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Goodness of Fit Testing for the Log-logistic Distribution Based on Type I Censored Data

Samah Ahmed
Qatar University

Ayman Baklizi
Qatar University

Reza Pakyari
Qatar University

ABSTRACT
A goodness of fit test procedure is proposed for the log-logistic distribution when the available data are subject to Type I censoring. The proposed test is based on transforming type 1 censored data into complete data from a suitably truncated distribution. A Monte Carlo power study is conducted to evaluate and compare the performance of the proposed method with the existing classical methods. An application based on a real dataset is considered for illustrative purposes.

Keywords: Goodness-of-fit testing. Empirical distribution function. log-logistic distribution. Type-I censoring.

1. Introduction

Right Censoring is the most common type of censoring and it is occurs when the survival time is incomplete at the right side of the follow up period. The most well-known kinds of right censoring are Type I, Type II and random censoring (Lawless, 2011). In Type I censoring, the censoring time c is assumed to be fixed. This type of censoring occurs when a study ends and the event of interest did not occur. The event is observed only if it is occurred before pre-specified time. In Type II censoring, when a specified number of events has occurred the study ends. Usually this type of censoring (Type II) is used in experiments that involve in testing lifetimes of equipments.

The Log-Logistic model is a continuous probability model defined on the non-negative real numbers. In the survival analysis, it is used as a parametric distribution for modeling time to occurrence of an event. It is well known that the Log-Logistic and Logistic distributions are equivalent statistical models. Any statistical technique developed for one distribution can be applied to the other distribution. If the random variable X follow Log-Logistic distribution, then $Y = log(X)$ is the Logistic random variable where $\mu = log(\alpha)$ and $\sigma = \frac{1}{\beta}$ (Lawless, 2011).

Based on the available literature, there is no study considered the problem of goodness of fit test for Log-Logistic distribution with type 1 censored data. However, there are several studies

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□ Ayman Baklizi (corresponding author, baklizi1@gmail.com), Samah Ahmed and Reza Pakyari are affiliated with the Statistics Program, Department of Mathematics, Statistics, and Physics, College of Arts and Sciences, Qatar University, Doha, Qatar.

focused on the goodness of fit tests (GOF) for the Logistic distribution using different methods. Meintanis (2004) investigated the GOF test for the Logistic distribution based on weighted integrals involving two methods of empirical transformations. Al-Subh et al. (2011) developed the GOF test for Logistic distribution based on Kullback-Leibler information. In addition, some authors investigated the problem of the GOF for the logistic distribution under Type I censoring. Bispo et al. (2012) studied the GOF test based on empirical distribution function (EDF) under Type I right censored samples for various lifetime models. Pakyari and Balakrishnan (2013) developed the GOF test for the Exponential model under Type I censored sample. The proposed method of Pakyari and Balakrishnan (2013) was based on considering the Type I censored sample of size (n) as order statistics from a complete sample of size $(d, d \leq n)$ from the exponential distribution with right truncation. Pakyari and Nia (2017) extended the method of Pakyari and Balakrishnan (2013) to the lognormal and Weibull distributions. In this paper, we extend the method of Pakyari and Balakrishnan (2013) to the log-logistic distribution under type I censored sample.

This paper is organized as follows, in sections 2 and 3, we present the classical and proposed goodness of fit tests procedure for the log-logistic model. Section 4, Monte Carlo simulations are used to study the performance of the proposed test and compare it with the classical tests in terms of power. Finally, in section 5, an example based on real data is presented.

2. The Classical goodness of fit tests

We are interested in testing the goodness of fit hypothesis that,

$$H_0 = (y, \mu, \sigma) = \frac{1}{1+exp\left(-\frac{y-\mu}{\sigma}\right)} \quad \text{vs.} \quad H_1: F(y, \mu, \sigma) \neq \frac{1}{1+exp\left(-\frac{y-\mu}{\sigma}\right)} \tag{1}$$

$$-\infty \leq y \leq \infty, -\infty \leq \mu \leq \infty. \ \sigma \geq 0,$$

where μ and σ are the unknown scale and shape parameters.

Let $c > 0$, be the censoring time and let $Y_1, \ldots, Y_d$ be the complete observations from a Type I censored sample of size n from the Logistic model. Let where $(\hat{\mu}, \hat{\sigma})$ are the MLE's of the Logistic distribution based on the type I censored data. Calculate

$$u_i = \frac{1}{1+exp\left(-\frac{y_i-\hat{\mu}}{\hat{\sigma}}\right)}, i = (1, \ldots, d), \tag{2}$$

The values $u_i, i = (1, \ldots, d)$ will be used to calculate the classical empirical distribution function statistics as described below.

Kolmogorov-Smirnov statistic is proposed by D'Agostino and Stephens (1986);

$$D_{n,p=} max_{1 \leq i \leq d} \left[max \left\{ \frac{i}{n} - u_i, u_i - \frac{i-1}{n} \right\} \right]. \tag{3}$$

Cramer-von Mises statistic is proposed by Pettitt and Stephens (1976);

$$W_{n,p}^2 = \sum_{i=1}^{d} (u_i - \frac{2i-1}{2n})^2 - \frac{d(4d-1)}{12n^2} + n\, u_d \left(\frac{d^2}{n^2} - u_d \frac{d}{n} + \frac{1}{3} u_d{}^2 \right) \tag{4}$$

Anderson–Darling statistic is proposed by Pettitt and Stephens (1976);

$$A_{n,p}^2 = \sum_{i=1}^{d}\left(\frac{2i-1}{n}\right)[log(1-u_i) - log(u_i)] - 2\sum_{i=1}^{d} log(1-u_i) + n\left[\frac{2d}{n} - \left(\frac{d}{n}\right)^2 - 1\right] log(1-u_d) + \frac{d^2}{n} log(u_d) - n\,u_d \quad (5)$$

where n is the size of Type I censored sample and d is the size of complete failure subjects and u_d is the value of the cdf of logistic distribution evaluated at c.

3. The Proposed goodness of fit tests

The main idea of the proposed method is based on the fact that, conditional on the number of failures in the type 1 censored sample, the failure times $Y_1, \dots, Y_d$ are order statistics arising from a sample of size d from the logistic distribution truncated at c; see Arnold et al. (1992) and David and Nagaraja (2003). Thus, the proposed test is based on considering the Type-I censored as order statistics from a complete sample of size d from the logistic distribution right-truncated at time c, and then transforming the order statistics to uniformity as will be explained below. The classical goodness of fit tests are then performed on the transformed data. See Pakyari and Balakrishnan (2013) for the exponential distribution case, and Pakyari and Nia (2017) for the Weibull and lognormal cases.

To test the null hypothesis (1) for the logistic distribution based on Type I censored sample $Y_{1:n}, \dots, Y_{d:n}$. We calculate the MLE's of (μ, σ), then we transform the Type I sample of the logistic distribution $Y_{1:n}, \dots, Y_{d:n}$ to order statistic from the uniform distribution by using the transformation

$$u_i = \frac{1+exp\left(-\frac{y_i-\hat{\mu}}{\hat{\sigma}}\right)}{1+e\quad\left(-\frac{c-\hat{\mu}}{\hat{\sigma}}\right)}, \text{ for } i = (1, \dots, d), \quad (6)$$

The EDF statistic are calculated from $u_1, u_2, \dots, u_d$. The null hypothesis will be rejected at significance level α if the test statistic exceeded the corresponding critical value (Pakyari and Balakrishnan, 2013). The proposed test statistics are:

Kolmogorov-Smirnov statistic Pakyari and Balakrishnan (2013);

$$D_d = max_{1\le i\le d}\left[max\left\{\frac{i}{d} - u_i, u_i - \frac{i-1}{d}\right\}\right], \quad (7)$$

Cramer-von Mises statistic Pakyari and Balakrishnan (2013);

$$W_d^2 = \sum_{i=1}^{d}(u_i - \frac{2i-1}{2d})^2 + \frac{1}{12d}, \quad (8)$$

Anderson-Darling statistic Pakyari and Balakrishnan (2013);

$$A_d^2 = -d - \frac{1}{d}\sum_{i=1}^{d}(2i-1)\{log(u_i) + log(1-u_{d+1-i})\}, \quad (9)$$

The EDF tests in equations (7, 8, and 9) are calculated based on u_i that obtained in equation (6). The following steps are used to determine the critical values of the proposed and classical tests statistics:

(1) Generate a Type I censored sample $Y_1, \dots, Y_n$ with a pre-chosen sample size n and termination time c from a standard logistic distribution ($\mu = 0, \sigma = 1$).
(2) Calculate the MLE's of ($\hat{\mu}, \hat{\sigma}$).
(3) For the classical tests calculate u_i as equation (2) and for the proposed test transform the sample using the order statistics from uniform distribution by using the transformation u_i as in equation (6).
(4) Calculate the classical EDF statistics using equations (3,4 and 5) and for calculate the proposed test statistics using equations (7,8, and 9). $u_i, i = 1, \dots, d$;
(5) Repeat Steps 1- 4 for a large number of times and determine the $(1-\alpha)^{th}$ quantile of the corresponding test statistic as the required critical value of that goodness of fit test.

In the next section, we assess the power of the proposed tests for the logistic distribution by means of a Monte Carlo simulation study based on various alternatives and different sample sizes.

4. Monte Carlo Simulation

In this section, we calculate the empirical significance level as well as the power of the proposed tests by means of Monte Carlo simulations. All the simulations were carried out in R using the pseudo-random generator in that software package. To calculate the empirical significance level of the proposed and classical tests, we generated 10,000 Type I censored random samples from the standard logistic distribution for different choices of sample sizes and observed proportion of failures $F(c)$ where $F(c)$ is the cdf of the logistic distribution. For evaluating the performance of the proposed and classical tests, we compare their power to those of the classical EDF statistics developed by Pettitt and Stephens (1976).

The powers of the tests are calculated by generating 10,000 Type I censored random samples from the alternative's distribution for different choices of sample sizes and observed proportion of failures $F(c)$ where $F(c)$ is the cdf of the alternative's distribution at c. The alternatives models that are considered for testing log-logistic representing monotone and non-monotone hazard function. The specific alternative models considered in this study are as follows.

The Gompertz distribution,

$$f(t, p, d) = d\ e^{pt} e^{\left[\frac{d}{p}(1-e^{pt})\right]}, \quad t > 0, p > 0, d > 0. \tag{10}$$

where $p > 0$ is the scale parameter and $d > 0$ is the shape parameter.

The Weibull distribution with density function

$$f(t, b, a) = abt^{b-1}\ e^{-at^b}, \quad t > 0, b > 0\ a > 0. \tag{11}$$

where $b > 0$ is the scale parameter and $a > 0$ is the shape parameter.

The Burr X distribution with density function

$$f(t, \nu, \theta) = \frac{2\nu t}{\theta^2}\left(e^{-\left(\frac{t}{\theta}\right)^2}\right)\left(1 - e^{-\left(\frac{t}{\theta}\right)^2}\right)^{\nu-1}, \quad t > 0, \nu, \theta > 0. \tag{12}$$

Where $\nu > 0$ is the scale parameter and $\theta > 0$ is the shape parameter.

The Exponential distribution with density function

$$f(t,\lambda) = \lambda\, e^{-\lambda t}, \quad t > 0\,, \lambda > 0. \tag{13}$$

where $\lambda > 0$ is the scale parameter.

For comparative purposes, the classical EDF statistics for the case of Type-I censored samples were also calculated from formulas (3)–(5) and for the proposed tests using the formulas from (7)–(9). Table 1 presents the empirical significance level at 10% nominal level. The values in the table reveal that the proposed tests maintain the level of significance at the nominal level and compete favorably with the classical tests considered here.

From Tables 2-4, the results show that the power values of the proposed and classical tests increase when the sample size and the proportion of failure increases. It is observed that in most of the cases the proposed tests outperform the classical tests. However, in some cases the classical KS and W tests have shown slightly higher power than the corresponding proposed tests at small and moderate proportions of failure. This suggests that the proposed KS and W test appears to lose in power due to the transformation performed in the samples. In addition, the results show that under different censoring conditions, the Anderson–Darling and Cramer–von Mises statistics for both proposed and classical methods show higher power levels than the Kolmogorov–Smirnov test. Thus, it seems advisable to use these two statistics when working with Type-I right-censored data.

Table 1. Estimated Empirical level for EDF tests at nominal level α = 0.10

				$F(c) = \left(1+\exp\left(-\frac{c-\mu}{\sigma}\right)\right)^{-1}$		
α	N	Method	Test statistics	0.40	0.60	0.80
0.10	35	Proposed	KS	0.098	0.097	0.101
			W	0.096	0.093	0.099
			AD	0.097	0.091	0.102
		Classical	KS	0.096	0.095	0.097
			W	0.093	0.093	0.101
			AD	0.097	0.093	0.099
	60	Proposed	KS	0.095	0.095	0.099
			W	0.096	0.098	0.103
			AD	0.093	0.103	0.102
		Classical	KS	0.094	0.093	0.094
			W	0.089	0.099	0.103
			AD	0.098	0.095	0.104
	90	Proposed	KS	0.102	0.103	0.098
			W	0.096	0.100	0.099
			AD	0.097	0.101	0.103
		Classical	KS	0.099	0.100	0.101
			W	0.098	0.105	0.092
			AD	0.100	0.095	0.098

Table 2. Estimated power for Exponential distribution with different sizes and proportion of failures F(c) at nominal level 0.10, n=35

			Proportion of Failures		
Alternative Model	**Method**	**Test statistic**	**0.40**	**0.60**	**0.80**
Gompertz (0.5, 1)	Proposed	KS	0.103	0.156	0.342
		W	0.113	0.180	0.393
		AD	0.127	0.203	0.418
	Classical	KS	0.115	0.149	0.234
		W	0.119	0.165	0.272
		AD	0.119	0.166	0.275
Weibull (1, 0.5)	Proposed	KS	0.094	0.121	0.228
		W	0.100	0.134	0.258
		AD	0.113	0.152	0.275
	Classical	KS	0.108	0.127	0.168
		W	0.108	0.122	0.162
		AD	0.110	0.139	0.191
Burr (0.7, 0.4)	Proposed	KS	0.111	0.194	0.429
		W	0.124	0.231	0.492
		AD	0.124	0.257	0.516
	Classical	KS	0.121	0.180	0.296
		W	0.129	0.213	0.349
		AD	0.128	0.208	0.355
Exponential (0.6)	Proposed	KS	0.097	0.121	0.227
		W	0.101	0.133	0.257
		AD	0.114	0.153	0.275
	Classical	KS	0.107	0.127	0.170
		W	0.106	0.121	0.160
		AD	0.108	0.138	0.193

Table 3. Estimated power for Exponential distribution with different sizes and proportion of failures F(c) at nominal level 0.10, n=60

			Proportion of Failures		
Alternative Model	**Method**	**Test statistic**	**0.40**	**0.60**	**0.80**
Gompertz (0.5, 1)	Proposed	KS	0.121	0.222	0.500
		W	0.134	0.262	0.572
		AD	0.146	0.281	0.579
	Classical	KS	0.118	0.183	0.326
		W	0.133	0.259	0.531
		AD	0.140	0.224	0.419
Weibull (1, 0.5)	Proposed	KS	0.107	0.157	0.317
		W	0.116	0.181	0.370
		AD	0.125	0.196	0.383
	Classical	KS	0.107	0.146	0.216
		W	0.114	0.180	0.316
		AD	0.122	0.169	0.268
Burr (0.7, 0.4)	Proposed	KS	0.139	0.290	0.623
		W	0.158	0.339	0.691
		AD	0.173	0.362	0.693
	Classical	KS	0.135	0.233	0.423
		W	0.156	0.340	0.669
		AD	0.157	0.287	0.529
Exponential (0.6)	Proposed	KS	0.109	0.157	0.318
		W	0.114	0.180	0.377
		AD	0.123	0.196	0.387
	Classical	KS	0.105	0.147	0.221
		W	0.110	0.178	0.298
		AD	0.119	0.170	0.282

Table 4. Estimated power for Exponential distribution with different sizes and proportion of failures F(c) at nominal level 0.10, n=90

Alternative Model	Method	Test statistic	Proportion of Failures		
			0.40	**0.60**	**0.80**
Gompertz (0.5, 1)	Proposed	KS	0.137	0.297	0.653
		W	0.150	0.336	0.722
		AD	0.167	0.355	0.727
	Classical	KS	0.138	0.232	0.450
		W	0.159	0.348	0.727
		AD	0.153	0.275	0.544
Weibull (1, 0.5)	Proposed	KS	0.114	0.201	0.423
		W	0.125	0.224	0.483
		AD	0.136	0.236	0.488
	Classical	KS	0.123	0.175	0.279
		W	0.130	0.228	0.461
		AD	0.127	0.196	0.351
Burr (0.7, 0.4)	Proposed	KS	0.166	0.392	0.781
		W	0.185	0.458	0.842
		AD	0.206	0.474	0.845
	Classical	KS	0.161	0.308	0.582
		W	0.196	0.473	0.850
		AD	0.184	0.377	0.688
Exponential (0.6)	Proposed	KS	0.117	0.201	0.426
		W	0.123	0.224	0.494
		AD	0.138	0.238	0.503
	Classical	KS	0.119	0.176	0.290
		W	0.125	0.176	0.453
		AD	0.123	0.197	0.368

5. Illustrative example

In this section, real data applications under Type I censored sample are considered. In order to see whether a given sample follows a Log-Logistic distribution by applying the proposed and classical methods. The data are times of breakdown of insulation fluid samples (in minutes) tested at 32 kV, see Table 5. Then, six observations are censored, an asterisk is used to mark the censored observations. This data has n=15, d=9 (complete failure observations) with 0.60 proportion of failure and censoring time or termination time c=27. Therefore, we are interested to test whether the times to breakdown follow the loglogistic distribution. To carry through, the parameters of the loglogistic are estimated using the MLE ($\hat{\alpha} = 11.957943$, $\hat{\beta} = 0.642404$), then the MLE's of Logistic distribution parameters are obtained ($\hat{\mu} = 2.481396$, $\hat{\sigma}$ =1.556653), these MLE's ($\hat{\mu}$, $\hat{\sigma}$) are used to calculate the value u_i for the proposed as equation (6) and for the classical as in equation (2) which are used to calculate the EDF statistics (KS,W, AD) for the proposed method follow equations (7,8, and 9) and for the classical method follow equations (3,4, and 5). Table 6 presents the EDF statistics with corresponding p-values and critical points for times to breakdown of an insulating fluid.

From Table 6 by comparing the proposed and classical tests statistics with the corresponding critical points, it appears that all values of the proposed and classical tests statistics less than the corresponding critical points. As well as, all the proposed and classical p-values are greater than the significant level $\alpha = 0.05$. This implies that, the Log-Logistic distribution have a good fit for the data and hence the sample follows the Log-Logistic model.

Table 5. Times to breakdown in minutes of an insulating fluid at 32 kV voltage level

0.27	0.40	0.69	0.79	2.75	3.91	9.88	13.95	15.93	27.80*
53.24*	82.85*	89.29*	100.58*	215.10*					

Table 6. EDF statistics with corresponding p-values and critical points for times to breakdown of an insulating fluid.

	Proposed method			**Classical method**		
	KS	W	AD	KS	W	AD
Test Statistics	0.20775	0.05822	0.38320	0.11805	0.01045	0.14295
Critical points	0.35577	0.20946	1.24055	0.17115	0.03885	0.34339
P-value	0.5628	0.6387	0.6479	0.4601	0.3875	0.4733

6. Conclusion

In this paper, goodness of fit tests were developed for the log-logistic distribution under Type I censored sample. The results revealed that the proposed method outperforms the classical method in most of the cases. The power values of the two methods increased when the sample size and the proportion of failure increased. Moreover, under various censoring conditions the AD and W statistics for both proposed and classical methods displayed higher power than the KS test. Hence, AD and W statistics are recommended over the KS test when working with Type I censored data.

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