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# A Modified Weighted Rayleigh Distribution and Its Bivariate Extension

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## ABSTRACT

In this paper, a new version of weighted Rayleigh distribution is constructed and studied. The statistical properties of the new distribution including the behavior of hazard and reversed hazard functions, moments, the central moments, moment generating function, mean, variance, coefficient of skewness, coefficient of kurtosis, median, mode, quantiles, stochastic ordering, exact information matrix and order statistics are also obtained, a simulation study and real data applications are performed. Furthermore, a bivariate extension of the new distribution called the bivariate modified Rayleigh (BMWR) distribution is introduced. The proposed bivariate distribution is of type Farlie–Gumbel–Morgenstern (FGM) copula. The BMWR distribution has modified weighted Rayleigh marginal distributions. The joint cumulative distribution function, the joint survival function, the joint probability density function, the joint hazard rate function and the statistical properties of the BMWR distribution are also derived.

***Keywords:*** Weighted Rayleigh distribution, maximum likelihood estimation, FGM copula, order statistic, moments; joint probability density function; joint hazard rate function.

## 1. Introduction

The principle of weighted distributions gives a collective entry for the problem of model specification and data interpretation. It presents a way for fitting models to the unknown weight function when samples may be taken both from the original distribution and the developed distribution. The weighted distributions arise frequently within the research related to the analysis of family data, reliability, analysis of intervention data, Meta-analysis, biomedicine, ecology, and other regions, for the improvement of the right statistical model. The concept of weighted distributions was provided by Fisher (1934) and Rao (1965).

To introduce the concept of weighted distribution, suppose $X$ is a non-negative random variable with pdf $(x)$ . The pdf of the weighted random variable $X_w$ denoted by $f_w(x)$is given by

$$f_w(x) = \frac{f(x).w(x)}{E[w(x)]}.$$

Where $E[w(x)] = \int_{-\infty}^{\infty} w(x)f(x)dx$ , and $w(x)$ be a non-negative weight function.

Note that $E[w(x)]$ is the normalizing factor obtained to make the total probability equal to unity by choosing $[w(x)] > 0$ , the random variable $X_w$ called the weighted version of $X$ , and its distribution is related to that of $X$ and is called the weighted distribution with weight function $w(x)$.

Note that the weight function $w(x)$ need not lie between zero and one, and actually may

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exceed unity. The use of weighted distributions as a tool in the selection of suitable models for observed data depends on the choice of the weight function.

A copula function is a convenient way to describe bivariate distributions. Copulas are of interest to statisticians for two main reasons: Firstly, as a way of studying scale-free measures of dependence, and secondly, as a starting point for constructing families of bivariate distributions.
Farlie–Gumbel–Morgenstern (FGM) copula, is a one of the most popular parametric families of copulas where $-1 < \theta < 1$ , that is defined by the following cumulative function

$$C(u, v) = uv(1 + \theta(1 - u)(1 - v))$$

and the density function

$$c(u, v) = 1 + \theta(1 - 2u)(1 - 2v)$$

Rayleigh distribution (RD) has been used in many areas of research, such as reliability, survival analysis and life testing. Modeling the lifetime of random phenomena has been every other vicinity of observation for which the Rayleigh distribution has been significantly used. Being first brought by Rayleigh (1880).
The Rayleigh distribution has the following pdf and cdf, respectively, with scale parameter $\sigma$ are given by,

$$f(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \quad ; x > 0 , \sigma > 0 . \tag{1}$$

$$F(x) = 1 - e^{-\frac{x^2}{2\sigma^2}} \quad ; x > 0 , \sigma > 0 . \tag{2}$$

## 2. The Transformation between the frequentist view and the Bayesian view

In this section, we construct the pdf of MWR distribution. Following Aleem et al (2013), a modified weighted version of Azzalini's (1985) approach was used as follows:
Let $f(x)$ be a pdf and $\bar{F}(x)$ be the corresponding survival (or reliability) function such as the cdf $F(x)$ exist.
The new weighted distribution is given by:

$$f_w(x) = \frac{[\bar{F}(\lambda x)]\, f(x)}{E[\bar{F}(\lambda x)]} \quad ,$$

$$f_w(x) = K\, f(x) \bar{F}(\lambda x) . \tag{3}$$

Where $f_w(x)$ the weighted probability distribution function, $\lambda$ is the scale parameter and K is the normalizing constant.

$$K = \frac{1}{E[\bar{F}(\lambda x)]}.$$

Now, make $\bar{F}(\lambda x)$ to represent the survival function of the Rayleigh distribution.

$$\bar{F}(x) = e^{-\frac{x^2}{2\sigma^2}} \quad , x > 0 , \sigma > 0 .$$

Then

$$\bar{F}(\lambda x) = e^{-\frac{\lambda^2}{2\sigma^2}x^2} \quad ; x > 0 , \sigma > 0 . \tag{4}$$

The normalizing constant.

$$E[\bar{F}(\lambda x)] = E[w(x)] = \frac{1}{(\lambda^2+1)}$$

Where $E[\bar{F}(\lambda x)] = \frac{1}{K}$

Then $K = (\lambda^2 + 1)$ (5)

Hence, The probability density function of modified weighted Rayleigh( MWR) distribution is given by substituting Eq. (1) , Eq. (4) and Eq. (5) into Eq. (3):

$$f_w(x) = \frac{(\lambda^2+1)}{\sigma^2}\, x\, e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2} \quad ; x > 0 , \sigma > 0 , \lambda \geq 0 . \tag{6}$$

The corresponding cumulative probability function of MWR distribution is given by:

$$F_w(x) = 1 - e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2} \tag{7}$$

Note that $\lim_{x\to\infty} F_w(x) = \lim_{x\to\infty} \left(1 - e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2}\right) = 1$

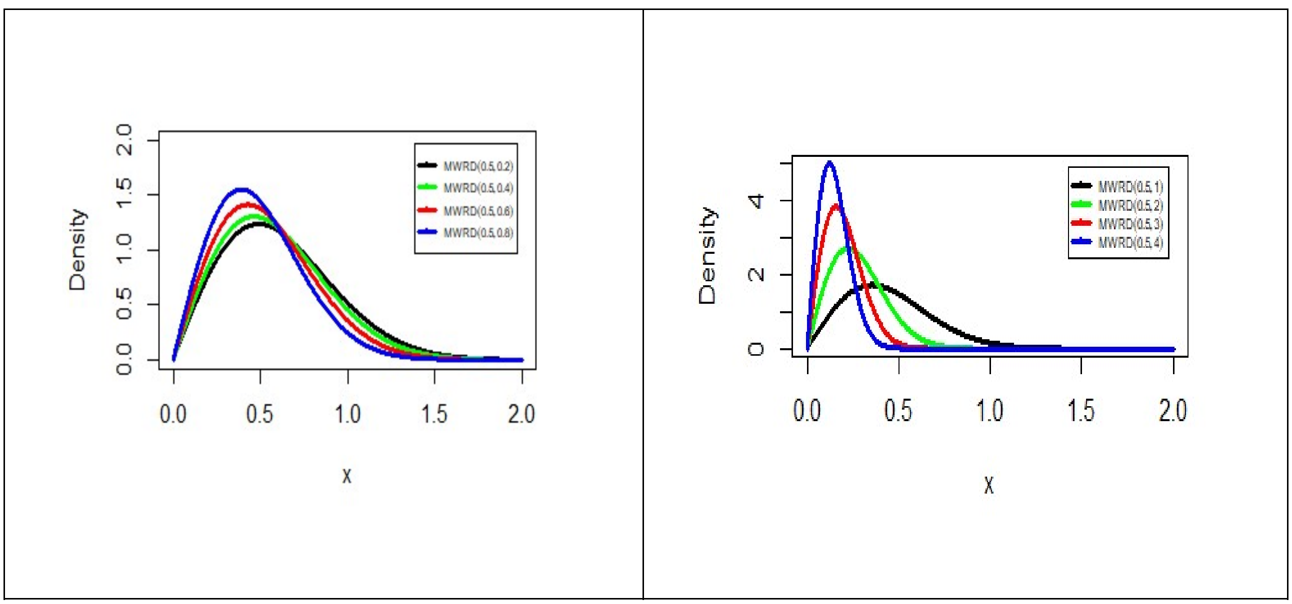



**Figure 1:** Probability Density Function of MWR $(\alpha, \lambda)$ Distribution.

The plots for the pdf of the MWR distribution in figure 1 indicate that as the increase of the value of additional parameter $\lambda$, the curve becomes more kurtosis (leptokurtic).

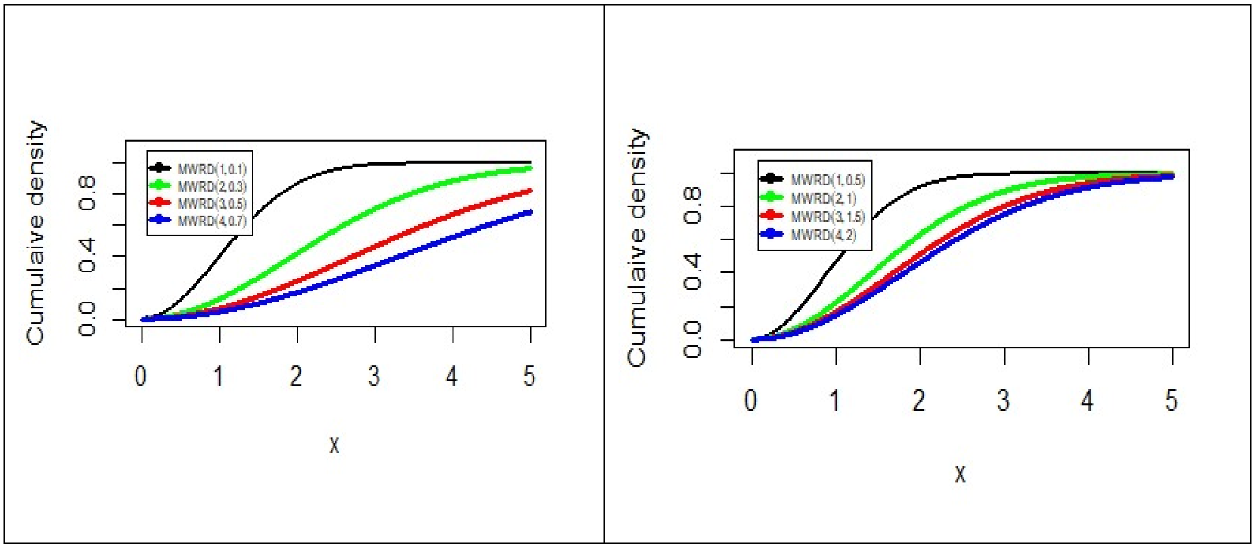



**Figure 2:** Cumulative Distribution Function of MWR$(\alpha, \lambda)$ Distribution.

The plots for the cdf of the MWR distribution in figure 2 indicate that as the increasing of the value of additional parameter $\lambda$, the curve is strictly increasing tends to one.

**Remark (1):**
When $\lambda = 0$ in Eq. (6), the modified weighted Rayleigh distribution reduces to give the Rayleigh distribution.

$$f(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} \quad ; x > 0 , \sigma > 0 .$$

**Remark (2):**
Suppose that $X$ follows MWR distribution $(\sigma, \lambda)$ and let $U = \frac{x^2}{2\sigma^2}$ , then $U$ follows a gamma distribution with parameters $(\lambda^2 + 1 , 1)$

**Remark (3):**
The MWR distribution $(\sigma, \lambda)$ distribution belongs to the exponential family. Therefore, $T = \sum_{i=0}^{n} X_i^2$ is a sufficient complete statistic, [see Ajami et al (2017)].

## 3. Statistical Properties of MWRD

We give some important statistical properties of the MWR distribution such as the $r^{th}$ moments, the central moments, the moment generating function, the quantile, the median, the mode, the reliability measures, the order statistic, and the stochastic order.

### 3.1. The $r^{th}$ Moments

The $r^{th}$ moment about the origin (i.e., moment about zero) of a continuous random variable

$X$ with density function $f(x)$ is given by:

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$$

Also, the moments about the origin are called non-central moments.

If $X$ is a random variable with density function Eq. (6), then The $r^{th}$ moment about the origin is given by

$$\mu'_r = E(X^r) = \Gamma\left(\frac{r}{2}+1\right)\frac{(2\,\sigma^2)^{\frac{r}{2}}}{(1+\lambda^2)^{\frac{r}{2}}} \tag{8}$$

Where $r$ is a positive integer.

The mean and other higher-order moments can be obtained from Eq. (8) as follows

i) The first moment about zero, $\mu = \frac{1}{2}\sqrt{\pi}\sqrt{\frac{2\,\sigma^2}{1+\lambda^2}}$

ii) The second moment about zero, $\mu'_2 = \frac{2\,\sigma^2}{1+\lambda^2}$

iii) The third moment about zero, $\mu'_3 = \frac{3}{4}\sqrt{\pi}\left(\frac{2\,\sigma^2}{1+\lambda^2}\right)^{\frac{3}{2}}$

iv) The fourth moment about zero, $\mu'_4 = 2\left(\frac{2\,\sigma^2}{1+\lambda^2}\right)^2$

v) The variance of the MWR distribution can be derived through the relation,

$$var(X) = \frac{\sigma^2\,(4-\pi)}{2(\,1+\lambda^2\,)}$$

vi) The coefficient of variation

$$CV(x) = \frac{the\ standard\ deviation}{the\ mean} = \sqrt{\frac{4-\pi}{\pi}} = 0.523$$

The CV of the MWR distribution is the same value as the Rayleigh distribution which is 0.523, this means that the standard deviation is 52% of the size of the mean.

## 3.2. Central Moments

Using the relationship between the central moments (the moment about the mean) and the noncentral moments (the moment about zero).

$$\mu_r = \sum_{j=0}^{r}\binom{r}{j}(-1)^{r-j}\,\mu'_j\,\mu^{r-j}.$$

Where $\mu_r$ is the $r^{th}$ central moment, $\mu$ is mean and $\mu'_j$ the $j^{th}$ noncentral moment.

i) The first four central moments of MWR distribution are obtained as:

$$\mu_1 = \frac{1}{2}\sqrt{\pi}\sqrt{\frac{2\,\sigma^2}{1+\lambda^2}}$$

$$\mu_2 = \frac{\sigma^2\,(4-\pi)}{2(\,1+\lambda^2\,)}$$

$$\mu_3 = \frac{(\sqrt{\pi^3}-3\sqrt{\pi})}{4}\left(\frac{2\,\sigma^2}{1+\lambda^2}\right)^{\frac{3}{2}}$$

and $$\mu_4 = \left(2-\frac{3}{16}\,\pi^2\right)\left(\frac{2\,\sigma^2}{1+\lambda^2}\right)^2$$

The coefficient of skewness of MWR distribution enables us to know whether the distribution is symmetric or not; it is defined by

$$\beta_1 = \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = 0.63$$

This means that is a skewed right distribution.

The coefficient of kurtosis of MWR distribution measures the flatness of the top, and it is defined by

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} = 3.25$$

This means that the distribution is leptokurtic, its tails are longer and fatter, and often its central peak is higher and sharper.

ii) The index of dispersion

$$D = \frac{the\ variance}{the\ mean} = \left(\frac{4-\pi}{2}\right)$$

## 3.3. Moment Generating Function

The moment generating function of the MWR distribution is given as follows

$$M_x(t) = \left[1 + \frac{\sqrt{\pi}\,\sigma^2\, t}{(\lambda^2+1)}\, e^{\frac{\sigma^2 t^2}{2(\lambda^2+1)}}\left[\text{erf}\left(\frac{\sigma^2\, t}{\lambda^2+1}\right) + 1\right]\right],$$

Where $\text{erf}(\varepsilon) = \frac{2}{\sqrt{\pi}} \int_0^{\varepsilon} e^{-x^2} dx$ , $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

## 3.4. Reliability Measures

The survival function of the modified weighted Rayleigh distribution is

$$\bar{F}_w(x) = e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2} \tag{9}$$

The hazard rate function computes the probability of a failure in the next instant given survival up to time . It is defined mathematically as:

$$h_w(x) = \frac{f_w(x)}{\bar{F}_w(x)}$$

The hazard function of the modified weighted Rayleigh distribution is given by :

$$h_w(x) = \frac{\frac{(\lambda^2+1)}{\sigma^2}x\, e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2}}{e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2}} = \frac{(\lambda^2+1)}{\sigma^2}\, x;\, \lambda \geq 0, \sigma > 0 \tag{10}$$

The reversed hazard rate of $X$ is defined by:

$$r_w(x) = \frac{f_w(x)}{F_w(x)} \quad ; \quad F_w(x) > 0.$$

The reversed hazard function of the new weighted Rayleigh distribution is given by:

$$r_w(x) = \frac{\frac{(\lambda^2+1)}{\sigma^2}x}{e^{\frac{(\lambda^2+1)}{2\sigma^2}x^2} - 1} = \frac{(\lambda^2+1)}{\sigma^2}\, x\left[e^{\frac{(\lambda^2+1)}{2\sigma^2}x^2} - 1\right]^{-1}.$$

(11)

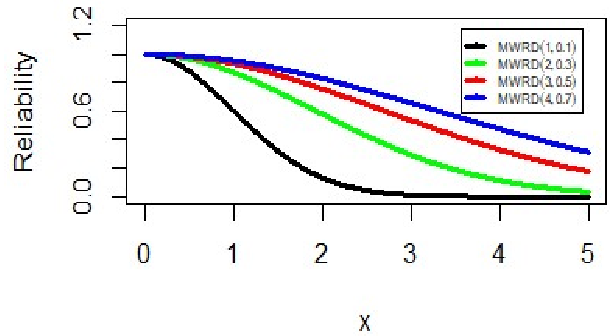



**Figure 3:** Reliability Function of MWR $(\alpha, \lambda)$ Distribution

Figure 3 shows the reliability behavior of the MWR distribution, It's clear that the reliability function decreases and approaches zero when $x \to \infty$, when the decrease in the value of additional parameter $\lambda$ leads to the curve decreases and approaches zero fastly.

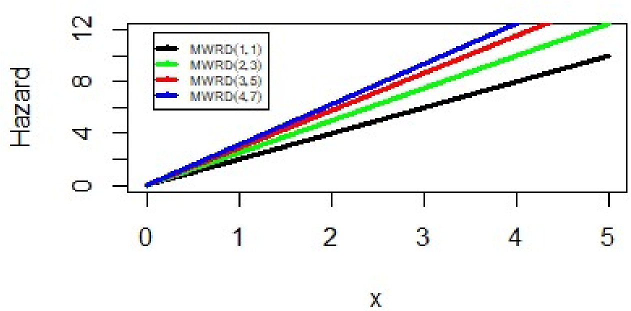



**Figure 4:** Hazard Function of MWR $(\alpha, \lambda)$ Distribution.

Figure 4 shows the hazard function of MWR distribution at various choices of $\lambda$. It's clear that the line increases with the increase in the value of parameter $\lambda$. The behavior of the hazard function is increasing function overall the value of the two parameters.

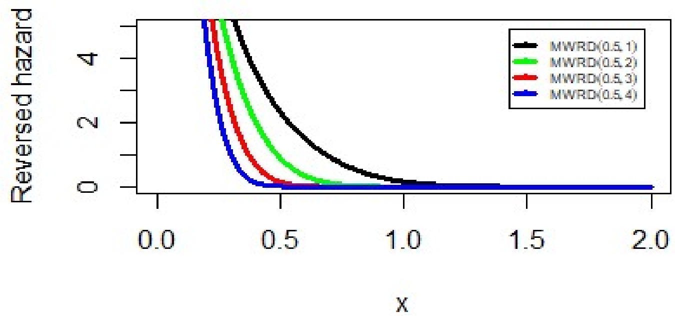



**Figure 5**: Reversed Hazard Function of MWR $(\alpha, \lambda)$ Distribution.

Figure 5 shows the reversed hazard function of MWR distribution at various choices of $\lambda$. It's clear that the reversed hazard function starting from a maximum value approaches zero after a long run.

## 3.5. Quantile, Median and Mode

The quantile $Q(u)$ of the modified weighted Rayleigh distribution is given by:

$$Q(u) = F_w^{-1}(u)$$

$$Q(u) = \sqrt{\frac{2\sigma^2}{(\lambda^2+1)} \log\left(\frac{1}{1-u}\right)} \tag{12}$$

where $U$ has the uniform $U(0,1)$ distribution.

The median is obtained directly by substituting $u = 0.5$ in the quantile $Q(u)$ in Eq.(12).

The median of the modified weighted Rayleigh distribution is given by:

$$X_{med} = \sqrt{\frac{2\sigma^2}{(\lambda^2+1)} \log(2)}$$

The first quantile of the MWR distribution is given by:

$$X_{0.25} = \sqrt{\frac{2\sigma^2}{(\lambda^2+1)} \log\left(\frac{4}{3}\right)}$$

The third quantile of the MWR distribution is given by:

$$X_{0.75} = \sqrt{\frac{2\sigma^2}{(\lambda^2+1)} \log(4)}$$

The mode of the MWR distribution is given by:

$$X_{mode} = \sigma\sqrt{\frac{1}{(\lambda^2+1)}}$$

$f_w(x)$ is increasing when $x \in (0, X_{mode})$ and is decreasing when $x \in (X_{mode}, \infty)$.

## 3.6. Order Statistic

We derive the pdf of the $i^{th}$order statistic of the MWR distribution, also, the first (the smallest order statistic), the largest (the last order statistic), and the joint of order statistic are obtained.

Let $X_1, X_2, X_3, \ldots \ldots, X_n$ be a sample from modified weighted Rayleigh distribution with pdf in Eq. (6) and cdf in Eq.(7), respectively.
Let $X_{(1)} < X_{(2)} < X_{(3)} < \cdots < X_{(n)}$ be the order statistic of $n$ independent observation from an NWR distribution
Where $0 < F_w(x) < 1$ , over the support $0 \le x < \infty$.

i) The probability density function of the first-order statistic $g_1(x) = min\,(X_1, X_2, \ldots \ldots, X_n)$ of the MWR distribution is given by:
$g_1(x) = n\ [1 - F_w(x)]^{n-1}\ [f_w(x)]$

Substituting Eq. (6) and Eq. (7) in $g_1(x)$ , we get:

$$g_1(x) = \frac{n\,(\lambda^2+1)}{\sigma^2}\ x\ \left[e^{-\frac{n\,(\lambda^2+1)}{2\sigma^2}x^2}\right]$$

The cumulative distribution function of the smallest order statistic $G_1(x)$ of the MWR distribution is given by:

$$G_1(x) = 1 - \left[e^{-\frac{n\,(\lambda^2+1)}{2\sigma^2}x^2}\right]$$

over the support $0 \le x < \infty$.

ii) The probability density function of the largest order statistic $g_n(x) = max\,(X_1, X_2, \ldots \ldots, X_n)$ of the MWR distribution is given by:
$g_n(x) = n\ [F_w(x)]^{n-1}\ [f_w(x)]$

Substituting Eq. (6) and Eq. (7) in $g_n(x)$ ,we get:

$$g_n(x) = \frac{n\,(\lambda^2+1)}{\sigma^2}\ x \left[1 - e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2}\right]^{n-1} \left[e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2}\right]$$

The cumulative distribution function of the largest order statistic $G_n(x)$ of the MWR distribution is given by:

$$G_n(x) = \left[1 - e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2}\right]^n$$

over the support $0 \le x < \infty$.

iii) the probability density function of the $i^{th}$ order statistic of the MWR distribution is:

$$g_i(x) = \frac{n!}{(i-1)!(n-i)!}\ [F_w(x)]^{i-1}\ [1 - F_w(x)]^{n-i}\ [f_w(x)]$$

Substituting Eq. (6) and Eq. (7) in $g_i(x)$ , we get:

$$g_i(x) = K\,x\,e^{-\frac{(\lambda^2+1)\,(n-i+1)}{2\sigma^2}x^2}\left[1 - e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2}\right]^{i-1},$$

Where $K = \frac{n!}{(i-1)!(n-i)!} \cdot \frac{(\lambda^2+1)}{\sigma^2}$ , over the support $0 \le x < \infty$.

iv) The joint probability density function of $X_{(i)}$ and $X_{(r)}$ $(for\ i < r)$ of the MWR distribution is given by:

Let $X_{(i)} = X$ and $X_{(r)} = Y$
Then

$$g_{i,r}(x,y) = \frac{n!}{(i-1)!(r-i-1)!(n-r)!} [F_w(x)]^{i-1} [F_w(y) - F_w(x)]^{r-i-1} \cdot [1 - F_w(y)]^{n-r} f_w(x) \quad f_w(y)$$

Substituting Eq. (6) and Eq. (7) in $g_{i,r}(x,y)$ , we get:

$$g_{i,r}(x,y) = Q\, x\, y \left[e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2}\right]\left[e^{-\frac{(\lambda^2+1)(n-r+1)}{2\sigma^2}y^2}\right] \cdot \left[1 - e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2}\right]^{i-1} \left[e^{-\frac{(\lambda^2+1)}{2\sigma^2}(x^2+y^2)}\right]^{\boldsymbol{r-i-1}}$$

Where $Q = \frac{\boldsymbol{n}!}{(\boldsymbol{i-1})!(\boldsymbol{r-i-1})!(\boldsymbol{n-r})!} \cdot \left(\frac{(\lambda^2+1)}{\sigma^2}\right)^{\boldsymbol{2}}$
over the support $0 \le x < \infty$.

## 3.6. Stochastic Ordering

The stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A continuous random variable $X$ is said to be smaller than a continuous random variable $Y$ in the

i) stochastic ordering ( $X \le_{st} Y$ ) if $F_X(x) \ge F_Y(x)$ for all $x$
ii) hazard rate order ( $X \le_{hr} Y$ ) if $h_X(x) \ge h_Y(x)$ for all $x$
iii) likelihood ratio order ( $X \le_{lr} Y$ ) if $\frac{f_X(x)}{f_Y(x)}$ decreases in $x$

The following stochastic ordering relationships due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering for continuous distributions

$$X \le_{lr} Y \Rightarrow X \le_{hr} Y$$
$$\Downarrow$$
$$X \le_{st} Y$$

The MWR distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

**Theorem (1)**
Let $X \sim MWR\ (\sigma_1, \lambda_1)$ and $Y \sim MWR\ (\sigma_2, \lambda_2)$, Then the following results hold true

i) If $\sigma_1 = \sigma_2$ and $\lambda_1 > \lambda_2$ then $X \le_{lr} Y$ , $X \le_{hr} Y$ and $X \le_{st} Y$
ii) If $\sigma_1 < \sigma_2$ and $\lambda_1 = \lambda_2$ then $X \le_{lr} Y$ , $X \le_{hr} Y$ and $X \le_{st} Y$

Proof

i) $$\frac{f_X(x)}{f_Y(x)} = \frac{\frac{(\lambda_1{}^2+1)}{\sigma_1{}^2} x\, e^{-\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2}x^2}}{\frac{(\lambda_2{}^2+1)}{\sigma_2{}^2} x\, e^{-\frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}x^2}} = \frac{\sigma_2{}^2\,(\lambda_1{}^2+1)}{\sigma_1{}^2\,(\lambda_2{}^2+1)}\; e^{-\,x^2\left(\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2} - \frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}\right)} \quad , x > 0$$

Now,

$$\log\left(\frac{f_X(x)}{f_Y(x)}\right) = \log\left(\frac{\sigma_2{}^2\,(\lambda_1{}^2+1)}{\sigma_1{}^2\,(\lambda_2{}^2+1)}\right) - x^2\left(\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2} - \frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}\right)$$

$$\frac{d}{dx}\log\left(\frac{f_X(x)}{f_Y(x)}\right) = -2x\left(\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2} - \frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}\right)$$

For $\sigma_1 = \sigma_2$ and $\lambda_1 > \lambda_2$

$$\frac{d}{dx}\log\left(\frac{f_X(x)}{f_Y(x)}\right) = -\frac{x}{\sigma_1{}^2}\left((\lambda_1{}^2+1) - (\lambda_2{}^2+1)\right)$$

$$\frac{d}{dx}\log\left(\frac{f_X(x)}{f_Y(x)}\right) < 0$$

This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$ , and $X \leq_{st} Y$
Similarly (ii) can be easily verified.

## 4. Estimation of Unknown Parameters of MWR Distribution

In this section, the methods of moments (MOM) and maximum likelihood (MLE) are derived to estimate the two scale parameters $\sigma$ and $\lambda$ and we derive the observed and Fisher information matrix.

### 4.1. Moment Estimates

The method of moments (MOM) is a technique commonly used in the field of parameter estimation. Let $X_1, X_2, \ldots, X_n$ represent a set of data be an independent and identically distributed random variables, with probability density function $f_w(x)$ and cumulative distribution function $F_w(x)$ of $(x, \phi)$, $\phi = (\sigma, \lambda)$.
The $r^{th}$ population moment about the zero equal the $r^{th}$ sample moment of random variable $X$.

$$E(X^r) = \frac{1}{n}\sum_{i=1}^{n} x_i^r$$

To construct moment estimators $\hat{\phi}$ for the unknown parameter $\phi$ we solve the set of the following equation

$$E(X) = \frac{\sum_{i=1}^{n} x_i}{n} = \bar{X},\; E(X^2) = \frac{\sum_{i=1}^{n} x_i^2}{n}$$

Then $\hat{\sigma} = \sqrt{\frac{2\,(1+\hat{\lambda}^2)}{\pi}}\,\bar{X}$ ,and $\hat{\lambda} = \sqrt{\frac{2\;n\,\hat{\sigma}}{\sum_{i=1}^{n} x_i^2} - 1}\,.$

## 4.2. Maximum Likelihood Estimates (MLE):

Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample of size n from $MWR(x, \phi)$. The likelihood function for the vector of parameters $\phi = (\sigma, \lambda)$ can be written

$$L(\underline{x}\,;\, \sigma, \lambda) = \prod_{i=1}^{n} f_w(x_i; \sigma, \lambda)$$

$$= \left(\frac{1+\lambda^2}{\sigma^2}\right)^n \prod_{i=1}^{n} x_i \; e^{-\frac{(\lambda^2+1)}{2\sigma^2}\sum_{i=1}^{n} x_i^2}$$

The log-likelihood function is

$$\log\left(L(\underline{x}\,;\, \sigma, \lambda)\right) = n\log(1+\lambda^2) - 2n\log(\sigma) + \sum_{i=1}^{n}\log(x_i) - \frac{(\lambda^2+1)}{2\sigma^2}\sum_{i=1}^{n} x_i^2$$

Taking the first derivatives w.r.t $\sigma, \lambda$ and equaling o zero, we get

$$\frac{\partial \log\left(L(\underline{x}\,;\sigma,\lambda)\right)}{\partial\sigma} = \frac{-2n}{\sigma} + \frac{(\lambda^2+1)}{\sigma^3}\sum_{i=1}^{n} x_i^2$$

$$, \frac{\partial \,\text{lo}\,\left(L(\underline{x}\,;\sigma,\lambda)\right)}{\partial\sigma}\Big|_{\sigma=\hat{\sigma}} = 0$$

then $$\hat{\sigma} = \sqrt{\frac{(\hat{\lambda}^2+1)}{2n}\sum_{i=1}^{n} x_i^2}$$

$$\frac{\partial \log\left(L(\underline{x}\,;\sigma,\lambda)\right)}{\partial\lambda} = \frac{2n\,\lambda}{(\lambda^2+1)} - \frac{\lambda}{\sigma^2}\sum_{i=1}^{n} x_i^2$$

$$, \frac{\partial \log\left(L(\underline{x}\,;\sigma,\lambda)\right)}{\partial\lambda}\Big|_{\lambda=\hat{\lambda}} = \mathbf{0}$$

and $$\hat{\lambda} = \sqrt{\frac{2n\,\hat{\sigma}^2}{\sum_{i=1}^{n} x_i^2} - 1}\,.$$

## 4.3. Fisher Information Matrix FIM

The Fisher information matrix is an important method of the amount of information the sample data can provide about parameters, and it is used to construct the confidence intervals with the asymptotic normality of maximum likelihood estimators.

To determine the FIM, we have a matrix of second partial derivatives:

$$I = -E\begin{bmatrix} \dfrac{\partial^2 \log\left(L(\underline{x}\,;\, \sigma,\lambda)\right)}{\partial\sigma^2} & \dfrac{\partial^2 \log\left(L(\underline{x}\,;\, \sigma,\lambda)\right)}{\partial\,\lambda\partial\sigma} \\ \dfrac{\partial^2 \log\left(L(\underline{x}\,;\, \sigma,\lambda)\right)}{\partial\sigma\partial\,\lambda} & \dfrac{\partial^2 \log\left(L(\underline{x}\,;\, \sigma,\lambda)\right)}{\partial\lambda^2} \end{bmatrix}$$

For MWR distribution with two parameters, the elements in the Fisher information matrix are

$$\frac{\partial^2 \log\left(L(\underline{x}\,;\, \sigma,\lambda)\right)}{\partial\sigma^2} = \frac{2n}{\sigma^2} - \frac{3(\lambda^2+1)}{\sigma^4}\sum_{i=1}^{n} x_i^2$$

$$\frac{\partial^2 \log\left(L(\underline{x}\,;\, \sigma,\lambda)\right)}{\partial\,\lambda\partial\sigma} = \frac{2\,\lambda}{\sigma^3}\sum_{i=1}^{n} x_i^2$$

$$\frac{\partial^2 \log\left(L(\underline{x}\,;\, \sigma,\lambda)\right)}{\partial\lambda^2} = \frac{2n\,(1-\lambda^2)}{(\lambda^2+1)^2} - \frac{1}{\sigma^2}\sum_{i=1}^{n} x_i^2$$

Since $E[\sum_{i=1}^{n} x_i^2] = \sum_{i=1}^{n}[E(x_i^2)] = n\,E[X^2] = n\,[\frac{2\,\sigma^2}{(\lambda^2+1\,)}] = \frac{2n\,\sigma^2}{(\lambda^2+1\,)}$

Hence the exact Fisher information matrix can be written as

$$I_n(\sigma,\lambda) = \begin{bmatrix} \frac{4n}{\sigma^2} & \frac{-4n\lambda}{\sigma\,(\lambda^2+1)} \\ \frac{-4n}{\sigma\,(\lambda^2+1)} & \frac{4n\,\lambda^2}{(\lambda^2+1)^2} \end{bmatrix}$$

Note that $\det(I) = \frac{4n}{\sigma^2} . \frac{4n\,\lambda^2}{(\lambda^2+1)^2} - \left[\frac{-4n\lambda}{\sigma\,(\lambda^2+1)}\right]^2 = 0$

Then, it is impossible to find the inverse of the information matrix.

## 5. Numerical Illustration

### 5.1. Real Data Analysis

We analyze a real-life data set to show the applicability of MWR distribution, These applications will show the flexibility of the MWR distribution in modeling positive data, we use lifetime data set and compared it with Rayleigh (R) distribution, Inverse Rayleigh (IR) distribution, exponential distribution, exponentiated inverse Rayleigh distribution(EIR) distribution, exponential distribution, Lindley (L) distribution,and Weibull distribution.

We use the fitdstrplus R package to fit the distributions.

For the comparison of the distributions, the criteria used are -lnL, Akaike information criterion (AIC) by Akaike (1974), Bayesian information criterion (BIC) by Schwarz (1978), and Kolmogorov Smirnov (K-S) Statistic. AIC estimates the performance of a model while comparing it with other models. The distribution with smaller values of -lnL, AIC and BIC is considered the best distribution. The specifications of these criteria are as follows:

$$AIC = 2(K) - 2\ln L$$
$$BIC = k\ln(n) - 2\ln L$$

where k = number of estimated parameters in the distribution.
n = the total number of observations.
ln L = maximized log-likelihood of the distribution under consideration

Some Relevant Distributions

1) Rayleigh (R) Distribution

$$f(x) = \frac{x}{\sigma^2}\,e^{-\frac{x^2}{2\sigma^2}} \;\; ; x > 0\,, \sigma > 0\,.$$

2) Inverse Rayleigh(IR) distribution introduced by Voda (1972)

$$f(x) = \frac{2\,\sigma^2}{x^3}\,e^{-\left(\frac{\sigma}{x}\right)^2};\; x,\sigma > 0.$$

3) Exponentiated inverse Rayleigh (EIR)distribution, introduced by Rao and Sauda. (2019)

$$f(x) = \frac{2\,\alpha\,\sigma^2}{x^3}\left(1 - e^{-\left(\frac{\sigma}{x}\right)^2}\right)^{\alpha-1} e^{-\left(\frac{\sigma}{x}\right)^2};\; x \ge 0, \alpha, \sigma > 0.$$

4) Exponential distribution

$$f(x) = \lambda\,e^{-\lambda x}\,; \qquad x \ge 0, \lambda > 0\,.$$

5) Lindley (L)distribution

$$f(x) = \frac{\theta^2}{(\theta+1)}\,(1+x)\,e^{-\theta} \quad ; x,\theta > 0.$$

6) Weibull distribution

$$f(x) = \alpha\, \beta\, x^{\beta-1}\, e^{-\alpha x^{\beta}} \quad ; x, \alpha, \beta > 0.$$

**Data Set (deep-groove ball bearings data)**

The comparison is carried out by taking the following data set.
We use a data set of 23 fatigue life for deep-groove ball bearings in Table 1, compiled by American Standards Association and reported in Lieblein and Zelen (1956) to illustrate the applicability of our proposed new weighted Rayleigh distribution, The data set (given in Table 1 is positively skewed (skewness = 0.79 and kurtosis = 3.14) with mean value 72.22, median 67.80 and is unimodal (mode at 50).

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 17.88 | 28.92 | 33.00 | 41.52 | 42.12 | 45.60 | 48.48 | 51.84 | 51.96 | 54.12 |
| 55.56 | 67.80 | 68.64 | 68.64 | 68.88 | 84.12 | 93.12 | 98.64 | 105.12 | 105.84 |
| 127.92 | 128.04 | 173.40 | | | | | | | |

**Table 1:** Data set of 23 fatigue life for deep-groove ball bearings

From Table 2, We find that the MWR distribution model is reasonably a good fit at the lower tail of the distribution as well. However, IR distribution, EIR distribution, exponential, L distribution, and Weibull seem to be fitting the data in the middle of the distribution. We made our decision of finding the best-fitting model based on the smallest log-likelihood value and smallest AIC and BIC values.

| Distributions | MLE's | K-S statistic | P-value | AIC | BIC | -ln L |
|---|---|---|---|---|---|---|
| MWR | $\hat{\sigma} = 65.26$ <br> $\hat{\lambda} = 0.48$ | 0.115 | 0.91 | 232.54 | 234.81 | 114.27 |
| R | $\hat{\sigma} = 58.48$ | 0.115 | 0.92 | 230.54 | 231.68 | 114.27 |
| IR | $\hat{\sigma} = 47.6$ | 0.14 | 0.7 | 235.9 | 237.1 | 116.98 |
| EIR | $\hat{\sigma} = 47.5$ <br> $\hat{\alpha} = 0.99$ | 0.13 | 0.8 | 237.97 | 240.24 | 116.98 |
| Exponential | $\hat{\lambda} = 0.013$ | 0.29 | 0.03 | 245.8 | 247 | 121.94 |
| L | $\hat{\theta} = 0.03$ | 0.18 | 0.4 | 234.62 | 235.76 | 116.3 |
| Weibull | $\hat{\alpha} = 0.002$ <br> $\hat{\beta} = 1.42$ | 0.2 | 0.32 | 237.5 | 239.82 | 116.7 |

**Table 2.** (deep-groove ball bearings data) fitting distributions to find the best fit model.

Based on these criteria, MWR distribution fits the guinea pig data best. From the results of the datasets proposed MWR distribution performs the best distribution according to the other distributions.

Although the K-S statistic of the MWR distribution equals the original distribution, the p-value of the MWR distribution is less than the original distribution and the AIC and BIC of the original distribution are better than the MWR distribution. Hence, MWR distribution may be the best of the original distribution in some phenomena.

## 5.2. Simulation Study

A simulation study is led to evaluate the performance of maximum likelihood (ML) for estimating the unknown two parameters of MWR distribution. The performance of the different estimators is evaluated in terms of mean, mean square error (MSE), biased and standard errors (SEs). The simulation is conducted by using R- software, 10000 random samples of MWR distribution $(\sigma, \lambda)$ was generated with values of different sample size (n) as $n = (30,50,100,125,150)$

Where; mean $\bar{x} = \frac{1}{n}\sum_{i=0}^{n} x_i$ , mean square error MSE $(x) = \frac{\sum_i^n (x - \bar{x})^2}{n}$ , $Bias(x) = |\hat{\boldsymbol{x}} - \boldsymbol{x}|$ ,
Standard errors $SEs(x) = \frac{sd(x)}{\sqrt{n}}$ and Relative biased_RBs(x) $= \frac{bias(x)}{x}$

From Tables (3:8), Bias values and MSE values are computed to compare considered ML estimators, from the results, we can note that the biases and the MSEs decrease as the sample size n increases.
A mutation occurs when (*n = 50,100*) in baise and MSE values for some values of the parameters

| N | | Mean | MSE | Bais | SEs | RBs |
|---|---|---|---|---|---|---|
| 30 | $\hat{\sigma}$ | 0.0144 | 0.0073 | 0.0855 | 0.000027 | 0.85 |
| | $\hat{\lambda}$ | 0.5951 | 0.1112 | 0.0951 | 0.000031 | 0.19 |
| 50 | $\hat{\sigma}$ | 0.01376 | 0.00744 | 0.08623 | 0.000026 | 0.86 |
| | $\hat{\lambda}$ | 0.39957 | 0.11009 | 0.10042 | 0.000031 | 0.2 |
| 100 | $\hat{\sigma}$ | 0.01374 | 0.00744 | 0.08625 | 0.000016 | 0.86 |
| | $\hat{\lambda}$ | 0.46026 | 0.025349 | 0.03973 | 0.000015 | 0.07 |
| 125 | $\hat{\sigma}$ | 0.01374 | 0.007442 | 0.08625 | 0.000016 | 0.86 |
| | $\hat{\lambda}$ | 0.46026 | 0.025349 | 0.03973 | 0.000015 | 0.07 |
| 150 | $\hat{\sigma}$ | 0.01374 | 0.00744 | 0.08627 | 0.000016 | 0.86 |
| | $\hat{\lambda}$ | 0.4350 | 0.035438 | 0.06498 | 0.000017 | 0.12 |

**Table 3:** ML Estimation of the Parameters of MWR distribution $\sigma = 0.1 , \lambda = 0.5$

| N | | Mean | MSE | Bais | SEs | RBs |
|---|---|---|---|---|---|---|
| 30 | $\hat{\sigma}$ | 0.333 | 0.0330 | 0.167 | 0.000713 | 0.33 |
| | $\hat{\lambda}$ | 0.371 | 0.0725 | 0.129 | 0.002362 | 0.25 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 50 | $\hat{\sigma}$ | 0.436 | 0.0163 | 0.0637 | 0.00111 | 0.12 |
| | $\hat{\lambda}$ | 0.964 | 0.4696 | 0.4643 | 0.00504 | 0.92 |
| 100 | $\hat{\sigma}$ | 0.334 | 0.0305 | 0.166 | 0.000549 | 0.33 |
| | $\hat{\lambda}$ | 0.312 | 0.1115 | 0.188 | 0.002760 | 0.37 |
| 125 | $\hat{\sigma}$ | 0.342 | 0.0275 | 0.158 | 0.000494 | 0.31 |
| | $\hat{\lambda}$ | 0.394 | 0.0770 | 0.106 | 0.002562 | 0.21 |
| 150 | $\hat{\sigma}$ | 0.343 | 0.0266 | 0.1572 | 0.000436 | 0.31 |
| | $\hat{\lambda}$ | 0.413 | 0.0569 | 0.0866 | 0.002222 | 0.17 |

**Table 4:** ML Estimation of the Parameters of MWR distribution $\sigma = 0.5, \lambda = 0.5$

| N | | Mean | MSE | Bais | SEs | RBs |
|---|---|---|---|---|---|---|
| 30 | $\hat{\sigma}$ | 1.272 | 199.82 | 0.7722 | 0.1411 | 1.54 |
| | $\hat{\lambda}$ | 0.3530 | 236.76 | 0.6469 | 0.1537 | 0.64 |
| 50 | $\hat{\sigma}$ | 1.144 | 11.52 | 0.6441 | 0.0333 | 1.28 |
| | $\hat{\lambda}$ | 1.772 | 16.81 | 0.7723 | 0.0402 | 0.77 |
| 100 | $\hat{\sigma}$ | 0.635 | 0.1816 | 0.1358 | 0.0040 | 0.27 |
| | $\hat{\lambda}$ | 0.7193 | 0.4956 | 0.2806 | 0.0064 | 0.28 |
| 125 | $\hat{\sigma}$ | 0.588 | 0.0130 | 0.0887 | 0.0007 | 0.17 |
| | $\hat{\lambda}$ | 0.625 | 0.1656 | 0.3746 | 0.0015 | 0.37 |
| 150 | $\hat{\sigma}$ | 0.688 | 0.0473 | 0.1889 | 0.00107 | 0.37 |
| | $\hat{\lambda}$ | 0.935 | 0.0316 | 0.0640 | 0.00165 | 0.06 |

**Table 5:** ML Estimation of the Parameters of MWR distribution $\sigma = 0.5, \lambda = 1$

| N | | Mean | MSE | Bais | SEs | RBs |
|---|---|---|---|---|---|---|
| 30 | $\hat{\sigma}$ | 1.2216 | 30.2651 | 0.22164 | 0.054971 | 0.22 |
| | $\hat{\lambda}$ | 0.06039 | 8.11372 | 0.13960 | 0.028451 | 0.69 |
| 50 | $\hat{\sigma}$ | 1.13159 | 14.1145 | 0.13159 | 0.03754 | 0.13 |
| | $\hat{\lambda}$ | 0.12193 | 3.96248 | 0.07806 | 0.01989 | 0.39 |
| 100 | $\hat{\sigma}$ | 1.05146 | 0.01576 | 0.05146 | 0.00114 | 0.05 |
| | $\hat{\lambda}$ | 0.18113 | 0.00466 | 0.01886 | 0.000656 | 0.09 |
| 125 | $\hat{\sigma}$ | 1.05304 | 0.01293 | 0.05304 | 0.0010 | 0.05 |
| | $\hat{\lambda}$ | 0.18244 | 0.00177 | 0.01755 | 0.0003 | 0.08 |
| 150 | $\hat{\sigma}$ | 1.0519 | 0.01097 | 0.05197 | 0.000909 | 0.05 |
| | $\hat{\lambda}$ | 0.1766 | 0.00315 | 0.02333 | 0.00051 | 0.11 |

**Table 6:** ML Estimation of the Parameters of MWR distribution $\sigma = 1, \lambda = 0.2$

| N | | Mean | MSE | Bais | SEs | RBs |
|---|---|---|---|---|---|---|
| | $\hat{\sigma}$ | 0.0777 | 0.01524 | 0.1222 | 0.00017 | 0.61 |

| | | | | | | |
|---|---|---|---|---|---|---|
| 30 | $\hat{\lambda}$ | 0.6557 | 0.06024 | 0.1442 | 0.00198 | 0.18 |
| 50 | $\hat{\sigma}$ | 0.9449 | 0.665 | 0.7449 | 0.0033 | 3.7 |
| | $\hat{\lambda}$ | 14.651 | 230.548 | 13.851 | 0.0622 | 1.7 |
| 100 | $\hat{\sigma}$ | 0.0809 | 0.0142 | 0.1190 | 0.00008 | 0.59 |
| | $\hat{\lambda}$ | 0.7339 | 0.0053 | 0.0660 | 0.00003 | 0.08 |
| 125 | $\hat{\sigma}$ | 0.0736 | 0.0160 | 0.1263 | 0.0001 | 0.63 |
| | $\hat{\lambda}$ | 0.3507 | 0.3596 | 0.4492 | 0.0039 | 0.56 |
| 150 | $\hat{\sigma}$ | 0.0791 | 0.0147 | 0.1208 | 0.0001 | 0.60 |
| | $\hat{\lambda}$ | 0.6583 | 0.0761 | 0.1416 | 0.0023 | 0.17 |

**Table 7:** ML Estimation of the Parameters of MWR distribution $\sigma = 0.2 , \lambda = 0.8$

| N | | Mean | MSE | Bais | SEs | RBs |
|---|---|---|---|---|---|---|
| 30 | $\hat{\sigma}$ | 0.538 | 0.0165 | 0.0616 | 0.0011 | 0.10 |
| | $\hat{\lambda}$ | 0.482 | 0.0462 | 0.1170 | 0.0018 | 0.19 |
| 50 | $\hat{\sigma}$ | 0.556 | 0.0122 | 0.0439 | 0.0010 | 0.07 |
| | $\hat{\lambda}$ | 0.559 | 0.00961 | 0.0406 | 0.0008 | 0.06 |
| 100 | $\hat{\sigma}$ | 0.595 | 0.0175 | 0.0047 | 0.0013 | 0.07 |
| | $\hat{\lambda}$ | 0.569 | 0.180 | 0.0303 | 0.0042 | 0.05 |
| 125 | $\hat{\sigma}$ | 0.547 | 0.0078 | 0.0527 | 0.0007 | 0.08 |
| | $\hat{\lambda}$ | 0.483 | 0.0431 | 0.1160 | 0.0017 | 0.19 |
| 150 | $\hat{\sigma}$ | 0.550 | 0.0067 | 0.0499 | 0.0006 | 0.08 |
| | $\hat{\lambda}$ | 0.503 | 0.0328 | 0.0966 | 0.0015 | 0.16 |

**Table 8:** ML Estimation of the Parameters of MWR distribution $\sigma = 0.6 , \lambda = 0.6$

## 6. The Bivariate Modified Weighted Rayleigh Distribution (BMWR) Distribution

In this section, Farlie–Gumbel–Morgenstern (FGM) copula and the univariate modified weighted Rayleigh distribution are used for creating the bivariate distribution which is called the BMWR distribution. Let $(X , Y)$ be two-dimensional random variables and support $\mathcal{R}^2$ where $\mathcal{R}$ is the real number, we introduce the joint pdf $(x, y)$ , the joint cdf $F(x, y)$ , the marginal distributions of $X$ and $Y$ , the conditional distribution, and the joint survival function $\bar{F}(x, y)$.

### 6.1. FGMBMWR Distribution

Suppose $X$ and $Y$ distributed as MWR distribution $(\sigma_1, \lambda_1)$ and $(\sigma_2, \lambda_2)$ with the distribution function $F_1(x)$ and $F_2(y)$ respectively. Then the bivariate vector $(X, Y)$ has a bivariate MWR distribution with the scale parameters $\sigma_1, \sigma_2, \lambda_1 and\ \lambda_2$ , we will denote the bivariate MWR distribution by BMWR distribution$(\sigma_1, \sigma_2, \lambda_1, \lambda_2, \theta)$.

According to the Farlie-Gumble-Morgenstern (FGM), the joint distribution function and the joint probability distribution function of BMWR distribution are as follows respectively

$$F(x,y)=\left(1-\ e^{-\frac{({\lambda_1}^2+1)}{2{\sigma_1}^2}x^2}\right)\left(1-\ e^{-\frac{({\lambda_2}^2+1)}{2{\sigma_2}^2}y^2}\right)\left[1+\theta e^{-\frac{({\lambda_1}^2+1)}{2{\sigma_1}^2}x^2-\frac{({\lambda_2}^2+1)}{2{\sigma_2}^2}y^2}\right] \tag{12}$$

The pdf of BMWR distribution can be expressed as

$$f(x,y)=\left(\frac{({\lambda_1}^2+1)}{{\sigma_1}^2}\ x\,e^{-\frac{({\lambda_1}^2+1)}{2\,{\sigma_1}^2}x^2}\right)\left(\frac{({\lambda_2}^2+1)}{{\sigma_2}^2}\ y\,e^{-\frac{({\lambda_2}^2+1)}{2\,{\sigma_2}^2}y^2}\right)\left[1+\theta\left(2e^{-\frac{({\lambda_1}^2+1)}{2{\sigma_1}^2}x^2}-1\right)\left(2e^{-\frac{({\lambda_2}^2+1)}{2{\sigma_2}^2}y^2}-1\right)\right] \tag{13}$$

Where $\sigma_1,\sigma_2,\lambda_1,\lambda_2\geq 0\ ,-1\leq\theta\leq 1$

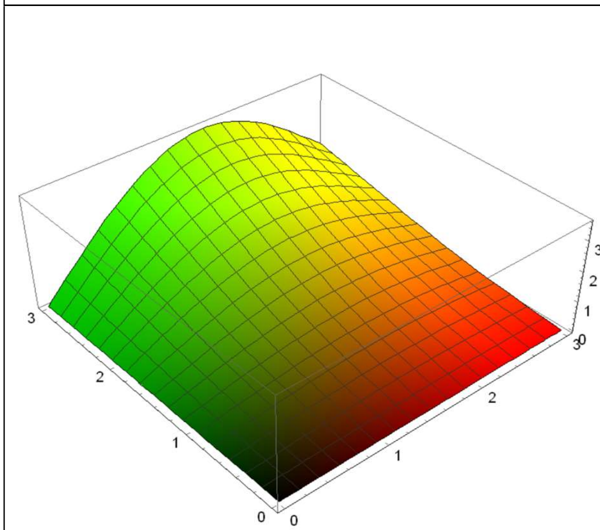
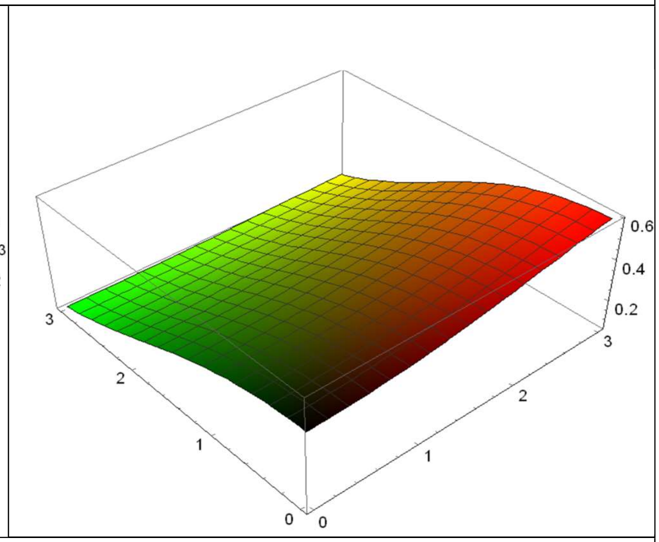
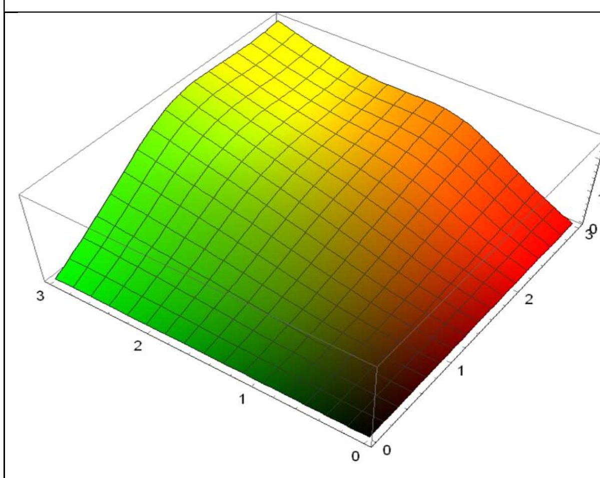
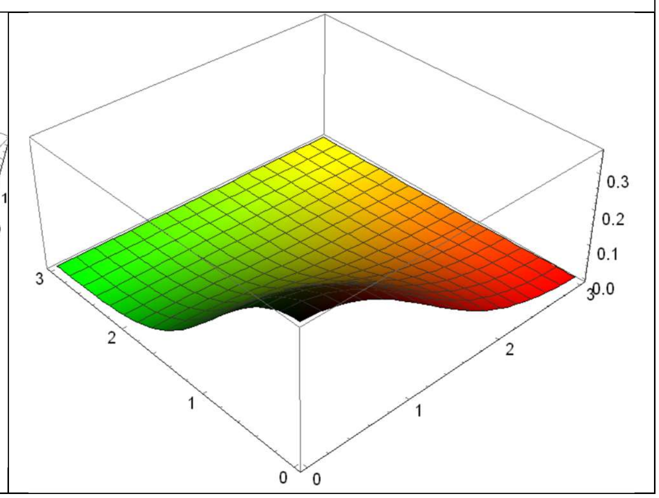
| Pdfs | Cdfs |
| --- | --- |
| **Set (1):** $\sigma_1=1.2\ ,\sigma_2=1.8\ ,$ | $\lambda_1=1.5\ ,\lambda_2=2\ ,\theta=0.25$ |
|  |  |
| **Set (2):** $\sigma_1=1\ ,\sigma_2=1.2\ ,$ | $\lambda_1=0.5\ ,\lambda_2=0.5\ ,\theta=-0.75$ |
|  |  |

**Figure 6:** Pdf and Cdf of FGM Bivariate Modified Weighted Rayleigh Distribution with The Parameters Values.

## 6.1. The Marginal Distribution

The marginal density of $(X\ ,Y)$ with BMWR distribution $(\sigma_1,\sigma_2,\lambda_1,\lambda_2,\theta)$ is given by

$$f(x,\sigma_1,\lambda_1) = \frac{({\lambda_1}^2+1)}{{\sigma_1}^2}\ x\ e^{-\frac{({\lambda_1}^2+1)}{2\,{\sigma_1}^2}x^2}\ ; x>0,\sigma_1,\lambda_1>0. \tag{14}$$

$$f(y,\sigma_2,\lambda_2) = \frac{({\lambda_2}^2+1)}{{\sigma_2}^2}\ y\ e^{-\frac{({\lambda_2}^2+1)}{2\,{\sigma_2}^2}y^2}\ ; y>0,\sigma_2,\lambda_2>0. \tag{15}$$

Respectively

## 6.2. The Conditional Distribution

$\forall y\ \in Y$ , The conditional probability distribution of $X$ given $Y=y$ is

$$f_{X\backslash Y}(x\setminus y) = c(x,y)\frac{({\lambda_1}^2+1)}{{\sigma_1}^2}\ x\ e^{-\frac{({\lambda_1}^2+1)}{2\,{\sigma_1}^2}x^2} \tag{16}$$

And the conditional cdf is

$$F_{X\backslash Y}(x\setminus y) = F_1(x)\big[1+\theta\big(1-F_1(x)\big)\big(1-2F_2(y)\big)\big] \tag{17}$$

Where $c(x,y) = 1+\ \theta(1-2F_1(x))(1-2F_2(y))$

## 6.3. The Moment Generating Function

If $(X,Y)\sim$ BMWR distribution $(\sigma_1,\sigma_2,\lambda_1,\lambda_2,\theta)$ with joint pdf in Eq. (13).

Then, the moment generating function of BMWR distribution is as follow:

$$\mu_{(x,y)}(t_1,t_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{{t_1}^n{t_2}^m}{n!\,m!}\ \frac{\Gamma\left(\frac{n}{2}+1\right)\ \Gamma\left(\frac{m}{2}+1\right)\ {\sigma_1}^n{\sigma_2}^m}{({\lambda_1}^2+1)^{\frac{n}{2}}\ ({\lambda_2}^2+1)^{\frac{m}{2}}}\ \left[\left(1-2^{\frac{n}{2}}\right)\left(1-2^{\frac{m}{2}}\right)\right]$$

## 6.4. The Product Moment

If the random vector $X=(X\ ,\ Y)$ is distributed as BMWR distribution, then it's the $r^{th}$ and $s^{th}$ moments about the zero is

$$\grave{\mu}_{r\,s} = \left(\frac{\sigma_1^2}{1+\lambda_1^2}\right)^{\frac{r}{2}}\left(\frac{\sigma_2^2}{1+\lambda_2^2}\right)^{\frac{s}{2}}\Gamma\left[\frac{r}{2}+1\right]\Gamma\left[\frac{s}{2}+1\right]\left[2^{\frac{r+s}{2}}(1+\theta)-\theta 2^{\frac{r}{2}}-\theta\ 2^{\frac{s}{2}}+\theta\right]$$

If the random vector $X=(X\ ,Y)$ is distributed as BMWR distribution, then the expectation of $x\ and\ \ y$ is

$$E(xy) = \sqrt{\frac{\pi\sigma_1^2\sigma_2^2}{(1+\lambda_1^2)(1+\lambda_2^2)}}\left[\frac{1}{2}+\frac{3}{4}\ \theta+\frac{3}{\sqrt{2}}\theta\right]$$

## 6.5. The Joint Hazard Rate Function

The expression of the joint survival function for copula FGM is as follows

$$\bar{F}(x,y) = \ \bar{C}\big(\bar{F}(x),\bar{F}(y)\big)$$

$$\bar{F}(x,y) = \bar{F}(x)+\bar{F}(y)-1+C(F(x),F(y))$$

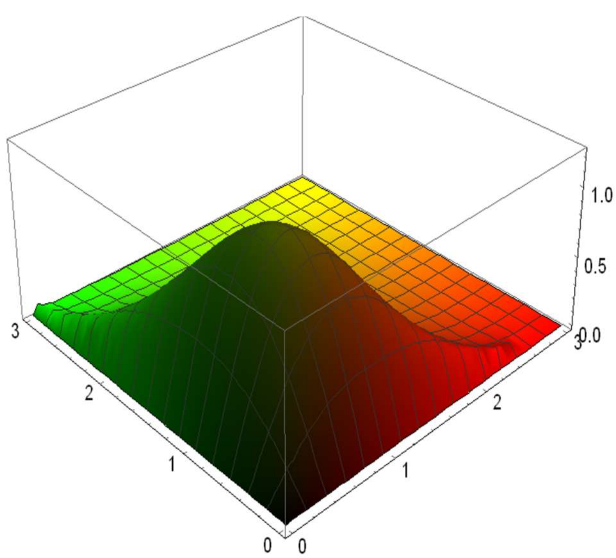
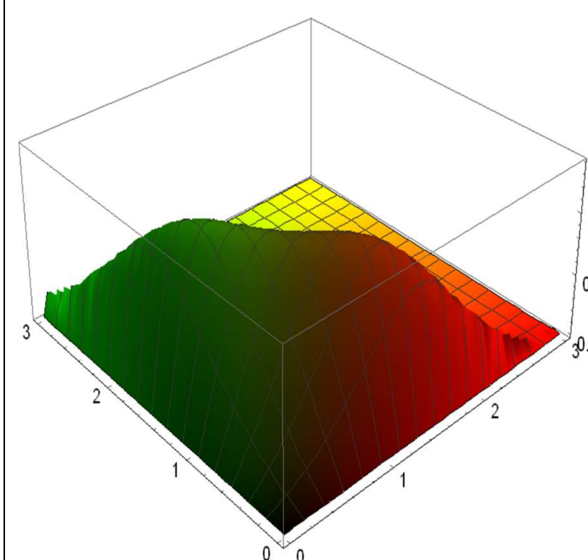
Then the joint reliability function of BMWR distribution is

$$\bar{F}(x,y)=\left(e^{-\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2}x^2}\right)\left(e^{-\frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}y^2}\right)-1+\left(1-e^{-\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2}x^2}\right)\left(1-e^{-\frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}y^2}\right)\left[1+\theta e^{-\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2}x^2-\frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}y^2}\right]$$

The joint hazard rate function is defined

$$h(x,y)=\frac{f(x,y)}{\bar{F}(x,y)}$$

$$h(x,y)=\frac{\left(\frac{(\lambda_1{}^2+1)}{\sigma_1{}^2}\,x\,e^{-\frac{(\lambda_1{}^2+1)}{2\,\sigma_1{}^2}x^2}\right)\left(\frac{(\lambda_2{}^2+1)}{\sigma_2{}^2}\,y\,e^{-\frac{(\lambda_2{}^2+1)}{2\,\sigma_2{}^2}y^2}\right)\left[1+\theta\left(2e^{-\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2}x^2}-1\right)\left(2e^{-\frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}y^2}-1\right)\right]}{\left(e^{-\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2}x^2}\right)\left(e^{-\frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}y^2}\right)-1+\left(1-e^{-\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2}x^2}\right)\left(1-e^{-\frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}y^2}\right)\left[1+\theta e^{-\frac{(\lambda_1{}^2+1)}{2\sigma_1{}^2}x^2-\frac{(\lambda_2{}^2+1)}{2\sigma_2{}^2}y^2}\right]}$$

| **Set (5):** $\sigma_1=1.2$, $\sigma_2=1.8$, $\lambda_1=1.5$, $\lambda_2=2$, $\theta=0.5$ | **Set (6):** $\sigma_1=1.2$, $\sigma_2=1.8$, $\lambda_1=1.5$, $\lambda_2=2$, $\theta=-0.5$ |
| --- | --- |
|  |  |

| **Set (7):** $\sigma_1 = 1.2\,, \sigma_2 = 1.4\,, \lambda_1 = 0.2\,,$ $\lambda_2 = 0.4\,, \theta = 0.25$ | **Set (8):** $\sigma_1 = 1.2\,, \sigma_2 = 1.4\,, \lambda_1 = 0.2$ $, \lambda_2 = 0.4\,, \theta = -0.25$ |
|---|---|

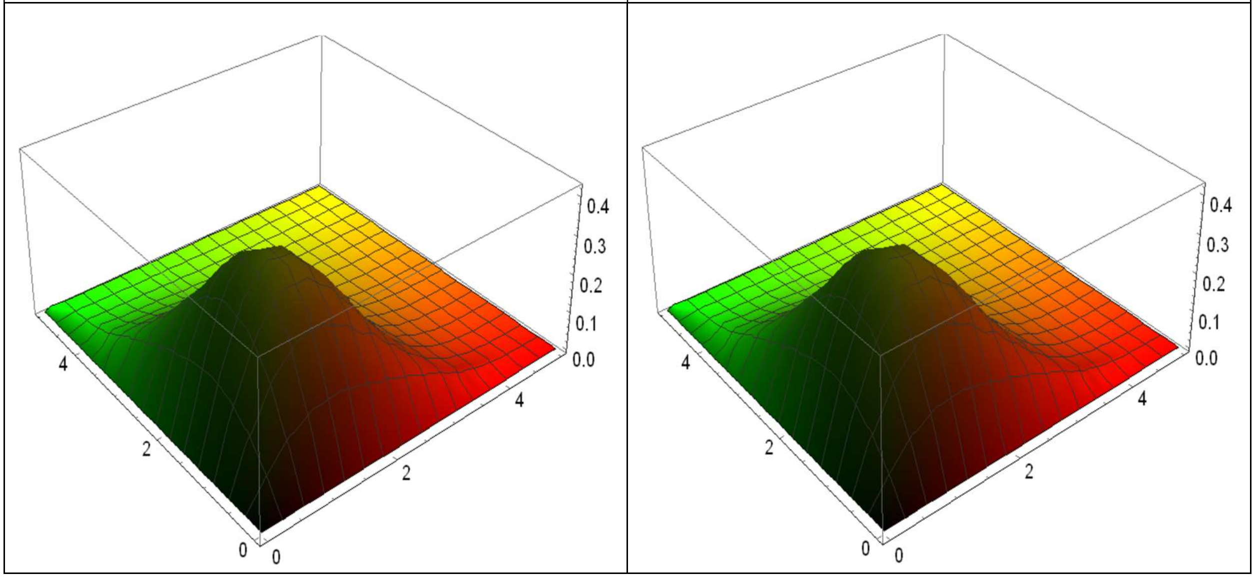



**Figure 7:** Joint Hazard Rate of FGM Bivariate Modified Weighted Rayleigh Distribution with The Parameters Values.

## 7. Application to Real Data for BMWR Distribution

In this section, we take a real data example from Al-Mutairi et al. (2011) which is the score of 25 first-graduate students in Probability I and Inference I of a premier institute in India. The data set is in Table (9).

| ***X*** | ***Y*** | ***X*** | ***Y*** | ***X*** | ***Y*** |
|---|---|---|---|---|---|
| 53 | 89 | 51 | 25 | 51 | 41 |
| 55 | 90 | 50 | 89 | 62 | 31 |
| 85 | 59 | 53 | 32 | 53 | 66 |
| 87 | 50 | 32 | 33 | 32 | 57 |
| 22 | 25 | 43 | 63 | 43 | 32 |
| 23 | 29 | 47 | 38 | 47 | 43 |
| 25 | 54 | 30 | 77 | 30 | 88 |
| 93 | 62 | 88 | 55 | 88 | 34 |
| 37 | 39 | | | | |

**Table 9:** The Score of 25 Students in Probability I And Inference I

We compare BMWR distribution with two model of the bivariate distributions, which are: bivariate weighted exponential based on the generalized exponential (BWEGE) distribution introduced by

Mahdavi et al.(2016) and bivariate weighted exponential (BWE) distribution, the comparison is based on the maximum likelihood (ML) estimates, Bayesian information criterion (BIC) and Akaike information criterion (AIC).

From Table 10, we find that the MWR distribution model is reasonably a good fit at the lower tail of the distribution as well. However, exponential distribution seems to be fitting the data in the middle of the distribution. We made our decision of finding the best-fitting model based on the smallest log-likelihood value and smallest AIC and BIC values.

| | **X** | | | | | | **Y** | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Distributions | MLE's | K-S statistic | P-value | AIC | BIC | -ln L | MLE's | K-S statistic | P-value | AIC | BIC | -ln L |
| MWR | $\hat{\sigma} = 42.7$ <br> $\hat{\lambda} = 0.37$ | 0.16 | 0.50 | 221.58 | 226.02 | 109.29 | $\hat{\sigma} = 40.2$ <br> $\hat{\lambda} = 0.14$ | 0.17 | 0.39 | 222.61 | 227.05 | 109.30 |
| Exponential | $\hat{\lambda} = 0.018$ | 0.35 | 0.003 | 250.33 | 251.55 | 124.16 | $\hat{\lambda} = 0.019$ | 0.38 | 0.001 | 249.6 | 250.8 | 123.8 |

**Table 10:** Goodness of Fit Test of MWR Distribution**.**

| Distributions | MLE's | AIC | BIC | -ln L |
|---|---|---|---|---|
| BMWR | $\hat{\lambda}_1 = 0.0548$ <br> $\hat{\lambda}_2 = 0.0334$ <br> $\hat{\sigma}_1 = 9.458$ <br> $\hat{\sigma}_2 = 10.245$ <br> $\hat{\theta} = 0.789$ | 430.125 | 435.36 | 198.365 |
| BWE | $\hat{\lambda}_1 = 0.0272$ <br> $\hat{\lambda}_2 = 0.0392$ <br> $\hat{\alpha} = 0.004$ <br> $\hat{\beta} = 1$ | 472.301 | 475.958 | 233.158 |
| BWEGE | $\hat{\lambda}_1 = 0.0495$ <br> $\hat{\lambda}_2 = 0.0615$ <br> $\hat{\alpha} = 0.0285$ <br> $\hat{\beta} = 7.430$ | 446.881 | 451.756 | 219.441 |

**Table 11.** Estimated Parameters, AIC, BIC, and – Loglikelihood for Score Data Set.

Based on these criteria, MWR distribution fits the degree of 25 students in Probability I and

Inference I data best. From this result of the data sets studied the MWR distribution performs the best distribution than the other distributions. As we can see from the results in Table 11, the smallest values of AIC and BIC are obtained for BMWR distribution.

## 8. Conclusion

In this paper, the new version of the weighted Rayleigh distribution based on a modified weighted version of Azzalini's (1985) and its bivariate extension is introduced. we introduced some important statistical properties of this distribution an empirical study was carried out to determine the effect of adding new parameters on the mean, variance, skewness and kurtosis of the distribution.

The applications of the modified distribution have been demonstrated using real-life data. Estimation of parameters is done using maximum likelihood estimation.

We introduced the bivariate extension of the new distribution named the bivariate modified Rayleigh distribution BMWR distribution is also introduced. The proposed bivariate distribution is of type Farlie–Gumbel–Morgenstern (FGM) copula. The BMWR distribution has modified weighted Rayleigh marginal distributions. The joint cumulative distribution function, the joint survival function, the joint probability density function, the joint hazard rate function and the statistical properties of the BMWR distribution are discussed.

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