu}{\sigma}\right)^{-j\xi}}{j!\,k!\,\Gamma(b-k)}\end{aligned}$$

$$\begin{aligned}f(x) &= ab\xi\sigma^{\xi}(x-\mu)^{-(\xi+1)} \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\left(1-\exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^{b-1} \\ &= ab\xi\sigma^{\xi}(x-\mu)^{-(\xi+1)} \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right) \\ &\quad \cdot \Gamma(b)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{k+j}(ak)^j \left(\frac{x-\mu}{\sigma}\right)^{-j}}{j!\,k!\,\Gamma(b-k)}\end{aligned}$$

$$= ab\xi\sigma^{\xi} \cdot \Gamma(b) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+j} (ak)^j \left(\frac{x-\mu}{\sigma}\right)^{-j\xi}}{j!\, k!\, \Gamma(b-k)} \cdot \exp\left(-a\sigma^{\xi} (x-\mu)^{-\xi}\right) \tag{2.26}$$

The expansion of c.d.f of the Kumaraswamy Fréchet distribution is

$$F(x) = 1 - \Gamma(b+1) \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(b+1-k)} \cdot \exp\left(-ak \left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right)$$

$$= 1 - \Gamma(b+1) \cdot \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\, \Gamma(b+1-k)} \cdot \sum_{i=0}^{\infty} \frac{(-1)^i (ak)^i \left(\frac{x-\mu}{\sigma}\right)^{-i\xi}}{i!}$$

$$= 1 - \Gamma(b+1) \cdot \sum_{k=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{k+i} (ak)^i \left(\frac{x-\mu}{\sigma}\right)^{-i\xi}}{i!\, k!\, \Gamma(b+1-k)} \tag{2.27}$$

2.2.6 The Moment of the Kumaraswamy Fréchet distribution

$$E(X^n) = \int_{\mu}^{\infty} x^n f(x) dx$$

$$= ab\xi\sigma^{\xi} \cdot \Gamma(b) \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{k+j} (\sigma^2 ak)^j}{j!\, k!\, \Gamma(b-k)} \cdot \int_{\mu}^{\infty} x^n (x-\mu)^{-(2j+\xi+1)} \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right) dx$$

Let $y = a\sigma^{\xi}(x-\mu)^{-\xi}$ therefore, $x = \mu + \left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}}$, we have

$$x^n = \left(\mu + \left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}}\right)^n$$

$$= \binom{n}{0}\mu^n + \binom{n}{1}\mu^{n-1}\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}} + \cdots + \binom{n}{n-1}\mu\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{(n-1)}{\xi}} + \binom{n}{n}\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{n}{\xi}}$$

$$dy = a\xi\sigma^{\xi}(x-\mu)^{-(\xi+1)}dx,, \ (x-\mu)^{-(\xi j)} = \left(\frac{y}{a\sigma^{\xi}}\right)^j$$

$$\int_{\mu}^{\infty} x^n (x-\mu)^{-(2j+\xi+1)} \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right) dx$$

$$= \frac{1}{a\xi\sigma^{\xi}} \int_{\mu}^{\infty} \left(\mu + \left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}}\right)^n \left(\frac{y}{a\sigma^{\xi}}\right)^j \exp(-y)\, dy$$

$n = 0,$

$$\int_{\mu}^{\infty} \left(\mu + \left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}}\right)^0 \left(\frac{y}{a\sigma^{\xi}}\right)^j \exp(-y)\, dy = \frac{\Gamma(j+1)}{a\sigma^{\xi}}$$

$n = 1,$

$$\int_{\mu}^{\infty}\left(\mu+\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}}\right)\left(\frac{y}{a\sigma^{\xi}}\right)^{j}\exp(-y)\,dy=\mu\frac{\Gamma(j+1)}{a\sigma^{\xi}}+\frac{\Gamma\left(j+1-\frac{1}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{1}{\xi}+j}}$$

$n=2,$

$$\int_{\mu}^{\infty}\left(\mu+\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}}\right)^{2}\left(\frac{y}{a\sigma^{\xi}}\right)^{j}\exp(-y)\,dy=\mu^{2}\frac{\Gamma(j+1)}{a\sigma^{\xi}}+\frac{2\mu\Gamma\left(j+1-\frac{1}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{1}{\xi}+j}}+\frac{\Gamma\left(j+1-\frac{2}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{2}{\xi}+j}}$$

$\vdots$

$n=n,$

$$\int_{\mu}^{\infty}\left(\mu+\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}}\right)^{n}\left(\frac{y}{a\sigma^{\xi}}\right)^{j}\exp(-y)\,dy$$

$$=\mu^{n}\frac{\Gamma(j+1)}{a\sigma^{\xi}}+\frac{\binom{n}{1}\mu^{n-1}\Gamma\left(j+1-\frac{1}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{1}{\xi}+1}}+\cdots+\frac{\binom{n}{n-1}\mu\Gamma\left(j+1-\frac{n-1}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{n-1}{\xi}+1}}+\frac{\Gamma\left(j+1-\frac{n}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{n}{\xi}+1}}$$

for $n<\xi$, we have

$$E(X)=\Gamma(b+1)\cdot\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+j}(\sigma^{2}ak)^{j}}{j!\,k!\,\Gamma(b-k)(a\sigma^{\xi})^{j}}\cdot\left(\mu\Gamma(j+1)+\frac{\Gamma\left(j+1-\frac{1}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{1}{\xi}}}\right)$$

$$E(X^{2})=\Gamma(b+1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+j}(\sigma^{2}ak)^{j}}{j!\,k!\,\Gamma(b-k)(a\sigma^{\xi})^{j}}\cdot\left(\mu^{2}\Gamma(j+1)+\frac{2\mu\Gamma\left(j+1-\frac{1}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{1}{\xi}}}+\frac{\Gamma\left(j+1-\frac{2}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{2}{\xi}}}\right)$$

$$E(X^{3})=\Gamma(b+1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+j}(\sigma^{2}ak)^{j}}{j!\,k!\,\Gamma(b-k)(a\sigma^{\xi})^{j}}\cdot\left(\mu^{3}\Gamma(j+1)+3\mu^{2}\frac{\Gamma\left(j+1-\frac{1}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{1}{\xi}}}+3\mu\frac{\Gamma\left(j+1-\frac{2}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{2}{\xi}}}+\frac{\Gamma\left(j+1-\frac{3}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{3}{\xi}}}\right)$$

$$E(X^{4})=\Gamma(b+1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+j}(\sigma^{2}ak)^{j}}{j!\,k!\,\Gamma(b-k)(a\sigma^{\xi})^{j}}\cdot\left\{\mu^{4}\Gamma(j+1)+4\mu^{3}\frac{\Gamma\left(j+1-\frac{1}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{1}{\xi}}}+6\mu^{2}\frac{\Gamma\left(j+1-\frac{2}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{2}{\xi}}}+4\mu\frac{\Gamma\left(j+1-\frac{3}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{3}{\xi}}}+\frac{\Gamma\left(j+1-\frac{4}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{4}{\xi}}}\right\}$$

$\vdots$

$$E(X^{n})=\int_{\mu}^{\infty}x^{n}f(x)dx$$

$$= \Gamma(b+1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{k+j}(\sigma^2 ak)^j}{j!\,k!\,\Gamma(b-k)(a\sigma^{\xi})^j}$$
$$\cdot\left\{\mu^n\Gamma(j+1)+\frac{\binom{n}{1}\mu^{n-1}\Gamma\left(j+1-\frac{1}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{1}{\xi}}}+\cdots+\frac{\binom{n}{n-1}\mu\Gamma\left(j+1-\frac{n-1}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{n-1}{\xi}}}+\frac{\Gamma\left(j+1-\frac{n}{\xi}\right)}{[a\sigma^{\xi}]^{-\frac{n}{\xi}}}\right\}$$
(2.28)

2.2.7 The Maximum Likelihood Estimation

We consider the maximum likelihood estimator (MLE) of the Kumarawamy Fréchet distribution. Let $x_1, x_2, \dots, x_n$ are the random sample of size n from the Kumaraswamy Fréchet distribution with 5 parameters(a, b, μ, σ, ξ). , where $\mu < x < \infty$, μ-location parameter $(-\infty < \mu < \infty)$, σ-scale parameter $(\sigma > 0)$ and ξ-shape parameter$(\xi > 0)$. The likelihood function for the vector of parameter $\boldsymbol{\theta} = (a, b, \mu, \sigma, \xi)^T$ can be expressed as

$$L(\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i)$$
$$= \left(ab\xi\sigma^{\xi}\right)^n \cdot \prod_{i=1}^{n}(x_i-\mu)^{-(\xi+1)} \cdot \prod_{i=1}^{n}\exp\left(-a\sigma^{\xi}\sum_{i=1}^{n}(x_i-\mu)^{-\xi}\right) \cdot \prod_{i=1}^{n}\left(1-\exp\left(-a\sigma^{\xi}(x_i-\mu)^{-\xi}\right)\right)^{b-1}$$
(2.29)

$$\ell(\boldsymbol{\theta}) = \log\left(L(\boldsymbol{\theta})\right)$$
$$= n(\ln a + \ln b + \ln \xi + \xi \ln \sigma) - (\xi+1)\sum_{i=1}^{n}\ln(x_i-\mu) - a\sigma^{\xi}\sum_{i=1}^{n}(x_i-\mu)^{-\xi} + (b-1)\sum_{i=1}^{n}\ln\left(1-e^{\left(-a\sigma^{\xi}(x_i-\mu)^{-\xi}\right)}\right)$$
(2.30)

The maximum likelihood estimators (MLE) $\widehat{\boldsymbol{\theta}} = \left(\hat{a}, \hat{b}, \hat{\mu}, \hat{\sigma}, \hat{\xi}\right)^T$ of $\boldsymbol{\theta} = (a, b, \mu, \sigma, \xi)^T$ can be obtained by differential evolution method from (2.30)

2.3 The Exponentiated Kumaraswamy Fréchet distribution

Gupta *et.al.* (1998) proposed the new family namely the Exponentiated Exponential distribution, then Gupta and Kundu (2001) discussed some properties of the Exponentiated Exponential family as an alternative to Gamma and Weibull distribution. Kakade *et.al.* (2008) clarified the exponentiated family of distribution reformed by using the cumulative distribution function (c.d.f) of the baseline distribution $G(x)$ in two ways.

(a) $F(x|\boldsymbol{\theta},\alpha) = \left(G(x)\right)^{\alpha}$

$$,\alpha > 0, \boldsymbol{\theta} \in \boldsymbol{\Theta}, -\infty < x < \infty$$

(b) $F(x|\boldsymbol{\theta},\alpha) = 1 - \left(1 - G(x)\right)^{\alpha}$

$$,\alpha > 0, \boldsymbol{\theta} \in \boldsymbol{\Theta}, -\infty < x < \infty$$

There are many papers about the exponentiated family of distribution. Gupta *et.al.* (1998) introduced the Exponentiated Exponential distribution using (a). Nadarajah and Kotz (2003) introduced the Exponentiated Fréchet distribution using (b). Cordeiro*et.al.* (2013) proposed the Exponentiated Generalized Class of distribution, the Exponentiated Generalized Fréchet, the Exponentiated Generalized Normal, Exponentiated Generalized Gamma and Exponentiated Generalized Gumbel using (b). Elbalat and Muhammed (2014) proposed the Exponentiated Generalized Inverse Weibull distribution using (b). We proposed the Exponentiated Kumaraswamy Fréchet distribution using (a). Therefore, the cumulative distribution function is

$$F(x) = \left(1 - \left(1 - \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^{b}\right)^{\alpha} \tag{2.31}$$

The probability density function (p.d.f) corresponding (7.1) is,

$$f(x) = \alpha a b \xi \sigma^{\xi}(x-\mu)^{-(\xi+1)} \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right) \cdot \left(1 - \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^{b-1} \cdot \left(1 - \left(1 - \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^{b}\right)^{\alpha-1} \tag{2.32}$$

2.3.1 Shape of the Exponentiated Kumaraswamy Fréchet distribution

The log of p.d.f. of the the Exponentiated Kumaraswamy Fréchet Distribution is,

$$\log\left(f(x)\right) = \log\left(\alpha a b \xi \sigma^{\xi}\right) - (\xi+1)\log(x-\mu) - a\sigma^{\xi}(x-\mu)^{-\xi} + (b-1)\log\left(1 - \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right) + (\alpha-1)\log\left(1 - \left(1 - e^{\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)}\right)^{b}\right) \tag{2.33}$$

The first derivative of log of f(x) we have,

$$\frac{\partial \log\left(f(x)\right)}{\partial x} = -\left(\frac{\xi+1}{x-\mu}\right) + a\sigma^{\xi}(\xi+1)(x-\mu)^{-(\xi+1)} - (b-1)\frac{a\sigma^{\xi}\xi(x-\mu)^{-(\xi+1)}}{(\exp(a\sigma^{\xi}(x-\mu)^{-\xi}) - 1)} + (\alpha-1)a\sigma^{\xi}\xi(x-\mu)^{-(\xi+1)} \cdot \frac{\exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)}{1 - (1 - \exp(-a\sigma^{\xi}(x-\mu)^{-\xi}))^{b}} \tag{2.34}$$

The second derivative of log of f(x) (lambda function) is,

$$\frac{\partial^2 \log\left(f(x)\right)}{\partial x^2} = (\xi+1)(x-\mu)^2\left(1 - a\xi\sigma^{\xi}(x-\mu)^{-\xi}\right) + (b-1)\frac{a\sigma^{\xi}\xi(\xi+1)(x-\mu)^{-(\xi+2)}}{(\exp(a\sigma^{\xi}(x-\mu)^{-\xi}) - 1)} - (b-1)\frac{\left(a\sigma^{\xi}\xi\right)^2(x-\mu)^{-2(\xi+1)}}{(1 - \exp(-a\sigma^{\xi}(x-\mu)^{-\xi}))} - (\alpha-1)a\sigma^{\xi}\xi(\xi+1)(x-\mu)^{-(\xi+2)} \cdot \frac{\exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)}{1 - (1 - \exp(-a\sigma^{\xi}(x-\mu)^{-\xi}))^{b}} + (\alpha-1)\left(a\sigma^{\xi}\xi\right)^2(x-\mu)^{-2(\xi+1)} \cdot \frac{\exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)}{1 - (1 - \exp(-a\sigma^{\xi}(x-\mu)^{-\xi}))^{b}} - (\alpha-1)\left(a\sigma^{\xi}\xi\right)^2(x-\mu)^{-2(\xi+1)}$$

$$\cdot \frac{\exp\left(-2a\sigma^{\xi}(x-\mu)^{-\xi}\right)}{(1-(1-\exp(-a\sigma^{\xi}(x-\mu)^{-\xi}))^{b})^{2}} \tag{2.35}$$

2.3.2 Survival Function of the Exponentiated Kumaraswamy Fréchet distribution

The survival function can be describes the relationship between the probability and events, as shown on the following form,

$$S(x) = P(X > x) = \int_{x}^{\infty} f(x)\,dx = 1 - F(x)$$

Therefore the survival function of the Exponentiated Kumaraswamy Fréchet Distribution is,

$$S(x) = 1 - \left(1 - \left(1 - \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^{b}\right)^{\alpha} \tag{2.36}$$

2.3.3 Hazard Rate Function of the Exponentiated Kumaraswamy Fréchet distribution

If X is a random variable with probability density function $\left(f(x)\right)$, cumulative distribution function$\left(F(x)\right)$, and survival function $\left(S(x)\right)$ then the hazard rate function $\left(h(x)\right)$, is defined by the ratio of $f(x)$ to $S(x)$

$$h(x) = \frac{f(x)}{S(x)} = \frac{f(x)}{1 - F(x)}$$

Therefore, the hazard rate function of the Exponentiated Kumaraswamy Fréchet Distribution is

$$h(x) = \alpha a b \xi \sigma^{\xi}(x-\mu)^{-(\xi+1)} \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right) \cdot \left(1 - \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^{b-1}$$
$$\cdot \left[\frac{\left(1 - \left(1 - \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^{b}\right)^{\alpha-1}}{1 - (1 - (1 - \exp(-a\sigma^{\xi}(x-\mu)^{-\xi}))^{b})^{\alpha}}\right] \tag{2.37}$$

The graph of some possible parameter sets of p.d.f, c.d.f, survival function and hazard rate function were shown in figure 2.11-2.14

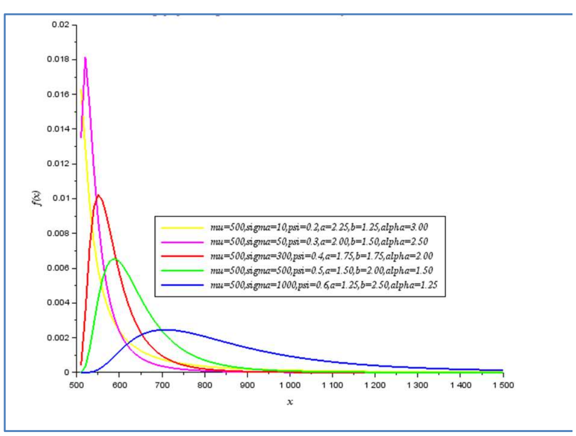

Figure 2.11 The graph of the p.d.f of the EKF distribution.

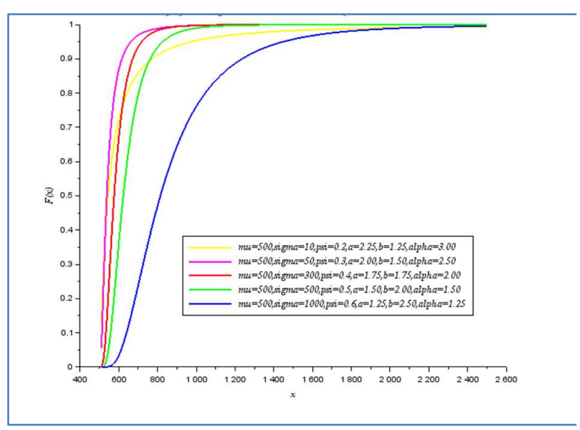

Figure 2.12 The graph of the c.d.f of the EKF distribution.

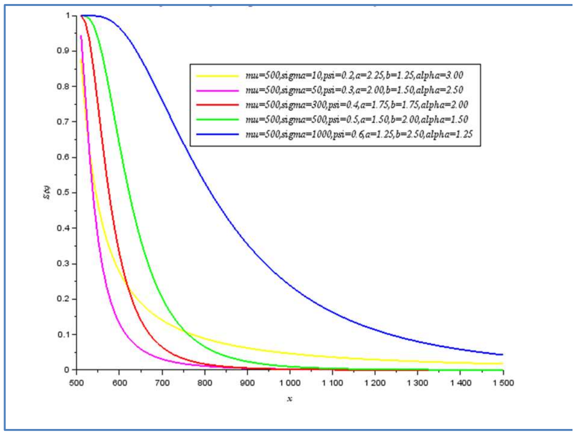

Figure 2.13 The graph of the survival functions of the EKF Distribution.

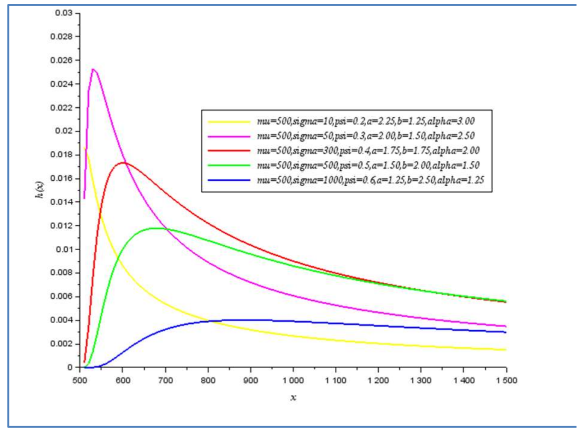

Figure 2.14 The graph of the hazard rate functions of the EKF distribution.

2.3.4 The Sub-model of the Exponentiated Kumaraswamy Fréchet distribution

The sub-models of the Exponentiated Kumaraswamy Fréchet distribution for selected values of the parameters are

1. When $\alpha = 1$, we obtain the Kumaraswamy Fréchet distribution with cdf:

$$F(x) = 1 - \left(1 - \exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)\right)^{b} \qquad \text{for } a, b, \sigma, \xi > 0, -\infty < \mu < \infty, x > \mu$$

2. When $\alpha = b = 1$, we obtain the Exponentiated Fréchet distribution with cdf:

$$F(x) = \exp\left(-a\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) \qquad \text{for } a, \sigma, \xi > 0, -\infty < \mu < \infty, x > \mu$$

3. When $a = b = 1$, we obtain the Exponentiated Fréchet distribution with cdf:

$$F(x) = \exp\left(-\alpha\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) \qquad \text{for } \alpha, \sigma, \xi > 0, -\infty < \mu < \infty, x > \mu$$

4. When $\alpha = a = b = 1$, we obtain the Fréchet distribution with cdf:

$$F(x) = \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-\xi}\right) \qquad \text{for } \sigma, \xi > 0, -\infty < \mu < \infty, x > \mu$$

5. When $\alpha = 1, \sigma = 2, \mu = 0$, we obtain the Kumaraswamy Inverse Rayleigh distribution with cdf:

$$F(x) = 1 - \left(1 - \exp\left(-a\frac{\theta}{x^2}\right)\right)^{b} \qquad \text{for } a, b, \theta = \sigma^2 > 0, \ , x > 0$$

6. When $\alpha = b = 1, \sigma = 2, \mu = 0$, we obtain the Exponentiated Inverse Rayleigh distribution with cdf:

$$F(x) = \exp\left(-a\frac{\theta}{x^2}\right) \qquad \text{for } a, \theta = \sigma^2 > 0, \ x > 0$$

7. When $\alpha = a = b = 1, \sigma = 2, \mu = 0$, we obtain the Inverse Rayleigh distribution with cdf:

$$F(x) = \exp\left(-\frac{\theta}{x^2}\right) \qquad \text{for} \theta = \sigma^2 > 0, x > 0$$

8. When $\alpha = 1, \sigma = 1, \mu = 0$, we obtain the Kumaraswamy Inverse Exponential distribution with cdf:

$$F(x) = 1 - \left(1 - \exp\left(-a\frac{\sigma}{x}\right)\right)^{b} \qquad \text{for } a, b, \sigma > 0, \ \ x > 0$$

9. When $\alpha = a = b = 1, \sigma = 1, \mu = 0$, we obtain the Inverse Exponential distribution with cdf:

$$F(x) = \exp\left(-\frac{\sigma}{x}\right) \qquad \text{for} \sigma > 0,\ x > 0$$

2.3.5 The Quantile Fucntion of the Exponentiated Kumaraswamy Fréchet distribution

The quantile function of the Exponentiated Kumaraswamy Fréchet Distribution is,

$$x_p = \mu + \sigma\left\{-\ln\left\{\left[1-\left(1-p^{\frac{1}{\alpha}}\right)^{1/b}\right]^{1/a}\right\}\right\}^{-1/\xi} \tag{2.37}$$

The median can be derived from p=1/2 then,

$$x_{0.5} = \mu + \sigma\left\{-\ln\left\{\left[1-\left(1-0.5^{\frac{1}{\alpha}}\right)^{1/b}\right]^{1/a}\right\}\right\}^{-1/\xi} \tag{2.38}$$

2.3.6 The Quantile Fucntion of the Exponentiated Kumaraswamy Fréchet distribution

We provided the simple expansions for the p.d.f and c.d.f of the Exponentiated Kumaraswamy Fréchet Distribution. The expansion of c.d.f of the Exponentiated Kumaraswamy Fréchet Distribution is

$$F(x) = \Gamma(\alpha+1)\cdot\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(bi+1)(-1)^{j+k+i}a^j\sigma^{\xi j}k^j(x-\mu)^{-j\xi}}{i!\,\Gamma(\alpha+1-i)k!\,\Gamma(bi+1-k)j!} \tag{2.39}$$

The expansion of p.d.f of the Exponentiated Kumaraswamy Fréchet Distribution is

$$\begin{aligned} f(x) &= \alpha ab\xi\sigma^{\xi}(x-\mu)^{-(\xi+1)}\exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right) \\ &\quad\cdot\Gamma(b)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{j+k}a^j\sigma^{\xi j}k^j(x-\mu)^{-j\xi}}{k!\,\Gamma(b-k)j!} \\ &\quad\cdot\Gamma(\alpha)\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{\Gamma(bi+1)(-1)^{j+k+i}a^j\sigma^{\xi j}k^j(x-\mu)^{-j\xi}}{i!\,\Gamma(\alpha-i)k!\,\Gamma(bi+1-k)j!} \\ &= \alpha ab\xi\sigma^{\xi}\Gamma(b)\Gamma(\alpha)\cdot\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{2j+2k+i}a^{2j}\sigma^{2\xi}\ k^{2j}\Gamma(bi+1)}{(j!)^2(k!)^2 i!\,\Gamma(b-k)\Gamma(\alpha-i)\Gamma(bi+1-k)} \\ &\quad\cdot(x-\mu)^{-(2j\xi+\xi+1)}\exp\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right) \end{aligned} \tag{2.40}$$

2.3.7 The Moment of the Exponentiated Kumaraswamy Fréchet distribution

$$\begin{aligned} E(X^n) &= \int_{\mu}^{\infty} x^n f(x)dx \\ &= \alpha ab\xi\sigma^{\xi}\Gamma(b)\Gamma(\alpha)\cdot\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{2j+2k+i}a^{2j}\sigma^{2\xi j}k^{2j}}{(j!)^2(k!)^2 i!}\cdot\frac{\Gamma(bi+1)}{\Gamma(b-k)\Gamma(\alpha-i)\Gamma(bi+1-k)} \\ &\quad\cdot\int_{\mu}^{\infty} x^n(x-\mu)^{-(2j\xi+\xi+1)}e^{\left(-a\sigma^{\xi}(x-\mu)^{-\xi}\right)}dx \end{aligned}$$

Let $y = a\sigma^{\xi}(x-\mu)^{-\xi}$ therefore, $x = \mu + \left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}}$, we have

$$x^n = \left(\mu + \left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}}\right)^n$$

$$= \binom{n}{0}\mu^n + \binom{n}{1}\mu^{n-1}\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}} + \cdots + \binom{n}{n-1}\mu\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{(n-1)}{\xi}} + \binom{n}{n}\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{n}{\xi}}$$

$$dy = a\xi\sigma^{\xi}(x-\mu)^{-(\xi+1)}dx,$$

$(x-\mu)^{-2j\xi} = \left(\frac{y}{a\sigma^{\xi}}\right)^{2j}$, then

$$\int_{\mu}^{\infty} x^n (x-\mu)^{-(2j\xi+\xi+\ )} e^{(-a\sigma^{\xi}(x-\mu)^{-\xi})} dx$$

$$= \frac{1}{a\xi\sigma^{\xi}} \int_0^{\infty} \left[\binom{n}{0}\mu^k + \binom{n}{1}\mu^{n-1}\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}} + \cdots + \binom{n}{n-1}\mu\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{(n-1)}{\xi}} + \binom{n}{n}\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{n}{\xi}}\right]\left(\frac{y}{a\sigma^{\xi}}\right)^{2j} \exp(-y)\, dy$$

Therefore,

$$E(X^n) = \int_{\mu}^{\infty} x^n f(x) dx$$

$$= \alpha b\Gamma(b)\Gamma(\alpha) \cdot \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{2j+2k+i} a^{2j}\sigma^{2\xi j} k^{2j}}{(j!)^2(k!)^2 i!} \cdot \frac{\Gamma(bi+1)}{\Gamma(b-k)\Gamma(\alpha-i)\Gamma(bi+1-k)}$$

$$\cdot \int_0^{\infty} \left[\binom{n}{0}\mu^n + \binom{n}{1}\mu^{n-1}\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}} + \cdots + \binom{n}{n-1}\mu\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{(n-1)}{\xi}} + \binom{n}{n}\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{n}{\xi}}\right]\left(\frac{y}{a\sigma^{\xi}}\right)^{2j} \exp(-y)\, dy$$

$$= \alpha\Gamma(b+1)\Gamma(\alpha) \cdot \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{2j+2k+i} a^{2j}\sigma^{2\xi j} k^{2j}}{(j!)^2(k!)^2 i!} \cdot \frac{\Gamma(bi+1)}{\Gamma(b-k)\Gamma(\alpha-i)\Gamma(bi+1-k)}$$

$$\cdot \int_0^{\infty} \left[\binom{n}{0}\mu^n\left(\frac{y}{a\sigma^{\xi}}\right)^{2j} + \binom{n}{1}\mu^{n-1}\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{1}{\xi}+2j} + \cdots + \binom{n}{n-1}\mu\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{(n-1)}{\xi}+2j} + \binom{n}{n}\left(\frac{y}{a\sigma^{\xi}}\right)^{-\frac{n}{\xi}+2j}\right] \exp(-y)\, dy$$

Let,

$$AA = \int_0^\infty \left[\binom{n}{0}\mu^n\left(\frac{y}{a\sigma^\xi}\right)^{2j} + \binom{n}{1}\mu^{n-1}\left(\frac{y}{a\sigma^\xi}\right)^{-\frac{1}{\xi}+2j} + \cdots + \binom{n}{n-1}\mu\left(\frac{y}{a\sigma^\xi}\right)^{-\frac{(n-1)}{\xi}+2j} + \binom{n}{n}\left(\frac{y}{a\sigma^\xi}\right)^{-\frac{n}{\xi}+2j}\right]\exp(-y)\,dy$$

for n=1,

$$AA = \int_0^\infty \left[\mu\left(\frac{y}{a\sigma^\xi}\right)^{2j} + \left(\frac{y}{a\sigma^\xi}\right)^{-\frac{1}{\xi}+2j}\right]\exp(-y)\,dy = \frac{\mu\Gamma(2j+1)}{(a\sigma^\xi)^{2j}} + \frac{\Gamma\left(2j+1-\frac{1}{\xi}\right)}{(a\sigma^\xi)^{-\frac{1}{\xi}+2j}}$$

for n=2,

$$AA = \int_0^\infty \left[\mu\left(\frac{y}{a\sigma^\xi}\right)^{2j} + \left(\frac{y}{a\sigma^\xi}\right)^{-\frac{1}{\xi}+2j} + \left(\frac{y}{a\sigma^\xi}\right)^{-\frac{2}{\xi}+2j}\right]\exp(-y)\,dy$$

$$= \frac{\mu^2\Gamma(2j+1)}{(a\sigma^\xi)^{2j}} + 2\frac{\mu\Gamma\left(2j+1-\frac{1}{\xi}\right)}{(a\sigma^\xi)^{\frac{1}{\xi}+2j}} + \frac{\Gamma\left(2j+1-\frac{2}{\xi}\right)}{(a\sigma^\xi)^{\frac{2}{\xi}+2j}}$$

⋮

for n=n,

$$AA = \frac{\mu^n\Gamma(2j+1)}{(a\sigma^\xi)^{2j}} + \binom{n}{1}\frac{\mu^{n-1}\Gamma\left(2j+1-\frac{1}{\xi}\right)}{(a\sigma^\xi)^{\frac{1}{\xi}+2j}} + \cdots + \binom{n}{n-1}\frac{\Gamma\left(2j+1-\frac{n-1}{\xi}\right)}{(a\sigma^\xi)^{\frac{n-1}{\xi}+2j}} + \frac{\Gamma\left(2j+1-\frac{n}{\xi}\right)}{(a\sigma^\xi)^{\frac{n}{\xi}+2j}}$$

for $n < (2j+1)\xi$, the moment function of the EKF is

$$E(X^n) = \Gamma(b+1)\Gamma(\alpha+1)\cdot\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{2j+2k+i}a^{2j}\sigma^{2\xi j}k^{2j}}{(j!)^2(k!)^2 i!}\cdot\frac{\Gamma(bi+1)}{\Gamma(b-k)\Gamma(\alpha-i)\Gamma(bi+1-k)}$$
$$\cdot\left\{\frac{\mu^n\Gamma(2j+1)}{(a\sigma^\xi)^{2j}} + \binom{n}{1}\frac{\mu^{n-1}\Gamma\left(2j+1-\frac{1}{\xi}\right)}{(a\sigma^\xi)^{-\frac{1}{\xi}+2j}} + \cdots + \binom{n}{n-1}\frac{\Gamma\left(2j+1-\frac{n-1}{\xi}\right)}{(a\sigma^\xi)^{-\frac{n-1}{\xi}+2j}} + \frac{\Gamma\left(2j+1-\frac{n}{\xi}\right)}{(a\sigma^\xi)^{-\frac{n}{\xi}+2j}}\right\} \quad (2.41)$$

Then first 4 moment function of X are

$$E(X) = \Gamma(b+1)\Gamma(\alpha+1)\cdot\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{2j+2k+i}a^{2j}\sigma^{2\xi j}k^{2j}}{(j!)^2(k!)^2 i!\,(a\sigma^\xi)^{2j}}\cdot\frac{\Gamma(bi+1)}{\Gamma(b-k)\Gamma(\alpha-i)\Gamma(bi+1-k)}$$

$$\cdot\left\{\mu\Gamma(2j+1)+\frac{\Gamma\left(2j+1-\frac{1}{\xi}\right)}{\left(a\sigma^{\xi}\right)^{-\frac{1}{\xi}}}\right\}$$

$$E(X^2)=\Gamma(b+1)\Gamma(\alpha+1)\cdot\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{2j+2k+i}a^{2j}\sigma^{2\xi}\ k^{2j}}{(j!)^2(k!)^2 i!\,(a\sigma^{\xi})^{2j}}$$
$$\cdot\frac{\Gamma(bi+1)}{\Gamma(b-k)\Gamma(\alpha-i)\Gamma(bi+1-k)}$$
$$\cdot\left\{\mu^2\Gamma(2j+1)+2\mu\frac{\Gamma\left(2j+1-\frac{1}{\xi}\right)}{\left(a\sigma^{\xi}\right)^{-\frac{1}{\xi}}}+\frac{\Gamma\left(2j+1-\frac{2}{\xi}\right)}{\left(a\sigma^{\xi}\right)^{-\frac{2}{\xi}}}\right\}$$

$$E(X^3)=\Gamma(b+1)\Gamma(\alpha+1)\cdot\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{2j+2k+i}a^{2j}\sigma^{2\xi j}k^{2j}}{(j!)^2(k!)^2 i!\,(a\sigma^{\xi})^{2j}}$$
$$\cdot\frac{\Gamma(bi+1)}{\Gamma(b-k)\Gamma(\alpha-i)\Gamma(bi+1-k)}$$
$$\cdot\left\{\mu^3\Gamma(2j+1)+3\mu^2\frac{\Gamma\left(2j+1-\frac{1}{\xi}\right)}{\left(a\sigma^{\xi}\right)^{-\frac{1}{\xi}}}+3\mu\frac{\Gamma\left(2j+1-\frac{2}{\xi}\right)}{\left(a\sigma^{\xi}\right)^{-\frac{2}{\xi}}}+\frac{\Gamma\left(2j+1-\frac{3}{\xi}\right)}{\left(a\sigma^{\xi}\right)^{-\frac{3}{\xi}}}\right\}$$

$$E(X^4)=\Gamma(b+1)\Gamma(\alpha+1)\cdot\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{2j+2k+i}a^{2j}\sigma^{2\xi j}k^{2j}}{(j!)^2(k!)^2 i!\,(a\sigma^{\xi})^{2j}}$$
$$\cdot\frac{\Gamma(bi+1)}{\Gamma(b-k)\Gamma(\alpha-i)\Gamma(bi+1-k)}$$
$$\cdot\left\{\mu^4\Gamma(2j+1)+4\mu^3\frac{\Gamma\left(2j+1-\frac{1}{\xi}\right)}{\left(a\sigma^{\xi}\right)^{-\frac{1}{\xi}}}+6\mu^2\frac{\Gamma\left(2j+1-\frac{2}{\xi}\right)}{\left(a\sigma^{\xi}\right)^{-\frac{2}{\xi}}}+4\mu\frac{\Gamma\left(2j+1-\frac{3}{\xi}\right)}{\left(a\sigma^{\xi}\right)^{-\frac{3}{\xi}}}\right.$$
$$\left.+\frac{\Gamma\left(2j+1-\frac{4}{\xi}\right)}{\left(a\sigma^{\xi}\right)^{-\frac{4}{\xi}}}\right\}$$

2.3.7 The Maximum Likelihood Estimation of the Exponentiated Kumaraswamy Fréchet distribution

We consider the maximum likelihood estimator (MLE) of the Kumarawamy Fréchet distribution. Let $x_1, x_2, \dots, x_n$ are the random sample of size n from the Exponentiated Kumaraswamy Fréchet Distribution with 6 parameters, $(\alpha, a, b, \mu, \sigma, \xi)$. , where$\mu < x < \infty$, μ-location parameter, $(-\infty < \mu < \infty)$, σ-scale parameter$(\sigma > 0)$ and ξ-shape parameter$(\xi > 0)$. The likelihood function for the vector of parameter $\boldsymbol{\theta} = (\alpha, a, b, \mu, \sigma, \xi)^T$ can be expressed as

$$L(\boldsymbol{\theta})=\prod_{i=1}^{n}f(x_i)$$

$$= \left(\alpha ab\xi\sigma^{\xi}\right)^{n}\prod_{i=1}^{n}(x_i-\mu)^{-(\xi+1)}\cdot e^{\left(-a\sigma^{\xi}\sum_{i=1}^{n}(x_i-\mu)^{-\xi}\right)}$$
$$\cdot\prod_{i=1}^{n}\left(1-\exp\left(-a\sigma^{\xi}(x_i-\mu)^{-\xi}\right)\right)^{b-1}$$
$$\cdot\prod_{i=1}^{n}\left(1-\left(1-e^{\left(-a\sigma^{\xi}(x_i-\mu)^{-\xi}\right)}\right)^{b}\right)^{\alpha-1}$$

(2.42)

$$\ell(\boldsymbol{\theta}) = \log\left(L(\boldsymbol{\theta})\right)$$
$$= n(\ln\alpha+\ln a+\ln b+\ln\xi+\xi\ln\sigma)-(\xi+1)\sum_{i=1}^{n}\ln(x_i-\mu)-a\sigma^{\xi}\sum_{i=1}^{n}(x_i-\mu)^{-\xi}$$
$$+(b-1)\sum_{i=1}^{n}\ln\left(1-e^{\left(-a\sigma^{\xi}(x_i-\mu)^{-\xi}\right)}\right)+(\alpha-1)$$
$$\cdot\sum_{i=1}^{n}\ln\left(1-\left(1-e^{\left(-a\sigma^{\xi}(x_i-\mu)^{-\xi}\right)}\right)^{b}\right)$$

(2.43)

The maximum likelihood estimators (MLE) $\widehat{\boldsymbol{\theta}} = \left(\hat{\alpha},\hat{a},\hat{b},\hat{\mu},\hat{\sigma},\hat{\xi}\right)^{T}$ of $\boldsymbol{\theta} = (\alpha,a,b,\mu,\sigma,\xi)^{T}$ can be obtained by differential evolution method from (2.43)

2.5 Differential Evolution Method

The log-likelihood can be maximized by numerical technique, in this study we used the differentiated evolution method. Differential Evolution (DE) is a parallel direct search method which utilizes NP D-dimensional parameter vectors. The "DE community" has been growing since the early DE years of 1994 – 1997. Stone R. and Price K. were published their papers about DE, they defined the original version of DE by the following constituents.

1) The population

$$P_{x,g} = \left(X_{i,g}\right),\quad i = 0,1,\dots,N_{p-1}, g = 0,1,\dots,g_{max},$$
$$X_{i,g} = \left(X_{j,i,g}\right),\quad j = 0,1,\dots,D-1$$

(2.44)

where Np denotes the number of population vectors, g defines the generation counter and D the dimensionality, i.e. the number of parameters.

2) The initialization of the population via

$$X_{j,i,g} = rand_j[0,1)\cdot\left(b_{j,U}-b_{j,L}\right)+b_{j,L}$$

(2.45)

The D-dimensional initialization vectors, $\mathbf{b}_L$ and $\mathbf{b}_U$ indicate the lower and upper bounds of the parameter vectors $\mathbf{x}_{i,j}$. The random number generator, $\mathrm{rand_j}[0,1)$, returns a uniformly distributed random number from within the range $[0,1)$, i.e., $0\le \mathrm{rand_j}[0,1) < 1$. The subscript, j, indicates that a new random value is generated for each parameter.

3) The perturbation of the base vector $\mathbf{y}_{i,g}$ by using a difference vector based mutation

$$\mathbf{v}_{i,g} = \mathbf{y}_{i,g} + F\cdot\left(\boldsymbol{X}_{r1,g}-\boldsymbol{X}_{r2,g}\right)$$

(2.46)

to generate a mutation vector $\mathbf{v}_{\mathrm{i,g}}$. The difference vector indices, r1 and r2, are randomly selected once per base vector. Setting $\mathrm{y}_{\mathrm{i,g}} = \mathrm{x}_{\mathrm{r0,g}}$ defines what is offen called classic DE where the base vector is also a randomly chosen population vector. The random indexes r0, r1, and r2 should be mutually exclusive. There are also described later. For example, setting the base vector to the current best vector or a linear combination of various vectors is also popular. Employing more than one difference vector for mutation has also been tried but has never gained a lot of popularity so far.

4) Diversity enhancement

The classic variant of diversity enhancement is crossover which mixes parameters of the *mutation vector*$\mathbf{v}_{\mathrm{i,g}}$ and the so-called *target vector*$\mathbf{x}_{\mathrm{i,g}}$ in order to generate the *trial vector*$\mathbf{u}_{\mathrm{i,g}}$. The most common form of crossover is uniform and is defined as

$$\mathbf{u}_{i,g} = u_{j,i,g} = \begin{cases} v_{j,i,g} & if\ \left(\mathrm{rand}_j\ [0,1) \leq C_r\right) \\ x_{j,i,g} & \text{otherwise} \end{cases} \tag{2.47}$$

In order to prevent the case at least one component is taken from the mutation vector, a detail that is not expressed in Eq.(4.46).

5) Selection

DE uses simple one-to-one survivor selection where the *trial vector*$\mathbf{u}_{\mathrm{i,g}}$ computes against the *target vector*$\mathbf{x}_{\mathrm{i,g}}$. The vector with the lowest objective function value survives into the next generation g+1.

$$\mathbf{x}_{i,g+1} = \begin{cases} \mathbf{u}_{i,g}\ \text{if}\ f\left(\mathbf{u}_{i,g}\right) \leq f\left(\mathbf{x}_{i,g}\right) \\ \mathbf{x}_{i,g}\text{otherwise} \end{cases} \tag{2.48}$$

Engelbercht A.P. (2007) introduced general differential evolution algorithm provided a generic implementation of the basic DE strategies. Initialization of the population is done by selecting random values for the elements of each individual from the bounds defined for the problem being solved as shown in following algorithm.

The algorithm of procedure for DEMLE

```
Set the generation counter, t = 0;
Initialize the control parameters, β and p_r;
Create and initialize the population, C(0), of n_s individuals;
while stopping condition(s) not true do
        for each individual, x_i(t) ∈ C(t) do
            Evaluate the fitness, f(x_i(t));
            Create the trial vector, u_i(t) by applying the mutation operator;
            Create an offspring, x'_i(t), by applying the crossover operator;
            If f(x'_i(t)) is better than f(x_i(t)) then
                Add x'_i(t) to C(t + 1) ;
            end
            else
                Add x_i(t) to C(t + 1) ;
            end
        end
end
Return the individual with the best fitness as the solution;
```

2.6 Goodness of fit test

The objective of goodness of fit test is to test a hypothesis:

$$H_0: F(x) = F_0(x)\ \forall\, x;\quad against \qquad H_1: F(x) \neq F_0(x)\ \exists\, x;$$

Where $F_0(x)$ is a known distribution function based on $X_1, X_2, \dots, X_n$, a random sample from the generalized extreme value distribution function $F(x)$. Kolmogorov-smirnov test is used for this hypothesis. The K-S test gives simultaneous confidence intervals for all the observations and provides a visual goodness of fit test. The K-S test provides bounds within which every observation should fall if the sample is actually drawn from the hypothesized distribution. The K-S statistics are calculated as follows. For each sample of the random numbers.

$$D^+ = \max\left[\frac{i}{n} - F_0\left(x_{(i)}\right)\right] \qquad i = 1,2,\dots,n$$

$$D^- = \max\left[F_0\left(x_{(i)}\right) - \frac{i-1}{n}\right] \qquad i = 1,2,\dots,n$$

$$D = \max[D^-, D^+]$$

where $F_0\left(x_{(i)}\right)$ is the c.d.f. of each distribution and $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$

2.7 Return Level

x_p is the return level with the return period $T = \left(\frac{1}{p}\right), p = \left(\frac{1}{T}\right)$, since to a reasonable degree of accuracy, the level x_p is expected to be exceeded on average once every $T = \left(\frac{1}{p}\right)$ years and more precisely, is exceeded by the maximum in any particular year with probability p.

2.8 Return Period

If P(z) is the probability of the level z being exceeded in a single year, then level z is often said to have a return period which is in inverse of P(z) years. For example, a daily discharge having a probability of being exceeded in a year in 0.01 is said to have a return period of T=1/0.01=100 years. A daily discharge that has a probability of being exceeded one in hundred years is called the 100-year return level.

The return period of flood mean the period that flood will occur once in T years. The return period for flood, T, is the years exceeded on flood once indicated as follows,

$$T = \frac{1}{1-p} \quad ; p = F(x)$$

Salvadori*et. al.* (2007) discussed the hazard ranked in four quantitative classed according to the rate of occurrence of the events, as follow

Hazard class	**Description**
High	Events occur more frequently than once every 10 years
Moderate	Events occur once every 10-100 years
Low	Events occur once every 100-1000 years
Very Low	Events occur les33333s frequently than every 1000 years

3 Simulation Study

To assess the performance of the methods, we generated sample size n=10, 20(small), 30, 50(moderate), 80 and 100 (large) from 6 parameter sets. In each case, the parameters were estimated by maximum likelihood estimation with NP=100 and Gmax=100000 of the differential evolution numerical method (DEMLE) using Scilab and 1000 iterations of the simulation. The

Bias, Variance, Mean Square Error (MSE) and Mean Absolute Percentage Error were calibrated. The lower value of these criteria is the better fit.

The study of the Fréchet Distribution, to assess the performance of the methods, we generated sample sizes n=10, 20,(small) 30, 50(moderate), 80 and 100 (large) from 6 parameter sets , [(50,10,2.5), (150,15,3.0), (200,50,2.5), (350,220,1.5), (500,200,1.75), (850,100,2.0)]. In each case, the parameters were estimated by maximum likelihood estimation with NP=100 and Gmax=100000 of the differential evolution numerical method (DEMLE) using Scilab and 10000 iterations of the simulation. The Bias, Variance, Mean Square Error (MSE) and Mean Absolute Percentage Error were calibrated. The lower value of these criteria is the better fit. The result of this part we observed that as n increase mostly of the bias decrease. Figure 3.1 shows the average MAPE of DEMLE of the Fréchet Distribution of all parameter sets.

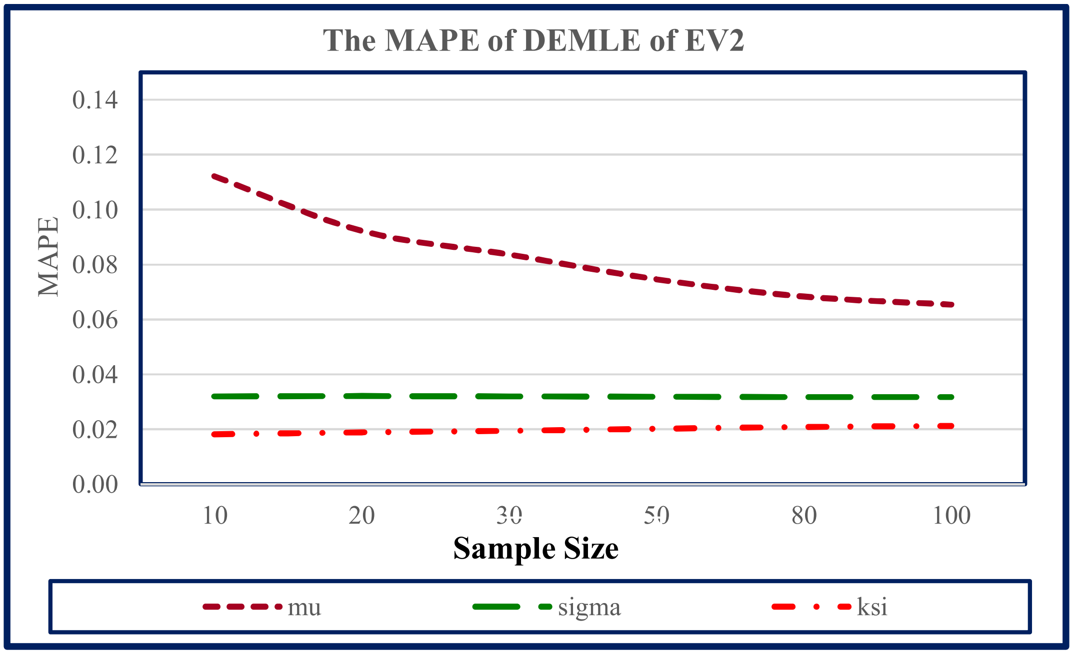

Figure 3.1 The graph of the average of MAPE of DEMLE of the Fréchet Distribution

The study of the Kumaraswamy Fréchet Distribution, to assess the performance of the methods, we generated sample sizes n=10, 20,(small) 30, 50(moderate), 80 and 100 (large) from 5 parameter sets [(1.25, 3.0, 100, 30, 1.25), (1.5, 2.0, 150, 50, 2.0), (2.0, 3.0, 300, 100, 1.25), (1.25, 2.5, 500, 150), (2.0,2.5,750,250,1.5), (1.5,2.5,1000,300,1.5)]. In each case, the parameters were estimated by maximum likelihood estimation with NP=100 and Gmax=100000 of the differential evolution numerical method (DEMLE) using Scilab with simulation 1000 time for each sample sizes each parameter set. The Bias, Variance, Mean Square Error (MSE) and Mean Absolute Percentage Error (MAPE) were calibrated. The lower value of these criteria is the better fit. The result of this part we observed that as n increase mostly of the bias decrease. Figure 3.2 shows the average MAPE of DEMLE of the Kumaraswamy Fréchet Distribution of all parameter sets.

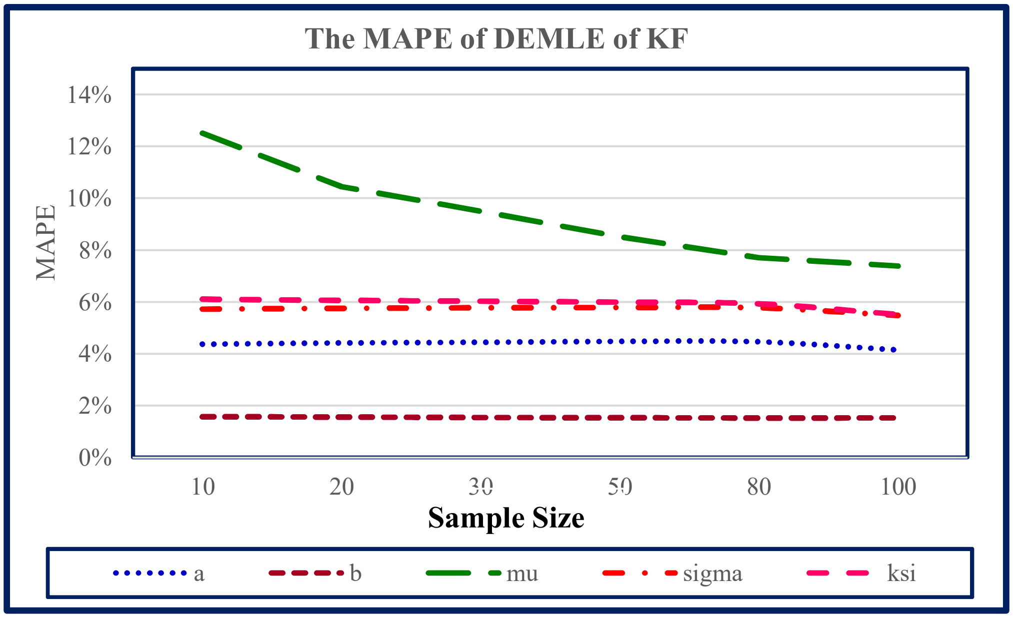

Figure 3.2 The graph of the MAPE of DEMLE of the Kumaraswamy Fréchet Distribution

The study of the Exponentiated Kumaraswamy Fréchet (EKF) Distribution, to assess the performance of the methods, we generated sample sizes n=10, 20,(small) 30, 50(moderate), 80 and 100 (large) from 5 parameter sets , [(100, 30, 1.25), (150, 50, 2.0), (300, 100, 1.25), (500, 150, 1.5), (750, 250, 1.5), (1000, 300, 1.5)]. In each case, the parameters were estimated by maximum likelihood estimation with NP=100 and Gmax=100000 of the differential evolution numerical method (DEMLE) using Scilab with simulation 1000 time for each sample sizes each parameter set. The Bias, Variance and Mean Square Error were calibrated. The result of this part we observed that as n increase mostly of the bias decrease. Figure 3.3 shows the average MAPE of DEMLE of the Exponentiated Kumaraswamy Fréchet Distribution of all parameter sets.

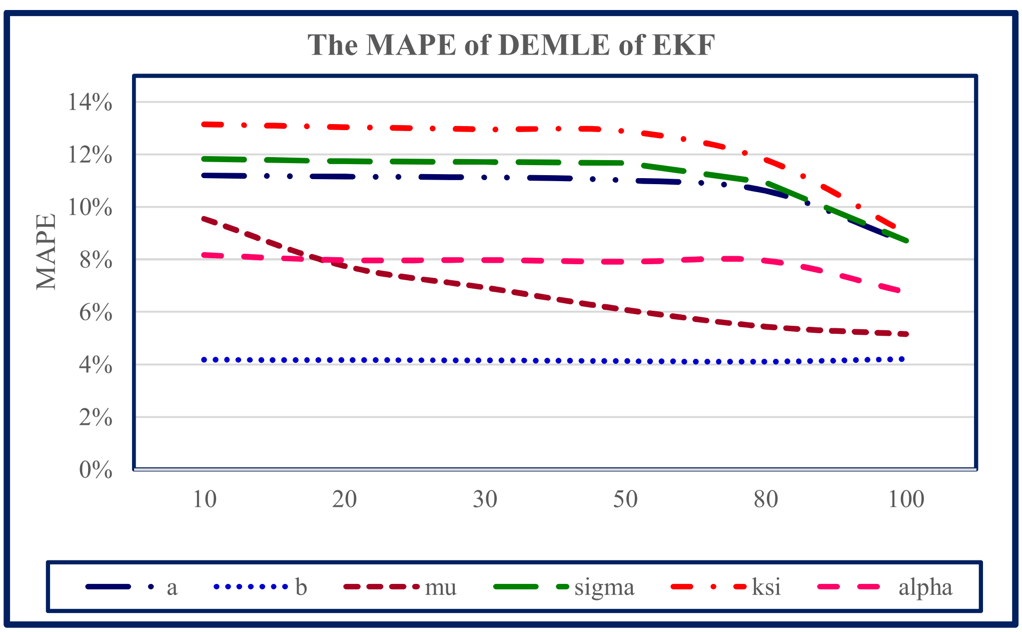

Figure 3.3 The graph of the MAPE of DEMLE of the Exponentiated Kumaraswamy Fréchet Distribution

4 Application for Flood Frequency in Upper Chaophraya River Basin.

In application part, we use 29 stream gauging stations in upper Chaophraya river basin to fit with the Fréchet distribution using differential evolution maximum likelihood estimation (DEMLE). The Kolmokorov-Smirnov test, MSE and MAPE, 80% of the stations followed the Fréchet distribution except P.1, N.64, N.13A and N.67. Thus, the flood peaks data of most stations follow the Fréchet Distribution. The return level of the Fréchet distribution different from the return level of the Generalized Extreme Value distribution 64.9%. Hazard class shows that most stream gauging stations in upper Chaophraya basin have high hazard class. The return period of the Fréchet distribution different from the return period of the Generalized Extreme Value distribution 51.03%.

We use 29 stream gauging stations in upper Chaophraya river basin to fit with the Kumaraswamy Fréchet distribution using differential evolution maximum likelihood estimation (DEMLE). The Kolmokorov-Smirnov test, MSE and MAPE, all stations followed the Kumaraswamy Fréchet distribution Thus, the flood peaks data of all stations follow the Kumaraswamy Fréchet Distribution. The return level of the Kumaraswamy Fréchet distribution different from the return level of the Generalized Extreme Value distribution 20.5%. Hazard class shows that most stream gauging stations in upper Chaophraya basin have high hazard class. The return period of the Kumaraswamy Fréchet distribution different from the return period of the Generalized Extreme Value distribution 26.22%.

To apply KF model for flood frequency, we found that the most of stations are not followed the EKF model except Y.16. The return period are definitely different but return level are quite fluctuated from GEV especially when T gathers than 50 years.

5 Conclusion and outlook for future work

In the simulation study, we can conclude that the DEMLE is good for estimation parameter with complex function (a lot of parameters in the distribution) and the DEMLE are mostly consistency.

In the application for flood frequency in upper Chaophraya river basin, we can conclude that the best fit model is the Kumaraswamy Fréchet distribution compared with the Fréchet distribution and the Exponentiated Kumaraswamy Fréchet distribution.

By the result of the study, we recommended some points for the future work to extend this study in the future. The future recommendations are,

(1) By the result of the computational part, the MSE and MAPE are not decreasing exactly when n increase. Therefore, we recommend the new parameter estimation. In this study we used original differential evolution maximum likelihood estimation, then in the future should apply advanced differential evolution methods such as self-adaptive differential evolution method, drift differential evolution method or opposition-Based differential evolution method etc. to compare with original differential evolution method.

(2) By the result in application part, the return level of each model increase rapidly when T-year increase. In fact, the data of flood frequency has the right boundary by the river bank. Therefore, we recommend the new type of model for flood frequency, the right truncated Fréchet distribution, the right truncated Kumaraswamy Fréchet distribution and the right truncated Exponentiated Kumaraswamy Fréchet distribution.

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