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A Candy Lover Learns to Optimize

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ABSTRACT
A container has two types of candies: Type A and Type B. Concerned about her child's well-being, a wise mom pronounces, "Each day, you can choose two candies from the container *random*. If they are of different types, you can eat them both. If they are of the same type, eat only one and return the other to the container." We analyze the expected number of days needed to eat all candies in the container and the proportion of days the child eats two candies. Several other variations are either worked out or left for readers to solve.

Keywords**:** random sampling without replacement, probability distribution, recursive relation, analytical expression, mathematical induction.

1. Introduction

We pose a simple probability problem understandable to high school graduates; and yet its solution poses a fairly hard challenge for mathematics graduate students. Nonetheless, one can develop a computer algorithm to compute the solution to a desired degree of accuracy, which suffices to make decisions. We invite mathematically inclined students to discover analytical solutions.

The son of a Professor of Mathematics (a specialist in Optimization Theory) loves to eat candies. The professor bought two kinds of candies: apple flavour (A); and butter toffee flavour (B). The two candies have the same shape and size; but Type A candies come wrapped in aqua wrapper and Type B in brown. Said the professor to himself, "Opening two containers to take out one candy from each is an inefficient process. So, every Sunday evening I will put 7 type A candies and 7 type B candies in one empty, opaque container. Each morning Johnny can take two candies out."

The professor's wife did not want her son to eat too many candies. She imposed a restriction hoping to ensure the candies will last a few days longer while Johnny will learn some self-discipline. She instructed the child thus: "Every day you will take out two candies at *random* from the container. If they are of mixed types AB, you can eat them both; if they are of type AA or type BB, you may eat only one candy and you must return the other to the container."

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Statement of the Problem

How effective was the mother's clever strategy? To answer, we raise the following questions:

(1) In how many days will the child finish eating all 14 candies? The answer is a random variable $X_{7,7}$ taking values 7, 8, 9,…, 13; but with what associated probabilities?

(2) What is the expected number of days the 14 candies will last; that is, what is $E[X_{7,7}]$?

(3) What is the expected proportion of days the child will eat two candies?

(4) How do the answers to Questions (1)-(3) change as the number $a = b$ of candies of each type increases *ad infinitum*?

2. Simulations

Suppose that initially the container has a type A candies and b type B candies. Without loss of generality, assume $a \leq b$. Clearly, if $a = 0$ or 1, then the candies will last exactly b days, with the child eating one type B candy every day. If $a = 1$, the type A candy will be eaten on Day i with probability $(b + 1 - i)/\binom{b+1}{2}$ for $i = 1,2,\ldots,b$. But if $1 < a \leq b$, then the number of days the candies will last is a random variable taking integer values b, $b + 1$, …, $(a + b - 1)$. We simulate the process of candy consumption using the freeware R [2].

- Step 1: Observe the random evolution of the contents in the container beginning with a type A and b type B candies on Day 0 until it becomes empty.
- Step 2: Replicate the above process multiple times to estimate the probability distribution of the number of days until the container is emptied.

Step 1. For illustration, we chose parameters $a = b = 7$, 14, 30, 100. The result of one iteration of Step 1 is called a **sample path**, which is depicted in Figure 1, together with the random number of days, $X_{a,b}$, the candies last. The child eats two candies on $d = a + b - X_{a,b}$ days.

Each sample path moves horizontally (going east) when one type A candy is eaten, vertically (going north) when one type B candy is eaten, and diagonally (going northeast) when two candies (one type A and one type B) are eaten. The starting point of each path is (0, 0) and the ending point is (a, b). The number of bullets along the path is the number of days needed to eat all candies. Note that if a path has d diagonal segments (where d ranges over 1 to $a = \min\{a, b\}$), then it has $(a - d)$ horizontal segments and $(b - d)$ vertical segments. In this case, the child eats two candies on d days out of $(a + b - d)$ days needed to eat all $(a + b)$ candies. Hence, the number of days needed to empty the container, $X_{a,b}$, *plus* the number of days two candies are eaten is always $(a + b)$. Therefore, the answers to questions (2) and (3) are linearly related with slope –1.

How many distinct sample paths are there? What are their probabilities? Let $N_{a,b}$ denote the number of distinct sample paths from (0, 0) to (a, b). Clearly, for all $a, b \geq 1$, $N_{a,b} = N_{b,a}$ and $N_{1,b} = b$. Thereafter, by conditioning on the initial segment going from (0, 0) to (1, 0), (1, 1), (0, 1), we have the recursive relation

$$N_{a,b} = N_{a-1,b} + N_{a-1,b-1} + N_{a,b-1} \tag{1}$$

Table 1 documents the number of sample paths for some values of (a, b). However, the crux of the problem lies in the fact that these paths are *not equally likely*, even when the paths have the same number of diagonal segments. We will explain how to calculate the probability of each sample path in Section 2.

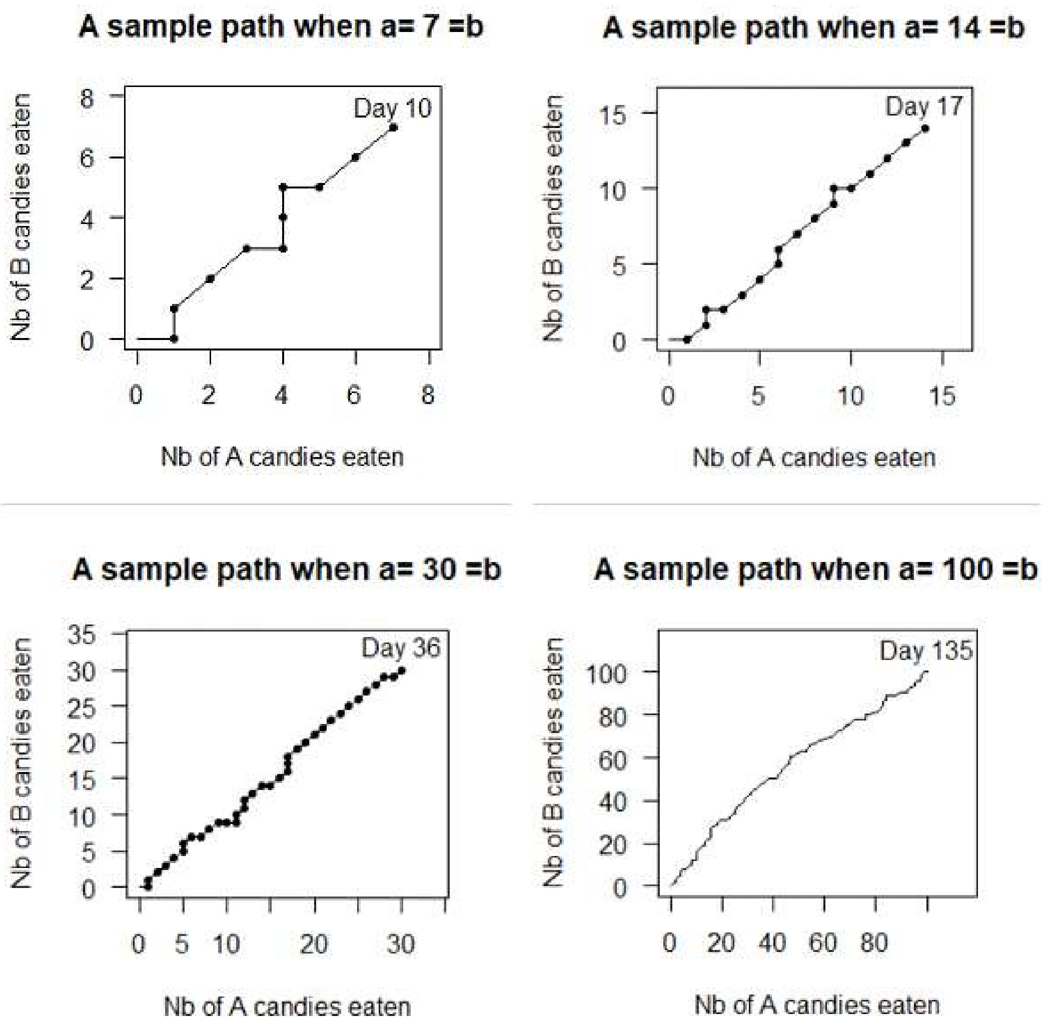

Figure 1. Sample paths for $a = b = 7$, 14, 30, and 100, showing the cumulative number of candies eaten until the container is empty. Each bullet represents a day.

It is worth mentioning that just as any path tracks the cumulative number of candies of each type eaten by a certain day, by subtraction, one can also track the number of candies of each type left in the container. Therefore, given the number of candies of each type already eaten, it is possible to calculate the probabilities of drawing candies of types AA, BB and AB on the following day.

Table 1. Number of distinct sample paths from (0, 0) to (a, b)

	b							(a, b)	$N_{a,b}$
	1	2	3	4	5	6	7		
1	1	2	3	4	5	6	7	(14, 14)	3.256957×10^{9}
2	2	5	10	17	26	37	50		
3	3	10	25	52	95	158	245	(30, 30)	3.980085×10^{21}
4	4	17	52	129	276	529	932		
5	5	26	95	276	681	1486	2947	(100, 100)	8.497919×10^{74}
6	6	37	158	529	1486	3653	8086		
7	7	50	245	932	2947	8086	19825		

Step 2. We chose different values of parameters (a, b) and varied the number of replications to demonstrate that the simulated probabilities have higher degree of accuracy as the number of replications increases. (The simulated probabilities are correct to at least m decimal places if the number of replications is 10^{2m}.) The results of Step 2 are shown in Table 2. Note that the simulated probabilities are close to the exact probabilities shown in the last column, which are computed using a technique described in the next section and documented in Table 3.

Table 2. Simulated probabilities for the number of days until the container becomes empty

a	b	replications	$X_{a,b}$	count	estimated probability	exact probability
2	2	10^3	2 days	664	0.664	2/3
			3 days	336	0.336	1/3
2	3	10^3	3 days	776	0.776	4/5
			4 days	224	0.224	1/5
2	3	10^5	3 days	79626	0.79626	4/5
			4 days	20284	0.20284	1/5
			3 days	39,961	0.39961	0.40
3	3	10^5	4 days	52,006	0.52006	0.52
			5 days	8033	0.08033	0.08
			4 days	227,880	0.227880	0.228571
4	4	10^6	5 days	541,815	0.541815	0.542041
			6 days	211,630	0.211630	0.210612
			7 days	18,675	0.018675	0.018776

3. From Manual Calculations to Computational Algorithms

Manual Calculations. A tree map, shown in Figure 2, depicts the evolution of the contents of the container for the case $a = b = 2$. Along each arrow, we write the conditional probability of that transition. We strongly urge readers to construct tree maps for cases a

= b = 3 and a = b = 4 and observe that the tree map for (a, b) is the mirror image of the tree map for (b, a).

$$
\begin{tikzcd}
 & & \text{2A, 2B} \arrow[dll, "\frac{1}{6}"'] \arrow[d, "\frac{2}{3}"'] \arrow[drr, "\frac{1}{6}"] & & \\
\text{1A, 2B} \arrow[d, "\frac{2}{3}"'] \arrow[dr, "\frac{1}{3}"] & & \text{1A, 1B} \arrow[d, "1"'] & & \text{2A, 1B} \arrow[dl, "\frac{1}{3}"'] \arrow[d, "\frac{2}{3}"] \\
\text{0A, 1B} \arrow[d, "1"'] & \text{1A, 1B} \arrow[d, "1"] & \text{0A, 0B} & \text{1A, 1B} \arrow[d, "1"'] & \text{1A, 0B} \arrow[d, "1"] \\
\text{0A, 0B} & \text{0A, 0B} & & \text{0A, 0B} & \text{0A, 0B}
\end{tikzcd}
$$

Figure 2. All possible sample paths for a container with 2 type A candies and 2 type B candies, together with conditional probabilities along each transition

If we multiply all conditional probabilities along a directed path, we will get the eventual prob- ability of traveling on that path until reaching the end configuration when the container becomes empty. Recall that the number of segments along each path equals the number of days $X_{a,b}$ to eat all candies. Adding up the probabilities of those paths that require equal number of days to finish all candies, we obtain the probability distribution of $X_{a,b}$. Note that the number of days when the child eats two candies is $a+b-X_{a,b}$. Hence, the answers to questions (2) and (3) are linearly related. The results of manual calculations are included in Table 3.

What is shown in Figure 2, can be generalized to explain how the contents of the container evolves if it starts at point (0, 0) with (a, b) candies of types A and B. From each node, at most three branches can emerge after a random draw of two candies:

(1) Draw AA; eat one type A candy, return the other to reach point (1, 0) leaving (a -1, b) candies in the container, with associated probability
$$P(AA) = \frac{\binom{a}{2}}{\binom{a+b}{2}} \tag{2}$$
(2) Draw AB; eat both candies to reach point (1, 1) leaving ($a-1$, $b-1$) candies with associated probability
$$P(AB) = \frac{a \cdot b}{\binom{a+b}{2}} \tag{3}$$
(3) Draw BB; eat one type B candy, return the other to reach point (0, 1) leaving (a, $b-1$)candies, with associated probability
$$P(BB) = \frac{\binom{b}{2}}{\binom{a+b}{2}} \tag{4}$$

Equations (2), (3), (4) are hypergeometric probabilities (see [4], for example) corresponding to the number of candies of each type in a *random sample without replacement* of size two from a container containing a type A and b type B candies. Starting from (a, b) candies on Day 0, and using conditional probabilities (2), (3), (4) of drawing two candies on each day, the resulting probability distribution of the number of days until the container becomes empty is shown in Table 3.

Table 3. The distribution of $X_{a,b}$, the number of days (a, b) candies will last, obtained by manual calculation using a tree map

a	b	probabilities
2	2	P(2 days) = 2/3
		P(3 days) = 1/3
3	3	P(3 days) = 2/5
		P(4 days) = 13/25
		P(5 days) = 2/25
4	4	P(4 days) = 8/35
		P(5 days) = 664/1225
		P(6 days) = 258/1225
		P(7 days) = 23/1225

Remark 1. Note that if the sample path reaches point $(a - 1, b - 1)$, then the container has in it only one candy of each type. Hence, the next day only type AB pair can be drawn, as justified by evaluating (2), (4) and (3) to be 0, 0 and 1, respectively.

Recursive Relation. From a tree map, we note that $X_{a,b}$ equals: (i) $1 + X_{a-1,b}$ with probability $\binom{a}{2} / \binom{a+b}{2}$; (ii) $1+X_{a-1,b-1}$ with probability $ab / \binom{a+b}{2}$; and (iii) $1+X_{a,b-1}$ with probability $\binom{b}{2} / \binom{a+b}{2}$. Since the total number of candies in the container is reduced by one or two each day, proceeding recursively, a computational algorithm can calculate the probabilities associated with permissible values of $X_{a,b}$. The algorithm stops when the boundary condition is met with either a or b or both reduced to 1. Recall that when the container has one type A candy and $b \geq 1$ type B candies, then the container will be emptied after exactly b days. The recursively computed probabilities and the run times are shown in Table 4.

Table 4. The probability distribution of $X_{a,b}$, the number of days (a, b) candies will last, is calculated using a recursive relation: The distributions of $X_{7,7}$ and $X_{14,14}$ are approximate.

a	b	Xa,b	$P(X_{a,b})$	Xa,b	$P(X_{a,b})$	Xa,b	$P(X_{a,b})$	Runtime (sec)
2	2	2 days	2/3	3 days	1/3			0.0000
3	3	3 days	2/5	4 days	13/25	5 days	2/25	0.0010
4	4	4 days	8/35	6 days	258/1225			0.0010
		5 days	664/1225	7 days	23/1225			
7	7	7 days	16/429	10 days	0.228096	12 days	8/1287	
		8 days	0.266362	11 days	0.056492	13 days	1/4056	9.4358
		9 days	0.405292					
		14 days	$4.084/10^4$	19 days	0.242942	24 days	$2.235/10^4$	
		15 days	0.011083	20 days	0.133501	25 days	$1.583/10^5$	
14	14	16 days	0.071645	21 days	0.048684	26 days	$6.444/10^7$	9560.6043
		17 days	0.195349	22 days	0.012009	27 days	$1.145/10^8$	
		18 days	0.282132	23 days	0.002007			

The documented run times in Table 4 indicate that the algorithm at this stage is not efficient: In particular, it took 2.45 hours to calculate the probabilities for $a = b = 14$. A more efficient algorithm is called for. Refer to [1] for efficient recursive computations.

Efficient Algorithm. To make the program more efficient, we represented the probability distri- bution as a vector. Let $p(x) = P\{X_{a,b} = x\}$ be the probability that the candies last for x days, where $x = 1, 2, \ldots, (a + b - 1)$. Then the vector of probabilities is denoted by

$$p_{a,b} = (p(1), p(2), \ldots, p(a + b - 1)) \tag{5}$$

Although some leading entries of the probability vector may be zero, we wish to keep those entries intact so that one can read off the index easily. Note that the index starts at 1, and not at 0. Using the vectors of probabilities, we rewrite the recursive relation as follows:

$$p_{a,b} = \frac{\binom{a}{2}}{\binom{a+b}{2}}(0, p_{a-1,b}) + \frac{a \cdot b}{\binom{a+b}{2}}(0, p_{a-1,b-1}) + \frac{\binom{b}{2}}{\binom{a+b}{2}}(0, p_{a,b-1}) \tag{6}$$

The probability vectors $p_{a-1,b}$, $p_{a,b-1}$, and $p_{a-1,b-1}$ on the right hand side are shifted one place to the right by prefixing a 0; and they can be calculated using a formula similar to (6). For example,

$$p_{2,2} = \frac{\binom{2}{2}}{\binom{4}{2}}(0, p_{1,2}) + \frac{2 \cdot 2}{\binom{4}{2}}(0, p_{1,1}) + \frac{\binom{2}{2}}{\binom{4}{2}}(0, p_{2,1})$$

Next, using the boundary conditions $p_{1,2} = p_{2,1} = (0, 1)$ and $p_{1,1} = (1, 0)$, we have

$$p_{2,2} = \frac{1}{6}(0,0,1) + \frac{4}{6}(0,1,0) + \frac{1}{6}(0,0,1) = \left(0, \frac{2}{3}, \frac{1}{3}\right)$$

This is how we write an efficient program: We define a function that implements the recursive formula (6); and we write a new algorithm that calls this function repeatedly. Also, we save the results in an Excel sheet so that the program can retrieve and return the result, if available, or perform the recursion until it reaches the closest $p_{u,v}$ available in the Excel file. This approach avoids continuing calculations all the way down to the boundary condition of $\min\{u, v\} = 1$, and thereby reduces the run time significantly. A small portion of the Excel sheet is shown in Table 5.

	b			
a	1	2	3	4
1	(1)	(0, 1)	(0, 0, 1)	(0, 0, 0, 1)
2		(0, 2, 1)/3	(0, 0, 4, 1)/5	(0, 0, 0, 64, 11)/75
3			(0, 0, 10, 13, 2)/25	(0, 0, 0, 300, 202, 23)/525
4				(0, 0, 0, 280, 664, 258, 23)/1225

On paper, we economize writing down the probability vector by eliminating the leading $(b - 1)$ zeros. Instead, we write, for example, "$p_{2,4}$ = ([4] 64/75, 11/75)", and read it as: "The first three entries are zeros; thereafter, starting from the fourth entry, the elements are 64/75, ... "

To answer Question (1) raised in Section 1, we must document $p_{7,7}$. However, empowered by our efficient algorithm, we also document $p_{14,14}$ and $p_{30,30}$.

$$\begin{aligned} p_{7,7} &= \left([7]\frac{16}{429}, \frac{1715752}{6441435}, \frac{117479654}{289864575}, \frac{88156007}{386486100}, \frac{131000081}{2318916600}, \frac{8}{1287}, \frac{1}{4056}\right) \\ &= \left([7]0.0373, 0.2664, 0.4053, 0.2281, 0.0565, \frac{6.216}{10^3}, \frac{2.466}{10^4}\right), \end{aligned}$$

$$\begin{aligned} p_{14,14} = \Big(&[14]\frac{4.084}{10^4}, 0.0111, 0.0716, 0.1953, 0.2821, 0.2429, 0.1335, 0.0487, \\ &0.0120, \frac{2.235}{10^3}, \frac{2.235}{10^4}, \frac{1.583}{10^5}, \frac{6.444}{10^7}, \frac{1.145}{10^8}\Big), \end{aligned}$$

$$\begin{aligned} p_{30,30} = \Big(&\frac{[30]9.079}{10^9}, \frac{1.080}{10^6}, \frac{3.176}{10^5}, \frac{4.094}{10^4}, \frac{2.925}{10^3}, 0.0132, 0.0403, 0.0890, 0.1469, \\ &0.1868, 0.1869, 0.1497, 0.0973, 0.0519, 0.0229, \frac{8.370}{10^3}, \frac{2.558}{10^3}, \frac{6.539}{10^4}, \\ &\frac{1.399}{10^4}, \frac{2.503}{10^5}, \frac{3.731}{10^6}, \frac{4.612}{10^7}, \frac{4.687}{10^8}, \frac{3.870}{10^9}, \frac{2.552}{10^{10}}, \frac{1.311}{1011}, \\ &\frac{5.055}{10^{13}}, \frac{1.374}{10^{14}}, \frac{2.792}{10^{16}}, \frac{1.895}{10^{18}}\Big). \end{aligned}$$

4. Expectation

Having calculated the probability vector $p_{a,b}$, we can immediately find the expected number of days needed to empty the candy container. For example, the distribution of $X_{4,4}$ is

$$p_{4,4} = \left(0,0,0, \frac{8}{35}, \frac{664}{1225}, \frac{258}{1225}, \frac{23}{1225}\right),$$

with indices ranging from Day 1 to Day 7. Hence,

$$E[X_{4,4}] = 4 \times \frac{8}{35} + 5 \times \frac{664}{1225} + 6 \times \frac{258}{1225} + 7 \times \frac{23}{1225} = 5.019592.$$

In fact, without first finding the probability distribution, we can directly calculate $\mu_{a,b} = E[X_{a,b}]$, the expected number of days the candies will last, by using a recursive relation similar to (6); namely, $\mu_{0,b} = b = \mu_{1,b}$, and thereafter for all $a, b \geq 1$, we have $\mu_{b,a} = \mu_{a,b}$ and

$$\mu_{a,b} = 1 + \frac{\binom{a}{2}}{\binom{a+b}{2}} \mu_{a-1,b} + \frac{a \cdot b}{\binom{a+b}{2}} \mu_{a-1,b-1} + \frac{\binom{b}{2}}{\binom{a+b}{2}} \mu_{a,b-1} \qquad (7)$$

To answer Question (2), raised in Section 1, we must report $\mu_{7,7}$. To compute it, we need $\mu_{i,j}$ for all $i \leq 7, j \leq 7$. We document these values in Table 6.

Table 6. Expected number of days to empty a container containing a type A candies and b type B candies

	b						
a	1	2	3	4	5	6	7
1	1	2.00	3.00	4.00	5.00	6.00	7.00
2	2	2.33	3.20	4.15	5.12	6.10	7.09
3	3	3.20	3.68	4.47	5.37	6.30	7.26
4	4	4.15	4.47	5.02	5.77	6.63	7.53
5	5	5.12	5.37	5.77	6.35	7.09	7.91
6	6	6.10	6.30	6.63	7.09	7.69	8.41
7	7	7.09	7.26	7.53	7.91	8.41	9.02

Having obtained $\mu_{a,b}$, it is now straight-forward to see that the child will eat two candies on average on $(a + b) - \mu_{a,b}$ days, out of the $\mu_{a,b}$ days the candies will last on average. Hence, on average the proportion of days the child will eat two candies is

$$\pi_{a,b} = (a + b)/\mu_{a,b} - 1.$$

In Table 7, we answer Questions (2) and (3) of Section 1 by documenting the expected number of days the candies will last and the expected proportion of days the child will eat two candies if he starts with (a, b) candies in the container on Day 0.

Table 7. Expected number of days candies will last and expected proportion of days the child will eat two candies

(a, b)	(7, 7)	(14, 14)	(30, 30)	$(10^2, 10^2)$	$(10^3, 10^3)$
expected # days candies last	9.02	18.34	39.66	132.97	1332.93
proportion of days eat two candies	0.552	0.527	0.513	0.50413	0.50045

Based on Table 7, we can now answer Question (4) of Section 1: As $a = b$ tends to infinity, the $2a$ candies will last on average $4a/3$ days; and the child will eat two candies on about 50% of the days and one candy on the remaining 50% of the days. This limiting result makes sense intuitively because when $a = b$ is large, then on the first day the child is almost equally likely to draw two candies of the same type as two candies of different types.

5. Finding Analytical Expressions: Open Problem

The results in Tables 2, 3, and 4 agree, confirming that the recursive relation yields the probability distribution for any (a, b). Using the computationally obtained results as reference, we hoped to find an analytical formula that will produce $p_{a,b}$ directly without using the values of $p_{u,v}$ for $u < a$ and/or $v < b$. We solved this problem only for $a = 2$. Indeed, $X_{2,b}$ (with $2 \leq b$) takes on only two values: b and $b + 1$. For $a = 2$, the probability of emptying the container after $(b + 1)$ days is given by $p_{2,2}[3] = 1/3$, and thereafter for $b \geq 3$, by the recursive relation

$$p_{2,b}[b+1] = \frac{1+\binom{b}{2}p_{2,b-1}[b]}{\binom{b+2}{2}} \tag{8}$$

since $p_{1,b}[b] = 1$ and $p_{1,b-1}[b] = 0$. Using mathematical induction, one can verify that the solution to (8) is given by

$$p_{2,b}[b+1] = \frac{2+\binom{b+2}{3}}{\binom{b+1}{2}\binom{b+2}{2}}$$

For $a = 2$, the probability of emptying the container after b days is $p_{2,b}[b] = 1 - p_{2,b}[b + 1]$. In particular, this implies that $\mu_{2,b} = b + p_{2,b}[b + 1]$, which we can verify from Table 6. However, for $a > 2$, the analytical expressions for the probability vector $p_{a,b}$, the expected number of days $\mu_{a,b}$ the candies last, and the expected proportion of days $\pi_{a,b}$ the child eats two candies remain unsolved. We invite interested readers to discover them.

6. Johnny Learns to Optimize

Variation 0: Suppose that Mom had imposed a more parsimonious rule: ”Everyday you will take out two candies *at random* from the container. If they are of mixed types AB, you can eat them both; if they are of type AA or BB, you should eat *no candy* and return them both to the container.” In such a case, Johnny would be quite unhappy; but as explained below, we, the problem solvers, would have a field day with a perfect description of $X_{a,b}$, the number of days Johnny needs to eat $2a$ candies (a type A and a type B candies), leaving in the container $(b - a)$ candies of type B.

Note that the first day Johnny would eat two candies of opposite types is a geometric($r_{a,b}$) random variable (see [4]), where $r_{a,b}$ is the probability of drawing a mixed type of candies AB; that is,

$$r_{a,b} = \frac{ab}{\binom{a+b}{2}} \tag{9}$$

Thereafter, the container will have left in it $(a - 1,\ b - 1)$ candies of the two types. Continuing in this manner, we see that $X_{a,b}$ is the sum of a independent geometric random variables with success probabilities $r_{a,b}, r_{a-1,b-1}, \ldots, r_{1,b-a+1}$, respectively. Consequently, the expected value of $X_{a,b}$ is

$$\gamma_{a,b} = \frac{1}{r_{a,b}} + \frac{1}{r_{a-1,b-1}} + \cdots + \frac{1}{r_{1,b-a+1}} \tag{10}$$

and the expected proportion of days Johnny will eat (two) candies is $a/\gamma_{a,b}$. These values are shown in Table 8 for some choices of (a, b).

In particular, if $a = b$, then (9) simplifies to $r_{a,a} = a/(2a-1)$, and (10) simplifies to $2a-H_a$, where $H_a = 1 + 1/2 + 1/3, \ldots, 1/a$ is the harmonic sum. For large a, the harmonic sum is approximately $\ln(a) + \gamma$, where $\gamma = 0.5772156649 \cdots$ is the Euler–Mascheroni constant. See [5]. Therefore, in the long-run as $a = b$ increases *ad infinitum*, Johnny will eat two candies about 50% of days.

Table 8. If both like candies must be returned to the container, then the expected number of days (a, b) candies will last and the expected proportion of days the child will eat (two) candies

(a, b)	(7, 7)	(14, 14)	(30, 30)	(10^2, 10^2)	(10^3, 10^3)
expected # days candies last	11.41	24.75	56.01	194.81	1992.52
proportion of days eat two candies	0.613	0.566	0.536	0.51331	0.50188
(a, b)	(7, 10)	(14, 18)	(30, 40)	(10^2, 120)	(10^3, 1100)
expected # days candies last	14.41	29.35	70.55	230.68	2229.67
proportion of days eat two candies	0.486	0.477	0.425	0.43349	0.44451

Interestingly, as the lower half of Table 8 shows, by adding several extra candies of type B, Mom can prolong Johnny's consumption of $2a$ candies (or equivalently, lower the proportion of days Johnny eats two candies) and still preserve the extra candies she has added! However, taking pity on little Johnny, let us keep this sinister plan hidden from Mom.

Variation 1: One day Johnny said to his mom, "Dad says he gave me a seven-day supply of candies. Why can't we finish all candies in 7 days?" Mom thought for a few minutes and said: "It is not good for you to eat two candies every day. Nonetheless, you can draw two candies at random each day. If they are of different types, eat them both. If they are of the same type, eat one and give the other to your sister. Then surely the container will be emptied in 7 days."
To evaluate $v_{a,b}$, the expected number of candies the child will eat if the container initially had (a, b) candies of two types, we use a recursive relation similar to (7). The boundary values are: $v_{0,b} = \lceil b/2 \rceil = v_{b,0}$; $v_{1,1} = 2$; and for $b \geq 2$, we have

$$v_{1,b} = 1 + \frac{\binom{b}{1}}{\binom{b+1}{2}}\left(1 + v_{0,b-1}\right) + \frac{\binom{b}{2}}{\binom{b+1}{2}} v_{1,b-2} = 1 + \frac{(b-1)v_{1,b-2} + 2(1 + v_{0,b-1})}{b+1}$$
$$= v_{b,1}$$

Thereafter, for $a, b \geq 2$, we have

$$v_{1,b} = 1 + \frac{\binom{a}{2}}{\binom{a+b}{2}} v_{a-2,b} + \frac{a \cdot b}{\binom{a+b}{2}}\left(1 + v_{a-1,b-1}\right) + \frac{\binom{b}{2}}{\binom{a+b}{2}} v_{a,b-2} = v_{b,a} \tag{11}$$

Table 9 lists the number of candies Johnny will eat, for $1 \leq a, b \leq 7$.

Table 9. Suppose that the child must share a candy of the same type with his sister. Expected number of candies the child will eat if a container initially contains *a* type A candies and *b* type B candies

	b						
a	1	2	3	4	5	6	7
1	2.00	2.67	3.00	3.80	4.00	4.86	5.00
2	2.67	3.33	4.20	4.60	5.43	5.71	6.56
3	3.00	4.20	4.80	5.71	6.14	7.00	7.33
4	3.80	4.60	5.71	6.29	7.22	7.67	8.55
5	4.00	5.43	6.14	7.22	7.78	8.73	9.18
6	4.86	5.71	7.00	7.67	8.73	9.27	10.23
7	5.00	6.56	7.33	8.55	9.18	10.23	10.77

Having obtained $v_{a,b}$, it is now straight-forward to see that the child will eat two candies on $v_{a,b} - \lceil (a + b)/2 \rceil$ days. Hence, on average the proportion of days the child will eat two candies is $v_{a,b}/\lceil (a + b)/2 \rceil - 1$. These values are shown in Table 10.

Table 10. Suppose that the child must share a candy of the same type with his sister. Expected number of candies the child will eat and expected proportion of days the child will eat two candies

(a, b)	(7, 7)	(14, 14)	(30, 30)	$(10^2, 10^2)$	$(10^3, 10^3)$
expected # days candies last	10.77	21.26	45.25	150.25	1500.25
proportion of days eat two candies	0.538	0.519	0.508	0.50251	0.50025

The long-run average number of candies the child eats per day is slightly lower if he must give the candy of the same type to his sister than if he were to return it to the container. Should he maximize his own candy consumption by following Mom's original instruction to put back in the container the second candy of the same type, or should he prefer the joy of sharing with his sister? Which optimization criteria will take precedence for Johnny?

In the long-run as $a = b$ increases *ad infinitum*, by giving the second like candy to his sister Johnny will eat two candies on about 50% of days and his sister will eat a candy on the remaining 50% of days. Again, this limiting result makes intuitive sense because on Day 1 the child is almost equally likely to draw candies of opposite types as of the same type. Also, because $a = b$, the days when Johnny and his sister eat one candy each, it is equally likely to be of type A or type B.

Variation 2: Days passed; Johnny's craving for candy did not subside. One day he asked his mom: "Can I draw three candies at random? I promise I will eat only one candy of each type I draw and I will return the rest to the container." If Mom approves Johnny's proposal, he will eat one type A candy if he draws AAA, one type B candy if he draws BBB, or he will eat one candy of each type if he draws AAB or ABB, with associated probabilities (from hypergeometric distribution)

$$P_{(a,b)}(AAA) = \frac{\binom{a}{3}}{\binom{a+b}{3}}$$

$$P_{(a,b)}(BBB) = \frac{\binom{b}{3}}{\binom{a+b}{3}}$$

$$P_{(a,b)}(AAA \text{ or } BBB) = \frac{\binom{a}{2}\cdot b + a\cdot\binom{b}{2}}{\binom{a+b}{3}} = \frac{\frac{ab(a+b-2)}{2}}{\binom{a+b}{3}} = \frac{3ab}{(a+b)(a+b-1)}$$

Of course, towards the end if only one candy is left in the container, the child will eat it the next day; and if two candies are left in the container and they are of different types, the child will eat them both on the same day; but if they are of the same type, the child will eat the two candies on two successive days.

We leave it to the reader to verify that if the child is allowed to draw $n = 3$ candies (or less if fewer candies remain in the container), then the expected number of days to empty a container originally filled with 7 candies each of two types is 7.68. Hence, the expected proportion of days the child will eat two candies is $14/7.68 - 1 = 0.823$.

Mom was not thrilled with the prospect of letting her son eat two candies on such a high percentage of days. Eventually, she relented and made a small concession: "Okay, I will give you a choice. To begin, draw two candies. If they are of different types eat them both. But if they are of the same type, either you eat one and return the other, or you may choose to draw a third candy provided you agree to the following rule: If the third candy is of a different type, you may eat it together with one of the first two candies drawn, and return the other. But if the third candy is also of the same type as the first two, then you will eat *no candy at all*, and you must return all three candies to the container. Deal? Or, no deal?"

If Johnny has learned any optimization by now, he should accept the deal: First, note that he should draw the third candy only if the container has more candies of the opposite type than of the same type that he has already drawn; otherwise, he should forgo his option to draw a third candy. In particular, starting with equal number a of candies of each type, Johnny should draw a third candy whenever he draws two candies of the same type, for in the container there are two more candies of the opposite type than the same type he has already drawn. Then the probability that he will eat two candies is

$$q_{a,a} = \frac{a^2 + 2\cdot\binom{a}{2}\frac{a}{2a-2}}{\binom{2a}{2}} = \frac{3a}{4a-2}$$

whenever $a \geq 2$. Of course, $q_{1,1} = 1$. The probability that he eats no candy is $(1 - q_{a,a})$. Therefore, starting with (a, a) candies, the number of days until he eats two candies is a geometric($q_{a,a}$) random variable, with mean $1/q_{a,a} = 4/3 - 2/(3a)$. Moreover, whether he eats two candies or none, the container still has an equal number of candies of each type. Hence, on the next day (and the next, and indeed *always*) he should exercise his option to draw the third candy whenever he draws two candies of the same type. With such a smart policy of *essentially* drawing three candies each day, he will finish eating all candies *on average* in

$$\lambda_{a,a} = \sum_{i=1}^{a} \frac{1}{q_{i,i}} = 1 + \sum_{i=2}^{a} \left\{\frac{4}{3} - \frac{2}{3i}\right\} = \frac{4a + 1 - 2H_a}{3}$$

days. Furthermore, he will eat two candies on $2a/\lambda_{a,a} - 1$ proportion of days. These values are shown in Table 11.

(a, b)	(7, 7)	(14, 14)	(30, 30)	(10^2, 10^2)	(10^3, 10^3)
expected # days candies last	7.94	16.83	37.67	130.21	1328.68
proportion of days eat two candies	0.764	0.663	0.593	0.53600	0.50526

Table 11. Suppose that the child always exercises the option to draw a third candy whenever the first two are of the same type. Then the expected number of days candies will last and the expected proportion of days the child will eat two candies

Thus, it turns out that Mom's small concession prompts the child to always draw essentially three candies each day; and it has not achieved her desire to significantly lower the proportion of days the child will eat two candies, at least for small values of a. Maybe Mom should modify her offer by adding: "Furthermore, you will forfeit drawing candies the following day." We leave it to the reader to work out the optimal strategy for the child under this further modification.

Variation 3. Interested readers may read the paper [3] which solved a different problem studying the evolution of a pill bottle that initially contains n long pills. Each day one pill is chosen at random: If it is a long pill, then it is split into halves, one half is eaten and the other half is returned to the bottle. If the selected pill is a half pill, then it is eaten and nothing is returned to the bottle. Assuming that all pills in the bottle are equally likely to be chosen (irrespective of size), the expected number of whole pills selected before the first half pill is chosen is

$$\frac{e^n}{n^{n-1}} \int_n^{\infty} t^{n-1} e^{-t} dt$$

which is of order $\sqrt{\pi n/2}$ as $n \to \infty$, and the expected number of half pills left in the bottle after the last whole pill is selected is the harmonic sum

$$H_n = 1 + 1/2 + 1/3, \ldots, 1/n.$$

On the other hand, assuming that whole pills are twice as likely to be chosen as half pills, the expected number of whole pills selected before the first half pill is selected equals the expected number of half pills left in the bottle after the last whole pill is selected, and both equal $2^{2n}/\binom{2n}{n} - 1$.

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