Journal of Probability and Statistical Science
21(1), 84-108, Feb. 2023

A Modified Weighted Uniform Distribution and Its Bivariate Extension

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ABSTRACT

In this paper, a new version of weighted uniform distribution is constructed and studied. The statistical properties of the new distribution including the behavior of hazard and reversed hazard functions, moments, the central moments, moment generating function, mean, variance, coefficient of skewness, coefficient of kurtosis, median, mode, quantile, stochastic ordering, and order statistics are also obtained, a simulation study and real data applications are performed. Moreover, a bivariate extension of the new distribution named the bivariate modified uniform (BMWU) distribution is introduced. The proposed bivariate distribution is of type Farlie–Gumbel–Morgenstern (FGM) copula. The BMWU distribution has modified weighted uniform marginal distributions. The joint cumulative distribution function, the joint survival function, the joint probability density function, the joint hazard rate function, and the statistical properties of the BMWU distribution are also derived.

Keywords: Weighted uniform distribution, maximum likelihood estimation, FGM copula, order statistic, moments, joint probability density function, joint hazard rate function

1. Introduction

The principle of weighted distributions gives a collective entry for the problem of model specification and data interpretation. It presents a way for fitting models to the unknown weight function when samples may be taken both from the original distribution and the developed distribution. The weighted distributions arise frequently within the research related to the analysis of family data, reliability, analysis of intervention data, Meta-analysis, biomedicine, ecology and other regions, for the improvement of a right statistical model. The concept of weighted distributions was provided by Fisher (1934) and Rao (1965).

To introduce the concept of weighted distribution, suppose X is a non-negative random variable with pdf (x) . The pdf of the weighted random variable X_w denoted by $f_w(x)$is given by

$$f_w(x) = \frac{f(x).w(x)}{E[w(x)]}.$$

Where $E[w(x)] = \int_{-\infty}^{\infty} w(x)f(x)dx$,

and $w(x)$ be a non-negative weight function.

Note that $E[w(x)]$ is the normalizing factor obtained to make the total probability equal to unity by choosing $E[w(x)] > 0$, the random variable X_w called the weighted version of X , and

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its distribution is related to that of X and is called the weighted distribution with weight function $w(x)$.

Note that the weight function $w(x)$ need not lie between zero and one, and actually may exceed unity. The use of weighted distributions as a tool in the selection of suitable models for observed data depends on the choice of the weight function.

A copula function is a convenient way to describe bivariate distributions. Copulas are of interest to statisticians for two main reasons: Firstly, as a way of studying scale-free measures of dependence, and secondly, as a starting point for constructing families of bivariate distributions. Farlie–Gumbel–Morgenstern (FGM) copula, is one of the most popular parametric families of copulas where $-1 < \theta < 1$, which is defined by the following cumulative function

$$C(u,v) = uv(1 + \theta(1-u)(1-v))$$

and the density function

$$c(u,v) = 1 + \theta(1-2u)(1-2v)$$

Uniform (U) distribution is a very simple distribution for a continuous random variable, it is particularly useful in theoretical statistics because it is convenient to deal with mathematically.

U distribution is regarded to the simplest probability function, it is bounded support and is related to all distributions by the fact that the cumulative distribution function, taken as a random variable, follows uniform distribution over $(0,1)$ and this result's basic to the inverse methodology of random variable generation. This distribution is additionally applied to determine the power functions of tests of randomness. There are also various applications in nonparametric inference, such as the Kolmogrov-Smirnov test for goodness of fit. The uses of uniform distribution to represent the distribution of round-off errors and the probability integral transformations are also well known.

The uniform distribution has the following pdf and cdf, respectively, with scale parameter σ are given by:

$$\boldsymbol{f(x) = 1 \quad ; 0 < x < 1}. \tag{1}$$

$$\boldsymbol{F(x) = x} \quad . \tag{2}$$

2. The Univariate Modified Weighted Uniform (MWU) Distribution

In this section, we construct the pdf of MWU distribution Following Aleem et al (2013), a modified weighted version of Azzalini's (1985) approach was used as follows:

Let $f(x)$ be a pdf and $\bar{F}(x)$ be the corresponding survival (or reliability) function such as the cdf $F(x)$ exist.

The modified weighted distribution is given by:

$$f_w(x) = \frac{[\bar{F}(\lambda x)]^{\alpha} f(x)}{E[\bar{F}(\lambda x)]^{\alpha}}, \quad x > 0 \tag{3}$$

Where $f_w(x)$ the weighted probability distribution function, λ is the scale parameter and α is the shape parameter.

Now, make $\bar{F}(\lambda x)$ to represent the survival function of the uniform distribution.

$$\bar{F}(x) = 1 - x \quad , 0 < x < 1 \ .$$

Then

$$[\bar{F}(\lambda x)]^{\alpha} = [1 - \lambda x]^{\alpha} \tag{4}$$

$$E[\bar{F}(\lambda x)]^{\alpha} = \int_0^1 [1 - \lambda x]^{\alpha} dx = \frac{1-(1-\lambda)^{\alpha+1}}{\lambda(\alpha+1)} \tag{5}$$

Hence, The pdf of the modified weighted uniform distribution is given by substituting Eq.(4) and Eq.(5) into Eq.(3):

$$f_{WU}(x) = \frac{\lambda(\alpha+1)}{1-(1-\lambda)^{\alpha+1}} (1 - \lambda\, x)^{\alpha}, \qquad 0 \le x \le \frac{1}{\lambda}, \alpha \ge 0, \lambda \ge 0 \quad .$$

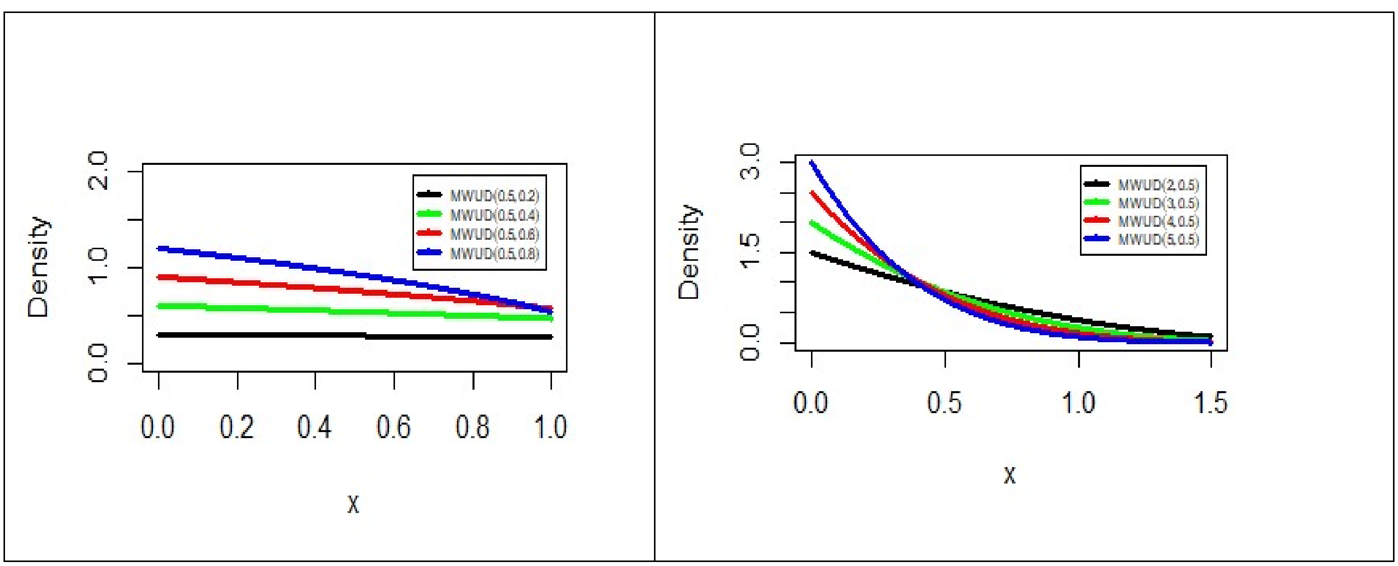
Since $f_{WU}(x)$ must be greater than or equal to zero, then $x \le \frac{1}{\lambda}$.

We can multiple $f_{WU}(x)$ by constant $1 - (1-\lambda)^{\alpha+1}$ to make the function is probability density function.

$$f_{WU}(x) = \lambda(\alpha + 1)\, (1 - \lambda\, x)^{\alpha}\ , 0 \le x \le \frac{1}{\lambda}\ , \alpha \ge 0, \lambda \ge 0 \tag{6}$$

The corresponding cdf is given by:

$$F_{WU}(x) = 1 - (1 - \lambda x)^{\alpha+1} \tag{7}$$



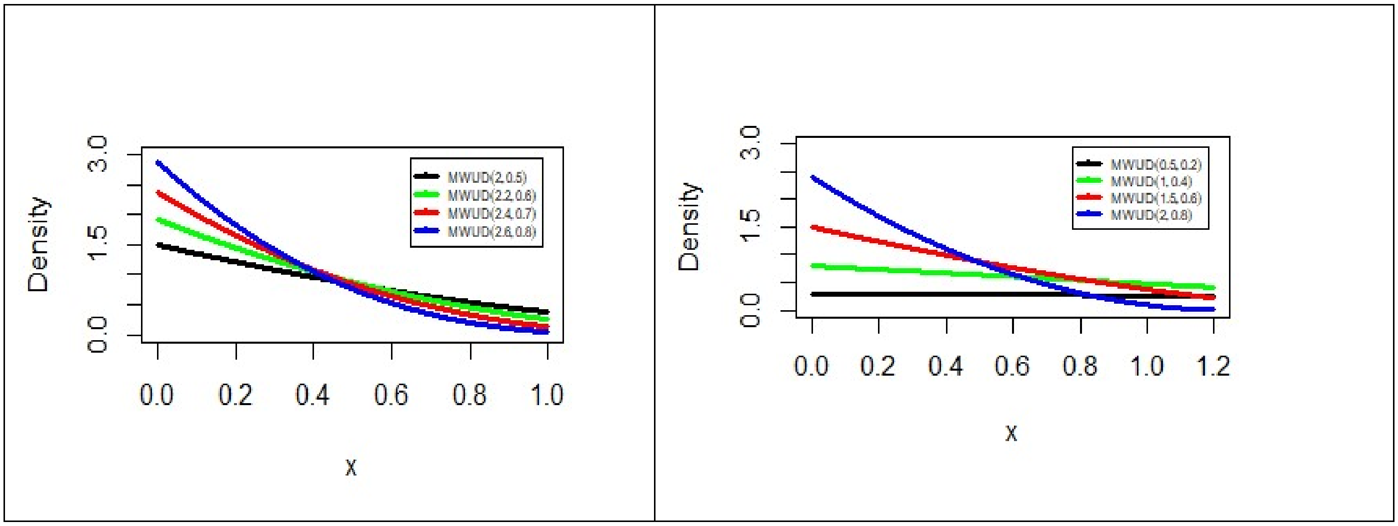


Figure 1 : Probability density function of MWU (α, λ) distribution.

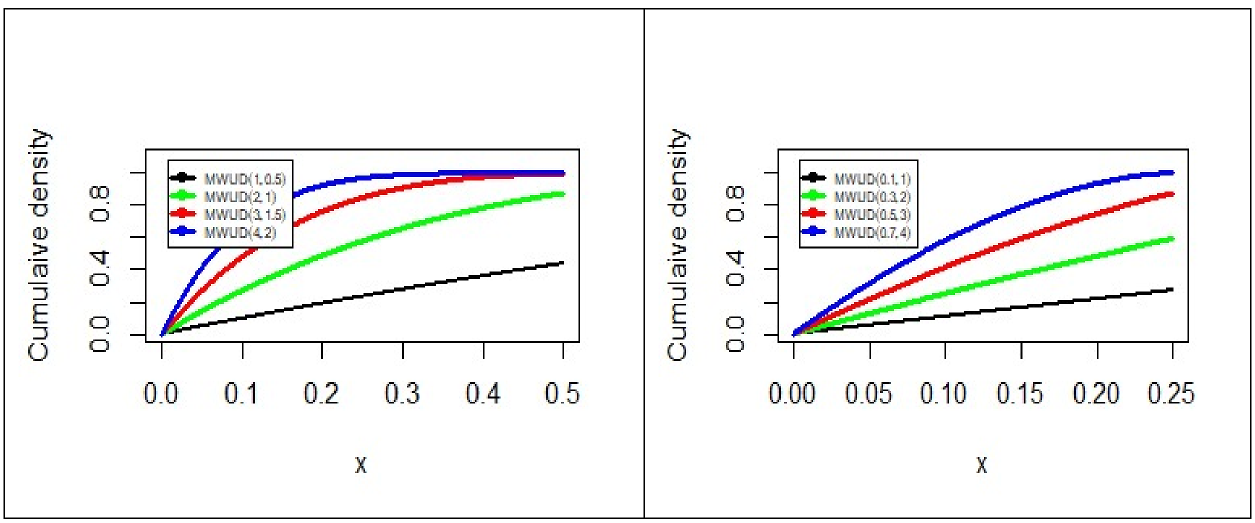


Figure 2 : Cumulative distribution function of MWU (α, λ) distribution.

Remark (1):
When $\lambda = 1$ and $\alpha = 0$ in Eq.(6), the MWU distribution becomes the uniform distribution $[0,1]$.

Remark (2):
When $\lambda = 1$ and $\alpha + 1 = b$ in Eq.(6), the MWU distribution becomes the Kumaraswamy distribution with parameters$(1, b)$.

3. Statistical Properties

3.1. The r^{th} Moments

The r^{th} moment about the origin (i.e., moment about zero) of a continuous random variable X with density function $f(x)$ is given by:

$$\mu'_r = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx$$

Also, the moments about the origin are called non-central moments.

If X is a random variable with density function Eq.(6), then The r^{th} moment about the origin is given by

$$\mu'_r = E(X^r) = \frac{(\alpha+1)\, B(r+1,\alpha+1)}{\lambda^r} \tag{8}$$

Where r is a positive integer and $B(r+1,\alpha+1)$ is a beta function.

The mean and other higher-order moments can be obtained from Eq.(8) as follows

i) The first moment about zero, $\mu' = \frac{1}{\lambda\,(\alpha+2)}$

ii) The second moment about zero, $\mu'_2 = \frac{2}{\lambda^2\,(\alpha+2)(\alpha+3)}$

iii) The third moment about zero, $\mu'_3 = \frac{6}{\lambda^3\,(\alpha+2)(\alpha+3)(\alpha+4)}$

iv) The fourth moment about zero, $\mu'_4 = \frac{24}{\lambda^4\,(\alpha+2)(\alpha+3)(\alpha+4)(\alpha+5)}$

v) The variance of the MWU distribution

$$Var(X) = \frac{(\alpha+1)}{\lambda^2\,(\alpha+2)^2(\alpha+3)}$$

vi) The coefficient of variation

$$CV(x) = \frac{the\ standard\ deviation}{the\ mean} = \sqrt{\frac{(\alpha+1)}{(\alpha+3)}}$$

3.2. Central Moments

Using the relationship between the central moments (the moment about the mean) and the noncentral moments (the moment about zero).

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} (-1)^{r-j}\, \mu'_j\, \mu^{r-j}$$

Where μ_r is the r^{th} central moment, μ is mean and μ'_j the j^{th} noncentral moment.

i) The first four central moments of MWR distribution are obtained as:

$$\mu_1 = \frac{1}{\lambda\,(\alpha+2)}\ ,$$

$$\mu_2 = \frac{(\alpha+1)}{\lambda^2\,(\alpha+2)^2(\alpha+3)}$$

$$\mu_3 = \frac{(\alpha+1)(\alpha+6)}{\lambda^3\,(\alpha+2)^2(\alpha+3)(\alpha+4)} \quad \text{and}$$

$$\mu_4 = \frac{9\alpha^3+24\alpha^2+57\alpha+}{\lambda^4\,(\alpha+2)^2(\alpha+3)(\alpha+4)(\alpha+5)}\ .$$

ii) The coefficient of skewness of MWU distribution enables us to know whether the distribution is symmetric or not, it is defined by:

$$\beta_1 = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{(\alpha+2)(\alpha+6)\sqrt{(\alpha+3)}}{(\alpha+3)\sqrt{(\alpha+1)}}$$

Since $\mu_3 > 0$ and $\beta_1 > 0$, This mean that is skewed right distribution.

iii) The coefficient of kurtosis of MWU distribution measures the flatness of the top, it is defined by:

$$\beta_2 = \frac{\mu_4}{(\mu_2)^2} = \frac{(9\alpha^3+24\alpha^2+57\alpha+12)(\alpha+2)^2(\alpha+3)}{(\alpha+1)^2(\alpha+4)(\alpha+5)}$$

Since $\beta_2 > 0$, the density of MWU distribution is more peaked around its center than the density of the normal.

3.3. Moment Generating Function

The moment generating function of a the MWU distribution is given as follows

$$M_x(t) = \frac{\lambda^\alpha}{t^{\alpha+1}} e^{\frac{t}{\lambda}} \gamma\left((\alpha+1), \frac{t(1-\lambda x)}{\lambda}\right)$$

Where $\gamma\left((\alpha+1), \frac{t(1-\lambda x)}{\lambda}\right)$ is the lower incomplete incomplete gamma function

3.4. Reliability Measures

The survival function $\bar{F}_{WU}(x) = 1 - F_{wu}(x)$.

Then the survival function of the modified weighted uniform distribution is

$$\bar{F}_{WU}(x) = (1-\lambda x)^{\alpha+1}, 0 \le x \le \frac{1}{\lambda}$$

The hazard function of the modified weighted uniform distribution is given by :

$$h_{WU}(x) = \frac{f_{WU}(x)}{\bar{F}_{WU}(x)}$$

$$h_{WU}(x) = \frac{\lambda(\alpha+1)}{(1-\lambda x)}$$

Since $\frac{d}{dx} h_{WU}(x) > 0$.

The reversed hazard rate of X is defined by:

$$r_{WU}(x) = \frac{f_{WU}(x)}{F_{WU}(x)} \quad ; \quad F_{wu}(x) > 0.$$

The reversed hazard function of the modified weighted uniform distribution is given by:

$$r_{WU}(x) = \frac{\lambda(\alpha+1)(1-\lambda x)^\alpha}{1-(1-\lambda x)^{\alpha+1}}$$

Since $\frac{d}{dx} r_{WU}(x) < 0$.

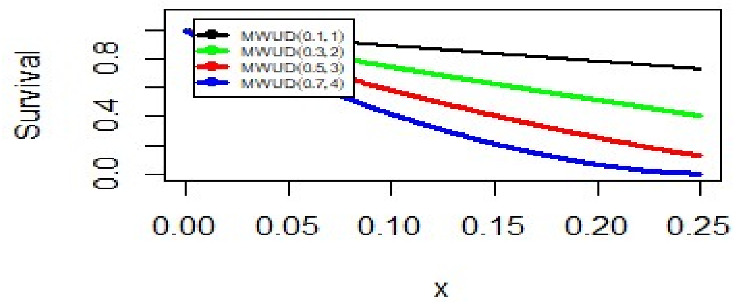


Figure 3: Reliability function of MWU (α, λ) distribution.

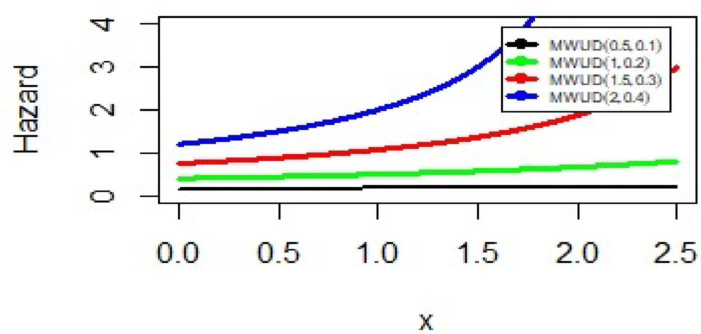


Figure 4 : Hazard function of MWU (α, λ) distribution

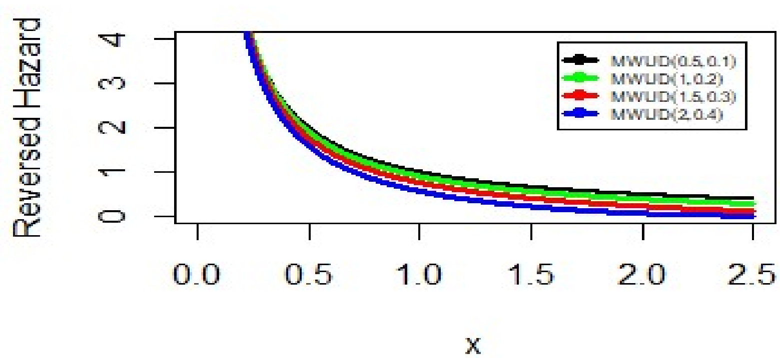


Figure 5 : Reversed hazard function of MWU (α, λ) distribution .

Then, the hazard function is always increasing function when the two parameters are nonnegative and this is shown in Fig.4.

Then, The reversed hazard function is always decreasing function when the two parameters are nonnegative and this is shown in Fig.5 .

3.5. Quantile, Median and Mode

We will derive the quantile, the median, the first quartile, the third quartile, and the mode. The quantile x_q of the modified weighted uniform distribution is given by:

$$x_q = F_w^{-1}(q)$$

$$x_q = \frac{1-(1-q)^{\frac{1}{\alpha+1}}}{\lambda} \tag{9}$$

The median is obtained directly by substituting $q = 0.5$ in the quantile x_q in Eq.(9). The median of the modified weighted uniform distribution is given by:

$$X_{med} = \frac{1-\left(\frac{1}{2}\right)^{\frac{1}{\alpha+1}}}{\lambda}$$

The first quartile of the modified weighted uniform distribution is given by substituting $q = 0.25$ in the quantile x_q in Eq.(9):

$$X_{0.25} = \frac{1-(0.75)^{\frac{1}{\alpha+1}}}{\lambda}$$

The third quartile of the modified weighted uniform distribution is given by substituting $q = 0.75$ in the quantile x_q in Eq.(9):

$$X_{0.75} = \frac{1-(0.25)^{\frac{1}{\alpha+1}}}{\lambda}$$

The mode of the modified weighted uniform distribution is given by:

$$X_{mode} = 0$$

It is clear from the Figures of the pdf of MWU (α, λ) distribution that the mode approaches to zero with different parametesr values.

3.6. Order Statistic

We derive the pdf of the i^{th} order statistic of the MWU distribution, also, the first (the smallest order statistic), the largest (the last order statistic), and the joint of order statistic are obtained.

Let $X_1, X_2, X_3, \ldots \ldots, X_n$ be a sample from a modified weighted uniform distribution with pdf in Eq.(6) and cdf in Eq.(7), respectively.
Let $X_{(1)} < X_{(2)} < X_{(3)} < \cdots < X_{(n)}$ be the order statistic of n independent observation from an NWU distribution
Where $0 < F_{WU}(x) < 1$, over the support $0 \le x \le \frac{1}{\lambda}$.

i) The probability density function of the first-order statistic $X_{(1)} = min\ (X_1, X_2\ ,\ \dots \dots ., X_n\ )$ of the MWU distribution is given by:

$$g_1(x) = n\,\lambda\,(\alpha + 1)(1 - \lambda x)^{n+\alpha n+} \quad \text{, where } x = x_{(1)}.$$

The cumulative distribution function of the smallest order statistic $G_1(x)$ of the MWU distribution is given by:

$$G_1(x) = 1 - \ [(1 - \lambda x)^{\alpha+1}]^n$$

ii) The probability density function of the largest order statistic $X_{(n)} = max\ (X_1, X_2\ , \dots\ \dots ., X_n\ )$ of the MWU distribution is given by:

$$g_n(x) = \ n\lambda\,(\alpha + 1)[1 - (1 - \lambda x)^{\alpha+1}]^{n-1}(1 - \lambda x)^{\alpha}, \text{where } x = x_{(n)}.$$

The cumulative distribution function of the largest order statistic $G_n(x)$ of the MWU distribution is given by:

$$G_n(x) = \ [1 - (1 - \lambda x)^{\alpha+1}]^n$$

iii) The probability density function of the i^{th} order statistic of the MWU distribution is:

$$g_i(x) = \frac{\lambda\,(\alpha+1)n!}{(i-1)!(n-i)!(\lambda-2)^n}\ [1 - (1 - \lambda x)^{\alpha+1}]^{i-1}\ (1 - \lambda x)^{\alpha n+n-\alpha i-i}$$

, where $x = x_{(i)}$, over the support $0 \le x_{(1)} < \dots < \ x_{(i)} < \dots < \ x_{(n)}\ \le \frac{1}{\lambda}$.

iv) The joint probability density function of $X_{(i)}$ and $X_{(r)}$ $(for\ i < r)$ of the MWU distribution is given by:

Let $X_{(i)} = X$ and $X_{(r)} = Y$

Then

$$g_{i,r}(x, y) = \frac{\lambda^2(\alpha + 1)^2\ n!}{(i - 1)!\ (r - i - 1)!\ (n - r)!}\ [1 - (1 - \lambda x)^{\alpha+1}]^{i-1}[(1 - \lambda x)^{\alpha+1} - (1 - \lambda y)^{\alpha+1}\ ]^{r-i-1}\ [(1 - \lambda y)^{\alpha+1}]^{n-r}(1 - \lambda x)^{\alpha+1}(1 - \lambda y)^{\alpha+1}$$

, where $x = x_{(i)}, y = x_{(r)}$, over the support $0 \le x_{(1)} < \dots < \ x_{(i)} < \dots < \ x_{(r)} < \dots < \ x_{(n)}\ \le \frac{1}{\lambda}$.

3.6. Stochastic Ordering

The stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A continuous random variable X is said to be smaller than a continuous random variable Y in the

i) stochastic ordering ($X\ \le_{st} Y$) if $F_X(x)\ \ge\ F_Y(x)$ for all x
ii) hazard rate order ($X\ \le_{hr} Y$) if $h_X(x)\ \ge\ h_Y(x)$ for all x
iii) likelihood ratio order ($X\ \le_{lr} Y$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x

The following stochastic ordering relationships due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering for continuous distributions

$$\begin{array}{c} X \le_{lr} Y \Rightarrow X \le_{hr} Y \\ \Downarrow \\ X \le_{st} Y \end{array}$$

The MWU distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem:

Theorem (1)

Let $X \sim MWU\ (\alpha_1, \lambda_1)$ and $Y \sim MWU\ (\alpha_2, \lambda_2)$, Then the following results hold true

i) If $\alpha_1 < \alpha_2$ and $\lambda_1 = \lambda_2$ then $X \le_{lr} Y$, $X \le_{hr} Y$ and $X \le_{st} Y$

ii) If $\alpha_1 = \alpha_2$ and $\lambda_2 > \lambda_1$ then $X \le_{lr} Y$, $X \le_{hr} Y$ and $X \le_{st} Y$

Proof (i)

$$\frac{f_Y(x)}{f_X(x)} = \frac{\lambda_2\,(\alpha_2+1)(1-\lambda_2 x)^{\alpha_2}}{\lambda_1\,(\alpha_1+1)(1-\lambda_1 x)^{\alpha_1}},$$

If $\lambda_1 = \lambda_2 = \lambda$ and $\alpha_1 < \alpha_2$

$$\frac{f_Y(x)}{f_X(x)} = \frac{(\alpha_2+1)}{(\alpha_1+1)}(1-\lambda x)^{\alpha_2-\alpha_1}$$

Then $\frac{f_Y(x)}{f_X(x)} > 0$ increasing if $x < \frac{1}{\lambda}$.

Then $X \le_{lr} Y$ implies that $X \le_{hr} Y$ and $X \le_{st} Y$

Proof (ii)

$$\frac{f_Y(x)}{f_X(x)} = \frac{\lambda_2\,(\alpha_2+1)(1-\lambda_2 x)^{\alpha_2}}{\lambda_1\,(\alpha_1+1)(1-\lambda_1 x)^{\alpha_1}},$$

If $\alpha_1 = \alpha_2 = \alpha$ and $\lambda_2 > \lambda_1$

$$\frac{f_Y(x)}{f_X(x)} = \frac{\lambda_2\,(1-\lambda_2 x)^{\alpha}}{\lambda_1\,(1-\lambda_1 x)^{\alpha}} = \frac{\lambda_2}{\lambda_1}\left(\frac{1-\lambda_2 x}{1-\lambda_1 x}\right)^{\alpha}$$

Taking the logarithm for both two sides, gets,

$$\ln\left(\frac{f_Y(x)}{f_X(x)}\right) = \ln(\lambda_2) - \ln(\lambda_1) + \alpha \ln\left(\frac{1-\lambda_2 x}{1-\lambda_2 x}\right)$$

$$\frac{d}{dx}\ln\left(\frac{f_Y(x)}{f_X(x)}\right) = \frac{\alpha}{\left(\frac{1-\lambda_2 x}{1-\lambda_1 x}\right)}\left(\frac{-(1-\lambda_1 x)\lambda_2+(1-\lambda_2 x)\lambda_1}{(1-\lambda_1 x)^2}\right) = \alpha\ \frac{\lambda_1(1-\lambda_2 x)-\lambda_2(1-\lambda_1 x)}{(1-\lambda_1 x)(1-\lambda_2 x)}$$

$$\frac{d}{dx}\ln\left(\frac{f_Y(x)}{f_X(x)}\right) = \alpha\left[\frac{\lambda_1}{(1-\lambda_1 x)} - \frac{\lambda_2}{(1-\lambda_2 x)}\right]$$

$\frac{f_Y(x)}{f_X(x)} > 0$ increasing in $x < \frac{1}{\lambda}$.

Then $X \le_{lr} Y$ implies that $X \le_{hr} Y$ and $X \le_{st} Y$

4 Estimation of Unknown Parameters of MWU Distribution

In this section, the namely, method of moment (MOM) and maximum likelihood method (MLE) are derived to estimate the two scale parameters σ and λ and we derive the observed and Fisher information matrix.

4.1. Moment Estimates :

The method of moments (MOM) is a technique commonly used in the field of parameter estimation. Let $X_1, X_2, \ldots\ldots, X_n$ represent a set of data by independent and identically distributed random variables, with probability density function $f_w(x)$ and cumulative distribution function $F_w(x)$ of (x, ϕ) , $\phi = (\sigma, \lambda)$.
The r^{th} population moment about the zero equal the r^{th} sample moment of random variable X.

$$E(X^r) = \frac{1}{n}\sum_{i=1}^{n} x_i^r$$

To construct moment estimators $(\hat{\alpha}, \hat{\lambda})$ for the unknown parameter (α, λ) we solve the set of the following equation

The mean of the MWU distribution exists, by equaling this to the sample mean $\bar{X}$

$$\bar{X} = \frac{1}{\lambda(\alpha+2)}$$

Then $\hat{\alpha} = \frac{1}{\hat{\lambda}\,\bar{X}} - 2$ (10)

The formula for $\hat{\lambda}$ may be found by the following argument, the probability that all n of X_i's are smallest than a particular value x .
Let $F_{X_{(n)}}(x)$ be the cumulative distribution function of the greatest sample value $X_{(n)}$.

$$F_{X_{(n)}}(x) = [1-(1-\lambda x)^{\alpha+1}]^n \ , \ x \le \lambda \ .$$

The corresponding density function is

$$f_{X_{(n)}}(x) = n\lambda\,(\alpha+1)(1-\lambda x)^{\alpha}[1-(1-\lambda x)^{\alpha+1}]^{n-1}$$

From the pdf of the greatest sample value, the expected value of $X_n$is:

$$E(X_{(n)}) = \int_0^{\frac{1}{\lambda}} x f_{X_{(n)}}(x)\, dX_{(n)} = \int_0^{\frac{1}{\lambda}} xn\lambda\,(\alpha+1)(1-\lambda x)^{\alpha}[1-(1-\lambda x)^{\alpha+1}]^{n-1}\, dx$$

, where $x = x_{(n)}$

let $u = (1-\lambda x)^{\alpha+1}$, $-\lambda\,(\alpha+1)(1-\lambda x)^{\alpha}dx = du$ and $u \in [0,1]$

$$E(X_{(n)}) = \int_0^1 \frac{n}{\lambda}\left(1-u^{\frac{1}{\alpha+1}}\right)[1-u]^{n-1}\, du = \frac{n}{\lambda}\left[\frac{1}{n} - B\left(n, \frac{1}{\alpha+1}\right)\right]$$

Equaling $X_{(n)}$ to $E(X_{(n)})$, the estimator of $\hat{\lambda}$ is found

$$X_{(n)} = \frac{1}{\hat{\lambda}}\left[1 - \frac{\Gamma(n+1)\Gamma\left(\frac{\hat{\alpha}+2}{\hat{\alpha}+1}\right)}{\Gamma\left(\frac{\hat{\alpha}+2}{\hat{\alpha}+1}+n\right)}\right] \qquad (11)$$

Solving Eq.(10) with Eq.(11) to find $\hat{\lambda}$ and $\hat{\alpha}$

$$\hat{\lambda} = \frac{1}{X_{(n)}}\left[1 - \frac{\Gamma(n+1)\Gamma\left(\frac{\hat{\alpha}+2}{\hat{\alpha}+1}\right)}{\Gamma\left(\frac{\hat{\alpha}+2}{\hat{\alpha}+1}+n\right)}\right],$$

and

$$\hat{\alpha} = \frac{X_{(n)}}{\bar{X}\left[1-\frac{\Gamma(n+1)\Gamma\left(\frac{\hat{\alpha}+2}{\hat{\alpha}+1}\right)}{\Gamma\left(\frac{\hat{\alpha}+2}{\hat{\alpha}+1}+n\right)}\right]} - 2$$

4.2. Maximum Likelihood Estimates (MLE)

Let $X_1, X_2, \ldots\ldots, X_n$ be a random sample of size n from MWU distribution(α, λ). The likelihood function for the parameter λ can be written

$$L(\underline{x};\ \alpha, \lambda) = \prod_{i=1}^{n} f_{WU}(x_i; \alpha, \lambda)$$
$$= (\lambda)^n(\alpha+1)^n \prod_{i=1}^{n}(1-\lambda\, x_i)^{\alpha}$$

The loglikelihood function is

$$\log\left(L(\underline{x};\ \alpha, \lambda)\right) = n\log(\lambda) + n\log(\alpha+1) + \alpha\sum_{i=1}^{n}\log(1-\lambda\, x_i)$$

Taking the first derivatives w.r.t α and equaling to zero, we get

$$\frac{\partial \log\left(L(\underline{x};\alpha,\lambda)\right)}{\partial\alpha} = \frac{n}{\alpha+1} + \sum_{i=1}^{n}\log(1-\lambda\, x_i)$$

Then $\hat{\alpha} = \frac{-n}{\sum_{i=1}^{n}\log(1-\hat{\lambda}\, x_i)} - 1$

The derivative w.r.t λ cannot be obtained in the usual way, since $\log\left(L(\underline{x};\ \alpha, \lambda)\right)$ is unbounded w.r.t λ. Since λ is the upper bound on the random variable, $\log\left(L(\underline{x};\ \alpha, \lambda)\right)$ must be maximized subject to the constraint

$$\max X_{(i)} \leq \hat{\lambda}$$

Then $\hat{\lambda} = \max X_{(i)} = X_{(n)}$.

5 Numerical Illustration

5.1. Real Data Analysis

In this section, we analyze two real-life data sets to show the applicability of MWU distribution.

We use the fitdstrplus R package to fit the distributions.

For the comparison of the distributions, the criteria used are -lnL, Akaike information criterion (AIC) by Akaike (1974), Bayesian information criterion (BIC) by Schwarz (1978), and Kolmogorov Smirnov (K-S) Statistic. AIC estimates the performance of a model while comparing it with other models. The distribution with smaller values of -lnL, AIC, and BIC is considered the best distribution. The specifications of these criteria are as follows:

$$AIC = 2(K) - 2\ln L$$
$$BIC = k\ln(n) - 2\ln L$$

where k = number of estimated parameters in the distribution.
n = the total number of observations.
ln L = maximized log-likelihood of the distribution under consideration

Some Relevant Distributions

1) Weibull distribution
$$f(x) = \alpha\,\beta\,x^{\beta-1}\,e^{-\alpha x^{\beta}} \quad ; x, \alpha, \beta > 0.$$

2) modified weighted Rayleigh distribution (MWR) distribution
$$f(x) = \frac{(\lambda^2+1)}{\sigma^2}\,x\,e^{-\frac{(\lambda^2+1)}{2\sigma^2}x^2} \quad ; x > 0\,, \sigma > 0\,, \lambda \geq 0\,.$$
3) Kumaraswamy distribution
$$f(x;a,b) = 2\,a\,b\;x^{a-1}\,(1-x^{a})^{b-1};\;\; 0 \leq x \leq 1\,, a, b > 0,$$
4) Topp-Leone distribution
$$f(x;\alpha) = 2\,\alpha\,x^{\alpha-1}\,(1-x)(2-x)^{\alpha-1};\;\; 0 \leq x \leq 1, \alpha > 0,$$

5.2 Data Set: Guinea Pig Data

we prove the flexibility of MWU distribution by analyzing the dataset, and its performance was compared with that of the Topp-Leone distribution (TL), Kumaraswamy distribution (Kw), modified weighted Rayleigh distribution (MWR), exponential distribution, and Weibull distribution.

We have used this data(about the ordered failure of 20 components) from Nigm et al. (2003), The data set with a mean value of 0.161, a median of 0.132, a variance of 0.0247 and the data is highly positively skewed with a coefficient of skewness of 1.44.

0.0009	0.004	0.0142	0.0221	0.0261	0.0418	0.0473	0.0834	0.1091	0.1252
0.1404	0.1498	0.175	0.2031	0.2099	0.2168	0.2918	0.3465	0.4035	0.6143

Table1: Data set of ordered failure of 20 components

Distributions	MLE's	K-S statistic	P-value	AIC	BIC	-ln L
MWU	$\hat{\alpha} = 12.02$ $\hat{\lambda} = 0.44$	0.109	0.94	-39.06	-37.07	-26.53
TL	$\hat{\alpha} = 0.511$	0.184	0.44	-29.23	-28.23	-15.61
Kw	$\hat{a} = 0.76$ $\hat{b} = 3.43$	0.102	0.97	-30.40	-28.41	-17.20

MWR	$\hat{\sigma} = 2.53$ $\hat{\lambda} = 16.06$	0.30	0.037	-1.49	0.5	-2.74
Exponential	$\hat{\lambda} = 6.20$	0.10	0.97	-30.98	-29.99	-16.49
Weibull	$\hat{\alpha} = 5.37$ $\hat{\beta} = 0.89$	0.119	0.90	-29.33	-27.34	-16.66

Table 2: Fitting distributions to find the best-fit model.

From Table 2, It can be seen that the MWU distribution has lower AIC and BIC values compared to the other distributions, which confirms that the MWU distribution is more suitable for this data. According to Genc (2017).

5.3 Simulation Study

A simulation study is led to estimate the performance of maximum likelihood (ML) for estimating the unknown two parameters of MWU distribution. The performance of the different estimators is evaluated in terms of mean, mean square error (MSE), biased and standard errors (SEs) The simulation is conducted by using R- software, 10000 random samples of MWU distribution (α, λ) was generated with different sample sizes (n) as $n = (50,100,125,150)$ Where**;** mean $\bar{x} = \frac{1}{n}\sum_{i=0}^{n} x_i$ **,** mean square error MSE $= \frac{\sum_i^n (x - \hat{x})^2}{n}$, $Bais(x) = |\bar{\boldsymbol{x}} - \boldsymbol{x}|$ Standard errors $SEs(x) = \frac{sd(x)}{\sqrt{n}}$ and Relative biased_RBs(x) $= \frac{bias(x)}{x}$

N		Mean	MSE	Bais	SEs	RBs
30	$\hat{\alpha}$	0.461	0.0045	0.0381	0.0005	0.7
	$\hat{\lambda}$	0.933	0.0110	0.0661	0.0008	0.6
50	$\hat{\alpha}$	0.537	0.0048	0.0375	0.0005	0.07
	$\hat{\lambda}$	1.251	0.178	0.2511	0.0033	0. 25
100	$\hat{\alpha}$	0.453	0.0043	0.0461	0.0004	0.09
	$\hat{\lambda}$	0.88	0.035	0.1116	0.0015	0.11
125	$\hat{\alpha}$	0.456	0.0034	0.0437	0.0004	0.08
	$\hat{\lambda}$	0.901	0.0247	0.0987	0.0012	0.09
150	$\hat{\alpha}$	0.460	0.0027	0.0396	0.00034	0.07
	$\hat{\lambda}$	0.923	0.0125	0.0769	0.0008	0.07

Table 3 : MLE estimation of the parameter of MWU distribution $\alpha = 0.5, \lambda = 1$

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N		Mean	MSE	Bais	SEs	RBs
30	$\hat{\alpha}$	0.649	0.129	0.3502	0.0008	0.35
	$\hat{\lambda}$	0.512	0.0206	0.0129	0.0014	0.02
50	$\hat{\alpha}$	0.639	0.134	0.360	0.00066	0.36
	$\hat{\lambda}$	0.431	0.9233	0.931	0.0023	1.86
100	$\hat{\alpha}$	0.666	0.1213	0.3330	0.0010	0.33
	$\hat{\lambda}$	0.5054	0.1263	0.0054	0.0035	0.01
125	$\hat{\alpha}$	0.651	0.128	0.3487	0.0008	0.34
	$\hat{\lambda}$	0.434	0.128	0.0658	0.0035	0.13
150	$\hat{\alpha}$	0.632	0.136	0.367	0.0004	0.36
	$\hat{\lambda}$	0.435	0.0262	0.0643	0.0014	0.12

Table 4: MLE estimation of the parameter of MWU distribution $\alpha = 1, \lambda = 0.5$

N		Mean	MSE	Bais	SEs	RBs
30	$\hat{\alpha}$	0.871	0.0166	0.0710	0.0010	0.08
	$\hat{\lambda}$	0.539	0.0445	0.1393	0.0015	0.34
50	$\hat{\alpha}$	0.8204	0.00628	0.0204	0.0007	0.02
	$\hat{\lambda}$	0.378	0.0134	0.0212	0.00114	0.05
100	$\hat{\alpha}$	0.828	0.0031	0.02803	0.00049	0.03
	$\hat{\lambda}$	0.4139	0.0010	0.01390	0.00028	0.03
125	$\hat{\alpha}$	0.8375	0.00372	0.03756	0.0004	0.04
	$\hat{\lambda}$	0.439	0.0071	0.0390	0.0007	0.09
150	$\hat{\alpha}$	0.846	0.0052	0.046	0.0005	0.05
	$\hat{\lambda}$	0.4611	0.01412	0.0611	0.0010	0.15

Table 5 : MLE of the parameter of MWU distribution $\alpha = 0.8$ $\lambda = 0.4$

N		Mean	MSE	Bais	SEs	RBs
30	$\hat{\alpha}$	0.3505	0.7235	0.8494	0.00043	0.70
	$\hat{\lambda}$	2.3804	0.1461	0.3804	0.00036	0.19
50	$\hat{\alpha}$	0.3119	0.7893	0.8880	0.00026	0.74
	$\hat{\lambda}$	2.070	0.0128	0.07081	0.00088	0.03
100	$\hat{\alpha}$	0.3222	0.7708	0.8777	0.00019	0.73
	$\hat{\lambda}$	2.1469	0.0241	0.1469	0.0005	0.07
	$\hat{\alpha}$	0.3078	0.7984	0.8921	0.00050	0.74

125	$\hat{\lambda}$	2.0143	0.1006	0.0143	0.00316	0.07
150	$\hat{\alpha}$	0.3536	0.7166	0.8463	0.00020	0.7
	$\hat{\lambda}$	2.3945	0.1561	0.394	0.00023	0.19

Table 6 : MLE estimation of the parameter of MWU distribution $\alpha = 1.2 , \lambda = 2$

N		Mean	MSE	Bais	SEs	RBs
30	$\hat{\alpha}$	0.7570	0.0776	0.25708	0.00107	0.51
	$\hat{\lambda}$	0.4946	0.01290	0.00536	0.00113	0.01
50	$\hat{\alpha}$	0.9273	0.1987	0.4273	0.00127	0.85
	$\hat{\lambda}$	0.9325	0.1992	0.4325	0.0011	0.86
100	$\hat{\alpha}$	0.7992	0.2239	0.2992	0.00366	0.59
	$\hat{\lambda}$	0.4734	0.36583	0.0265	0.0060	0.05
125	$\hat{\alpha}$	0.71113	0.04548	0.2111	0.00030	0.42
	$\hat{\lambda}$	0.3304	0.02903	0.1695	0.00016	0.33
150	$\hat{\alpha}$	0.7554	0.06816	0.2554	0.00054	0.51
	$\hat{\lambda}$	0.4919	0.00679	0.00802	0.00082	0.1

Table 7 : MLE estimation of the parameter of MWU distribution $\alpha = 0.5, \lambda = 0.5$

N		Mean	MSE	Bais	SEs	RBs
30	$\hat{\alpha}$	0.680	0.0948	0.2803	0.00127	0.70
	$\hat{\lambda}$	1.4195	0.31531	0.5195	0.00213	0.57
50	$\hat{\alpha}$	0.4610	0.01067	0.06108	0.00083	0.15
	$\hat{\lambda}$	0.6342	0.1091	0.26575	0.0019	0.29
100	$\hat{\alpha}$	0.4840	0.01071	0.0840	0.0006	0.21
	$\hat{\lambda}$	0.7217	0.06439	0.17820	0.0018	0.19
125	$\hat{\alpha}$	0.6073	0.0453	0.2073	0.00048	0.51
	$\hat{\lambda}$	1.1977	0.0934	0.2977	0.00068	0.33
150	$\hat{\alpha}$	0.709	0.0999	0.3095	0.00064	0.77
	$\hat{\lambda}$	1.522	0.3991	0.6221	0.001	0.69

Table 8 : MLE estimation of the parameter of MWU distribution $\alpha = 0.4 , \lambda = 0.9$

In conclusion from Tables 3:8, we note that the greater the sample size, the greater the efficiency of the estimator in terms of lower values of MSE and Bais.

6 . The Bivariate Modified Weighted Uniform Distribution BMWU Distribution

In this section, Farlie–Gumbel–Morgenstern (FGM) copula and the univariate modified weighted uniform distribution are used for creating the bivariate distribution which is called BMWU distribution. Let $(X\ ,Y)$ be a two dimensional random variables and support $\mathcal{R}^2$ where $\mathcal{R}$ is the real number, we introduce the joint pdf $f(x,y)$, the joint cdf $F(x,y)$, the marginal distributions of X and Y , the conditional distribution and the joint survival function $\bar{F}(x,y)$.

6.1. FGMBMWR Distribution

In this section, the univariate modified weighted uniform distribution is used for creating the bivariate distribution by using Farlie–Gumbel–Morgenstern (FGM) copula which is called BMWU distribution.

Suppose X and Y distributed as MWU distribution (α_1,λ_1) and (α_2,λ_2) with distribution function $F_1(x)$ and $F_2(y)$ respectively. Then the bivariate vector (X,Y) has a bivariate MWU distribution with the scale parameters $\lambda_1 and\ \lambda_2$ and shape parameters α_1 and α_2, we will denote the bivariate modified weighted uniform distribution by BMWU distribution$(\alpha_1,\alpha_2,\lambda_1,\lambda_2,\theta)$

The bivariate FGM system of distributions is written as:

$$F(x,y)=F_1(x)F_2(y)[1+\ \theta(1-F_1(x))(1-F_2(y))] \quad -1\le\theta\le 1$$

Its pdf can be obtained by a direct differentiation as

$$f(x,y)\ =f_1(x)f_2(y)[1+\ \theta(2F_1(x)-1)(2F_2(y)-1)]$$

The cdf of the BMWU distribution can be expressed as

$$F(x,y)=[1-(1-\lambda_1x)^{\alpha_1+1}][1-(1-\lambda_2y)^{\alpha_2+1}][1+\theta(1-\lambda_1x)^{\alpha_1+1}(1-\lambda_2y)^{\alpha_2+1}] \tag{12}$$

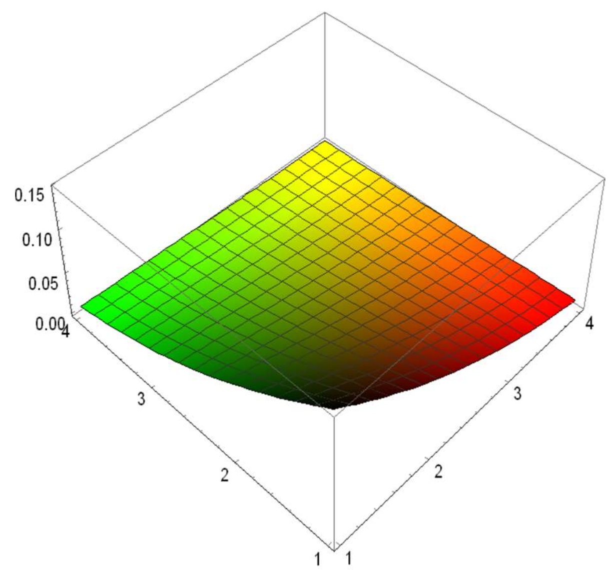
The pdf of the BMWU distribution can be expressed as

$$f(x,y)=\lambda_1\lambda_2(\alpha_1+1)(\alpha_2+1)(1-\lambda_1x)^{\alpha_1}(1-\lambda_2y)^{\alpha_2}$$
$$[1+\theta(1-2(1-\lambda_1x)^{\alpha_1+1})(1-2(1-\lambda_2y)^{\alpha_2+1})] \tag{13}$$

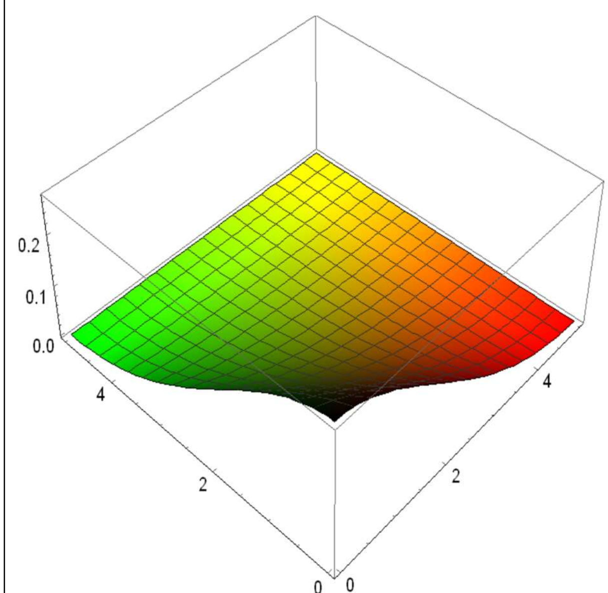
Where $0<x<\frac{1}{\lambda_1}$ and $0<y<\frac{1}{\lambda_2}$

Where $\int_0^\infty\int_0^\infty f(x,y)\,dx\,dy=1$

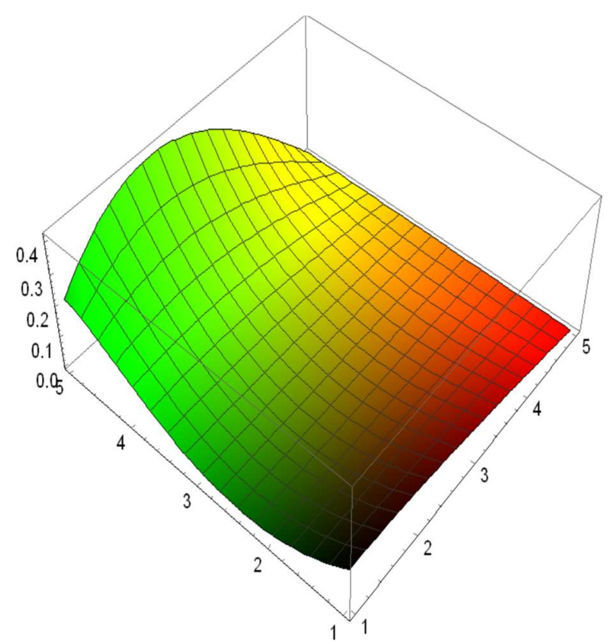
Set (1): (2 , 2 , 0.2 0.2 ,0.25)



Set (2): (2 ,2 , 0.2 0.2 ,-0.25)



Set (3): (1 , 2 , 0.2, 0.4, 0.5)



Set (4) : (1 , 2 , 0.2, 0.4, -0.5)

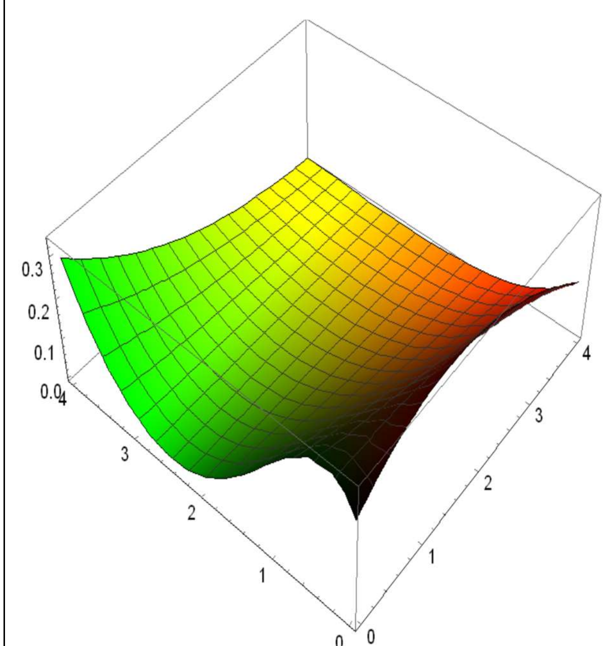


Figure 6: Pdf of FGM bivariate modified weighted uniform distribution with the parameters values $(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta)$

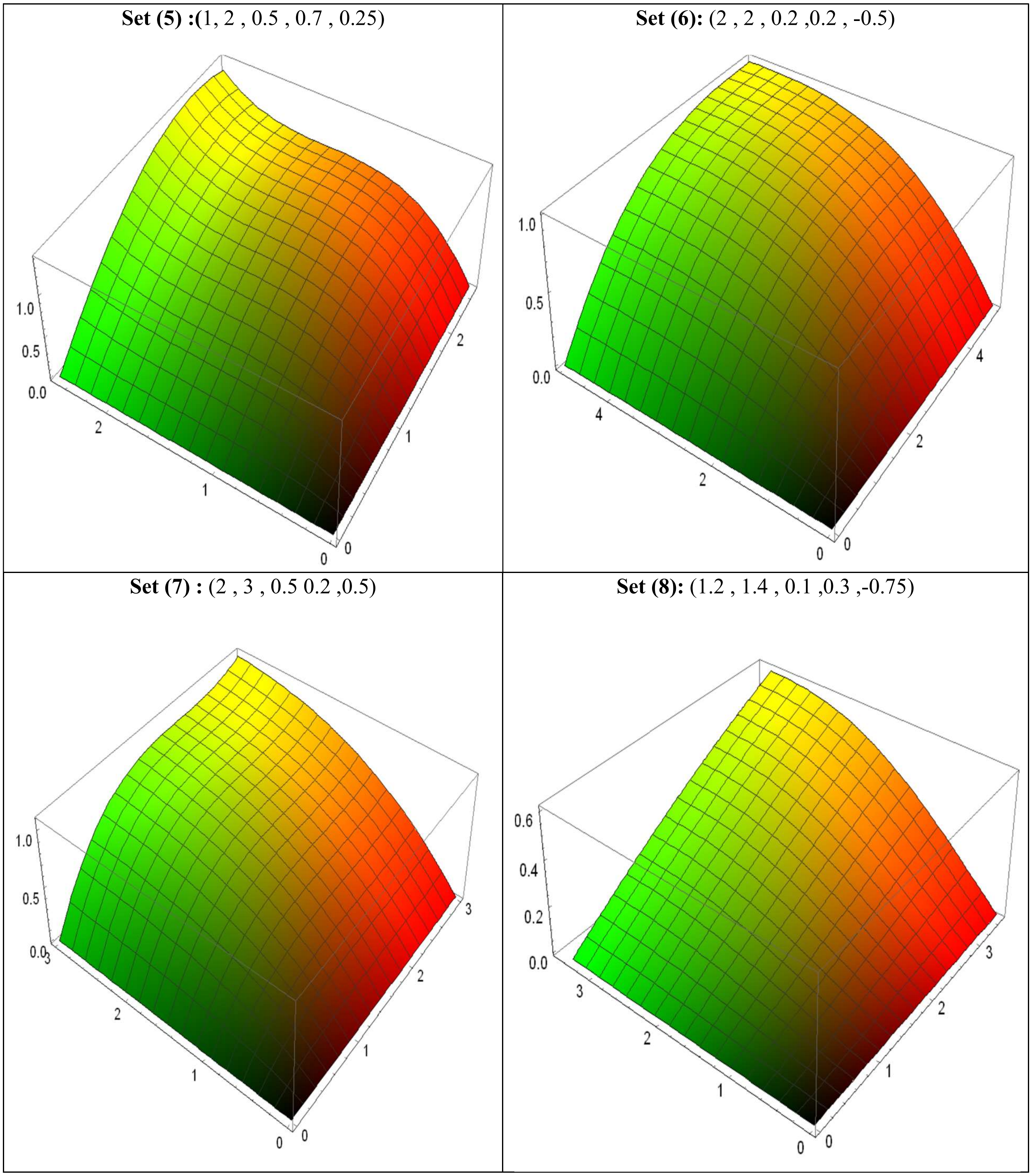


Figure 7: Cdf of FGMB modified weighted uniform distribution with the parameters values $(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta)$

6.1. The Marginal Distribution

Let (X, Y) ~ BMWU distribution $(\alpha_1, \alpha_2 \lambda_1, \lambda_2, \theta)$, Then $X\sim$ MWU distribution (α_1, λ_1) and
$Y\sim$ MWU distribution (α_2, λ_2)

Then $f(x; \alpha_1, \lambda_1) = f_1(x) = \lambda_1\ (\alpha_1 + 1)(1 - \lambda_1 x)^{\alpha_1}$
$f(y; \alpha_2, \lambda_2) = \lambda_2\ (\alpha_2 + 1)(1 - \lambda_2 x)^{\alpha_2}$.
Respectively

6.2. The Conditional Distribution

$\forall y\ \in Y$, Let (X, Y) ~ BMWU distribution $(\alpha_1, \alpha_2 \lambda_1, \lambda_2, \theta)$, Then the conditional distribution of X given $Y = y$ is given by

$$f(X \setminus Y \leq y) = \lambda_1\ (\alpha_1 + 1) c(x, y)(1 - \lambda_1 x)^{\alpha_1}$$

Where $c(x, y) = [1 +\ \theta(1 - 2F_1(x))(1 - 2F_2(y))]$

6.3. The Moment Generating Function

Let (X, Y) denote a random variable with a probability density function

$$f(x, y) = f_1(x) f_2(y)[1 + 4\theta F_1(x) F_2(y) - 2\theta F_1(x) - 2\theta F_2(y) + \theta]$$

Then the moment generating function follows as:

$$\begin{aligned}\mu_{(x,y)}(t_1, t_2) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{{t_1}^n {t_2}^m}{n!\,m!} \frac{(\alpha_1 + 1)(\alpha_2 + 1)}{{\lambda_1}^n {\lambda_2}^m} &[(1 + \theta) B(\,n + 1, \alpha_1 + 1) B(\,m + 1, \alpha_2 + 1) \\ &+ 4\theta[B(\,n + 1, \alpha_1 + 1) - B(\,n + 1,2\alpha_1 + 2)].\,[B(\,m + 1, \alpha_2 + 1) - B(m + 1,2\alpha_2 + 2)\,] \\ &- 2\theta B(\,m + 1, \alpha_2 + 1)\big(B(\,n + 1, \alpha_1 + 1) - B(\,n + 1,2\alpha_1 + 2)\big) \\ &- 2\theta B(\,n + 1, \alpha_1 + 1)(B(\,m + 1, \alpha_2 + 1) - B(m + 1,2\alpha_2 + 2)\,]\end{aligned} \tag{14}$$

6.4. The Product Moment

If the random vector $X =\ (X\ ,\ Y)$ is distributed as BMWU distribution, then it's the r^{th} and s^{th} moments about the zero is

$$\acute{\mu}_{r\,s} = \left[(1+\theta)\frac{(\alpha_1+1)\,B(\,r+1,\alpha_1+1)(\alpha_2+1)\,B(\,s+1,\alpha_2+1)}{{\lambda_1}^r{\lambda_2}^r}\right] + 4\theta\left[\frac{(\alpha_1+1)(\alpha_2+1)[B(\,r+1,\alpha_1+1)-B(\,r+1,2\alpha_1+2)][B(s+1,\alpha_2+1)-B(\,s+1,2\alpha_2+2)]}{{\lambda_1}^r{\lambda_2}^r}\right] - 2\theta\left[\frac{(\alpha_1+1)\,B(\,r+1,\alpha_1+1)(\alpha_2+1)[B(s+1,\alpha_2+1)-B(\,s+1,2\alpha_2+2)]}{{\lambda_1}^r{\lambda_2}^r}\right] - 2\theta\left[\frac{(\alpha_1+1)\,[B(\,r+1,\alpha_1+1)-B(\,r+1,2\alpha_1+2)](\alpha_2+1)\,B(\,s+1,\alpha_2+1)}{{\lambda_1}^r{\lambda_2}^r}\right]$$

If the random vector $X = (X\,,Y)$is distributed as BMWU distribution, then the expectation of $x\ and\ \ y$ is

$$E(xy) = \frac{1}{\lambda_1\,\lambda_2(\alpha_1+2)(\alpha_2+2)}\left[1+\theta+\theta\,\frac{(3\alpha_1+4)(3\alpha_2+4)}{(2\alpha_1+3)(2\alpha_2+3)}-\theta\,\frac{(3\alpha_2+4)}{(2\alpha_2+3)}-\theta\,\frac{(3\alpha_1+4)}{(2\alpha_1+3)}\right]$$

6.5. The Joint Hazard Rate Function

The reliability function is discussed by Osmetti and Chiodini (2011) that is more important to express a joint survival function as a copula of its marginal survival function, If X and Y be a random variables with survival functions $\bar{F}(x)$ and $\bar{F}(y)$ as following
The survival function of the marginal distributions is defined as
The expression of the joint survival function for copula FGM is as follows

$$\bar{F}(x,y) = \bar{C}\big(\bar{F}(x),\bar{F}(y)\big)$$
$$\bar{F}(x,y) = \bar{F}(x)+\bar{F}(y)-1+C(F(x),F(y))$$

The survival function of the marginal distributions is defined as

$$\bar{F}(z;\alpha_i,\lambda_i) = 1-F(z;\alpha_i,\lambda_i) = (1-\lambda_i\,z)^{\alpha_i+1}\quad ;z\epsilon[0,\tfrac{1}{\lambda_i}]\,,\alpha_i,\lambda_i>0, i=1,2\,.$$

The reliability function of FGM modified weighted uniform distribution is

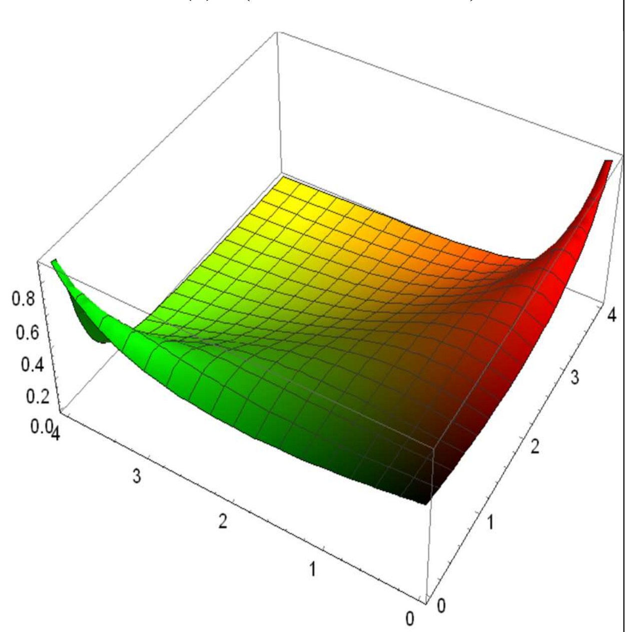
$$\bar{F}(x,y) = (1-\lambda_1\,x)^{\alpha_1+1}+(1-\lambda_2\,y)^{\alpha_2+1}-1 +\big[[1-(1-\lambda_1 x)^{\alpha_1+1}][1-(1-\lambda_2 y)^{\alpha_2+1}][1+\theta(1-\lambda_1 x)^{\alpha_1+1}(1-\lambda_2 y)^{\alpha_2+1}]\big]$$

The joint hazard rate function of FGM modified weighted uniform distribution is

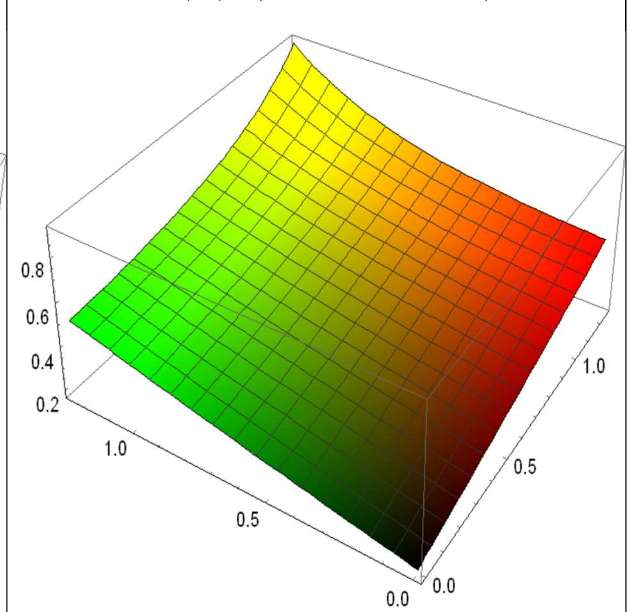
$$h(x,y) = \frac{f(x,y)}{\bar{F}(x,y)}$$

$$h(x,y) = \frac{\lambda_1\lambda_2(\alpha_1+1)(\alpha_2+1)(1-\lambda_1 x)^{\alpha_1}(1-\lambda_2 y)^{\alpha_2}\big[1+\theta\big(1-2(1-\lambda_1 x)^{\alpha_1+1}\big)\big(1-2(1-\lambda_2 y)^{\alpha_2+1}\big)\big]}{(1-\lambda_1\,x)^{\alpha_1+1}+(1-\lambda_2\,y)^{\alpha_2+1}-1+\big[[1-(1-\lambda_1 x)^{\alpha_1+1}][1-(1-\lambda_2 y)^{\alpha_2+1}][1+\theta(1-\lambda_1 x)^{\alpha_1+1}(1-\lambda_2 y)^{\alpha_2+1}]\big]}$$

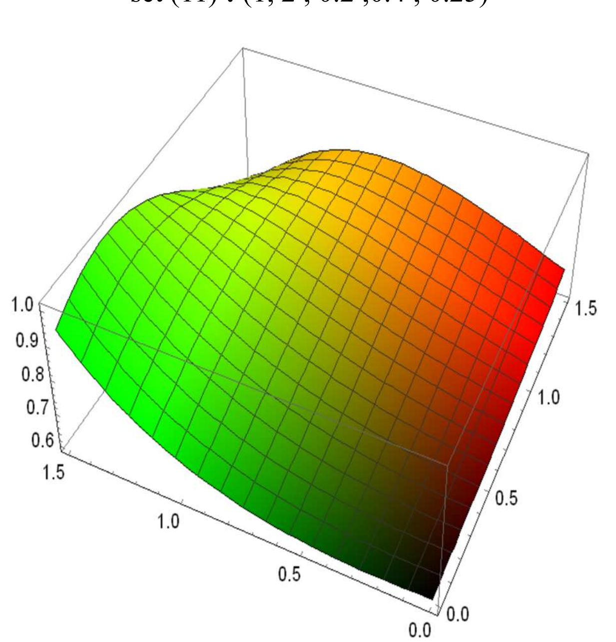
set (9) : (2 , 2 , 0.2 ,0.2 , 0.5)



set (10) : (2 , 2 , 0.2 ,0.2 ,-0.5)



set (11) **:** (1, 2 , 0.2 ,0.4 , 0.25)



set (12) **:** (1, 2 , 0.2 ,0.4 , -0.25)

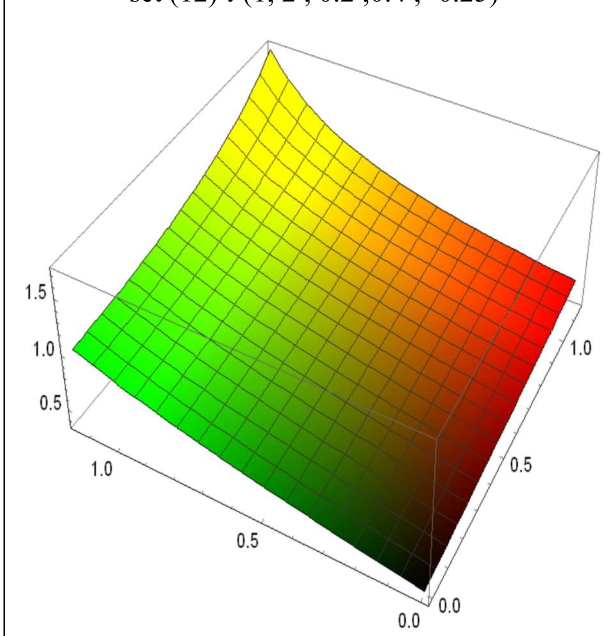


Figure 8: Joint hazard rate of FGM bivariate modified weighted uniform distribution with the parameters values $(\alpha_1, \alpha_2, \lambda_1, \lambda_2, \theta)$.

7. Application to Real Data for BMWU Distribution

In this section, we used the procedures for real data sets Meintanis (2007). For our first example, we apply football (soccer) data. Suppose that matches where (i) there was at least one goal in the match scored by the home team, and (ii) there was at least one goal in the match scored directly from a kick (foul kick, other kick or penalty kick) by any team. Suppose X be the time (in minutes) of the first kick goal scored by any team, and Y be the time (in a minute) of the first goal of any type scored by the home team as shown in Table 9, we divided all data by 100, because of all the data belong to the interval $(0, 1)$, the time of any professional football match is 90 minute. We, therefore, model the proportions of the matching time that any team scored the first kick goal and that home team scored the first goal of any type.

2005-2006	X	Y	2004-2005	X	Y
Lyon–Real Madrid 3–0	26	20	Internazionale–Bremen 2–0	34	34
Milan–Fenerbahce 3–1	63	18	Real Madrid–Roma 4–2	53	39
Chelsea–Anderlecht 1–0	19	19	Man. United–Fenerbahce 6–2	54	7
Club Brugge–Juventus 1–2	66	85	Bayern–Ajax 4–0	51	28
Fenerbahce–PSV 3–0	40	40	Moscow–PSG 2–0	76	64
Internazionale–Rangers 1–0	49	49	Barcelona–Shakhtar 3–0	64	15
Panathinaikos–Bremen 2–1	8	8	Leverkusen–Roma 3–1	26	48
Ajax–Arsenal 1–2	69	71	Arsenal–Panathinaikos 1–1	16	16
Man. United–Benfica 2–1	39	39	Dynamo Kyiv–Real Madrid 2–2	44	13
Real Madrid–Rosenborg 4–1	82	48	Man. United–Sparta 4–1	25	14
Villarreal–Benfica 1–1	72	72	Bayern–M. Tel–Aviv 5–1	55	11
Juventus–Bayern 2–1	66	62	Bremen–Internazionale 1–1	49	49
Club Brugge–Rapid 3–2	25	9	Anderlecht–Valencia 1–2	24	24
Olympiacos–Lyon 1–4	41	3	Panathinaikos–PSV 4–1	44	30
Internazionale–Porto 2–1	16	75	Arsenal–Rosenborg 5–1	42	3
Schalke–PSV 3–0	18	18	Liverpool–Olympiacos 3–1	27	47
Barcelona–Bremen 3–1	22	14	M. Tel–Aviv–Juventus 1–1	28	28
Milan–Schalke 3–2	42	42			
Bremen–Panathinaikos 5–1	2	2			
Rapid–Juventus 1–3	36	52			

Tabel 9: UEFA Champion's League Data.

We compare the BMUR distribution with the bivariate Kumaraswamy distribution based on the maximum likelihood estimates and Akaike information criterion (AIC)

From Table10 , the values of AIC and BIC are less for the MWU distribution Kumaraswamy (Kw) distribution, which reflect the better fit of the MWU distribution to this data.

	X						**Y**					
Distributions	MLE's	K-S	P-	AIC	BIC	-ln L	MLE's	K-S	P-	AIC	BIC	-ln L

		statistic	value					statistic	value			
MWU	$\hat{\lambda} = 1.063$ $\hat{\alpha} = 1.093$	0.15	0.32	-7.22	-5.615	-4.61	$\hat{\lambda} = 1.12$ $\hat{\alpha} = 0.673$	0.08	0.94	-12.914	-9.692	-8.457
Kw	$\hat{\alpha} = 1.64$ $\hat{\lambda} = 2.42$	0.09	0.86	-6.92	-3.70	-5.46	$\hat{\alpha} = 1.14$ $\hat{\lambda} = 2.43$	0.09	0.92	-12.725	-9.503	-8.362

Table 10: Goodness of fit test of MWU distribution

For the bivariate modified weighted uniform (BMWU) distribution. The log-likelihood value and AIC values are founded as (7.145 and -9.64) respectively. For the bivariate Kumaraswamy distribution. The log-likelihood value and AIC values are founded as (5.744 and -7.4889) respectively . Therefore, based on the goodness-of-fit statistic AIC, we observe that the BMWU distribution performs better in this way of discussion.

8. Conclusion

In this paper, the new version of weighted uniform distribution based on modified weighted version of Azzalini's(1985) and its bivariate extension is introduced. we introduced some important statistical properties of this distribution an empirical study was carried out to determine the effect of adding new parameter on the mean, variance, skewness and kurtosis of the distribution. The applications of the modified distribution has been demonstrated using real life data. Estimation of parameters is done using maximum likelihood estimation .

We introduced the bivariate extension of the new distribution named the bivariate modified uniform distribution BMWU distribution is also introduced. The proposed bivariate distribution is of type Farlie–Gumbel–Morgenstern (FGM) copula. The BMWU distribution has modified weighted uniform marginal distributions. The joint cumulative distribution function, the joint survival function, the joint probability density function, the joint hazard rate function and the statistical properties of the BMWU distribution are discussed.

Acknowledgment

The authors would like to appreciate the anonymous reviewers for providing useful comments and suggestions for the improvement of the work.

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