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The Gompertz-Topp-Leone-G Family of Distributions with Applications

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ABSTRACT

We develop a new family of distributions called the Marshall-Olkin-Type II-Topp-Leone-G (MO-TII-TL-G) distribution, which is an infinite linear combination of the exponential-G family of distributions. The statistical properties of the new distributions are studied and its model parameters are estimated using the maximum likelihood method. A simulation study is carried out to determine the performance of the maximum likelihood estimates and lastly, real data examples are provided to demonstrate the usefulness of the proposed model in comparison to several other models.

Keywords: Gompertz-G Distribution, Topp-Leone-G Distribution, Maximum Likelihood Estimation, Exponentiated-G Distribution.

Mathematics Subject Classifications: 62E99; 60E05

1. Introduction

There are various lifetime distributions that are discussed in the literature on statistical modelling. There are various lifetime distributions that are discussed in the literature on statistical modelling. Researchers are developing new generalized and extended distributions either by adding parameters to the existing ones or by combining well known distributions. These e orts attempt to extend existing classical distributions in order to enhance their goodness-of-fit and achieve more versatility in modelling data

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The Gompertz distribution (Gompertz [10]) is one of the distributions that have found applications in different areas including but not limited to biology and marketing scienceLately, several authors introduced extensions of the Gompertz distribution so that the new extensions are more flexible and can be used to model different real data sets. Generalizations of the Gompertz include beta-Gompertz distribution by Jafari et al. [11], odd generalized exponential Gompertz distribution by El-Damcese et al. [7], Gompertz-G family of distributions by Alizadeh et al. [2], Transmuted Gompertz-G family of distributions by Reyad et al. [17], Marshall-Olkin exponential Gompertz distribution by Khaleel et al. [12] and Marshall-Olkin Gompertz-G family of distributions by Chipepa and Oluyede [6].

The Gompertz-G (Gom-G) family of distributions has cumulative distribution function (cdf) and probability distribution function (pdf) given by

$$F(x;\lambda,\gamma,\zeta)=1-\exp\left\{\frac{\lambda}{\gamma}[1-(1-G(x;\zeta))^{-\gamma}]\right\}$$

and

$$f(x;\lambda,\gamma,\zeta)=\lambda g(x;\zeta)(1-G(x;\zeta))^{-\gamma-1}\exp\left\{\frac{\lambda}{\gamma}[1-(1-G(x;\zeta))^{-\gamma}]\right\},$$

respectively, for $\lambda,\gamma>0$, where ζ is the parameter vector from the baseline distribution. In this note, we take $\lambda=1$.

The Topp-Leone (TL) distribution plays a key role in modeling lifetime data in areas such as insurance and finance. The distribution was proposed by Topp and Leone [20]. It is has a bounded J-shape that has attracted several statisticians. Al-Shomrani et al. [1] proposed the Topp-Leone generated family of distributions with cdf and pdf given by

$$F(x;b,\zeta)=[1-\bar{G}(x;\zeta)^2]^b$$

and

$$f(x;b,\zeta)=2bg(x;\zeta)\bar{G}(x;\zeta)[1-\bar{G}(x;\zeta)^2]^{b-1},$$

respectively, for $b>0$, where $G(x;\zeta)$ is the baseline cdf depending on a parameter vector ζ.

We are motivated by the usefulness of the Gompertz-G and Topp-Leone-G family of distributions to propose a new family of distributions which is a combination of these two distributions. The general objectives of developing the new family of distributions are; to obtain special models which exhibits monotonic and non-monotonic hazard rate functions, to construct heavy-tailed distributions for modelling various real data sets and to provide consistently better fits than other generalized distributions with the same underlying model.

The paper is organized as follows. We develop the new family of distributions and provide expansion of the density function in Section 2. We present some of the special cases of the Gompertz-Topp-Leone-G (Gom-TL-G) family of distributions in Section 3. Section 4 contains the statistical properties of the Gom-TL-G family of distributions. Monte Carlo simulation results are given in Section 5. Applications of the proposed model to real data sets are given in Section 6 and concluding remarks in Section 7.

2. The Gom-TL-G Model

We introduce the Gom-TL-G family of distributions using the generalization by Alizadeh et al. [2], and taking the baseline distribution to be the Topp-Leone-G family of distributions. Therefore, the cdf and pdf of the Gom-TL-G family of distributions is given by

$$F(x;\gamma,b,\zeta) = 1-\exp\left\{\tfrac{1}{\gamma}[1-(1-[1-\bar{G}^2(x;\zeta)]^b)^{-\gamma}]\right\} \tag{1}$$

and

$$\begin{aligned} f(x;\gamma,b,\zeta) &= 2bg(x;\zeta)\bar{G}(x;\zeta)[1-\bar{G}^2(x;\zeta)]^{b-1}(1-[1-\bar{G}^2(x;\zeta)]^b)^{-\gamma-1} \\ &\times \exp\left\{\tfrac{1}{\gamma}[1-(1-[1-\bar{G}^2(x;\zeta)]^b)^{-\gamma}]\right\}, \end{aligned} \tag{2}$$

respectively, for $\gamma, b > 0$ and ζ is the parameter vector. The hazard rate function (hrf) is given by

$$f(x;\gamma,b,\zeta) = 2bg(x;\zeta)\bar{G}(x;\zeta)[1-\bar{G}^2(x;\zeta)]^{b-1}(1-[1-\bar{G}^2(x;\zeta)]^b)^{-\gamma-1}.$$

1.1 Expansion of Density Function

We express the pdf of Gom-TL-G family as a mixture of an Exponentiated-G (Exp-G) distribution which is useful in presenting the mathematical properties of the Gom-TL-G family. The pdf in equation (2) can be expressed as follows

$$\begin{aligned} f(x;\gamma,b,\zeta) = &\sum_{k,l,m,p,q=0}^{\infty} \frac{(-1)^{l+m+p+q}1^k 2b}{\gamma^k k!\,(q+1)}\binom{k}{l}\binom{-\gamma(l+1)-1}{m} \\ &\times \binom{b(m+1)-1}{p}\binom{2p+1}{q}(q+1)g(x;\zeta)G^q(x;\zeta) \\ &= \sum_{q=0}^{\infty} v_{q+1}\, g_{q+1}(x;\zeta), \end{aligned}$$

where

$$\begin{aligned} v_{q+1} &= \sum_{k,l,m,p=0}^{\infty} \frac{(-1)^{l+m+p+q}1^k 2b}{\gamma^k k!\,(q+1)}\binom{k}{l}\binom{-\gamma(l+1)-1}{m} \\ &\times \binom{b(m+1)-1}{p}\binom{2p+1}{q}, \end{aligned}$$

and $g_{p+1}(x;\zeta) = (q+1)g(x;\zeta)G^q(x;\zeta)$ is an Exp-G distribution with power parameter $(q+1)$. Therefore, the pdf of the Gom-TL-G family of distributions can be expressed as an infinite linear combination of the Exp-G distributions. See the appendix for the derivations.

2. Some Special Cases

We present three special cases of the Gom-TL-G family of distributions in this section. We consider cases when the baseline distributions are Burr XII, Weibull and Lindley distributions.

2.1 Gompertz-Topp-Leone-Burr XII (Gom-TL-BXII) Distribution

If we take the Burr XII distribution as the baseline distribution, with cdf and pdf $G(x;c,k) = 1-(1+x^c)^{-k}$ and $g(x;c,k) = kcx^{c-1}(1+x^c)^{-k-1}$, respectively, for $c,k>0$, we obtain the Gom-TL-BXII distribution with cdf and pdf given by

$$F(x;\gamma,b,c,k) \quad = \quad 1-\exp\left\{\frac{1}{\gamma}[1-(1-[1-(1+x^c)^{-2k}]^b)^{-\gamma}]\right\}$$

and

$$\begin{aligned} f(x;\gamma,b,c,k) &= 2bkcx^{a-1}(1+x^c)^{-2(k+1)}[1-(1+x^c)^{-2k}]^{b-1} \\ &\times \ (1-[1-(1+x^c)^{-2k}]^b)^{-\gamma-1} \\ &\times \ \exp\left\{\frac{1}{\gamma}[1-(1-[1-(1+x^c)^{-2k}]^b)^{-\gamma}]\right\}, \end{aligned}$$

respectively for $\gamma,b,c,k>0$. When $k=1$, we obtain the Gompertz-Topp-Leone-Log-logistic (Gom-TL-LLoG) distribution. When $c=1$, we obtain the Gompertz-Topp-Leone-Lomax (Gom-TL-Lomax) distribution. The pdf of the Gom-TL-BXII distribution can handle data that is left or right-skewed and symmetric shaped. Also, the hrf of the distribution exhibit increasing, decreasing, bathtub, J and reverse-J shapes.

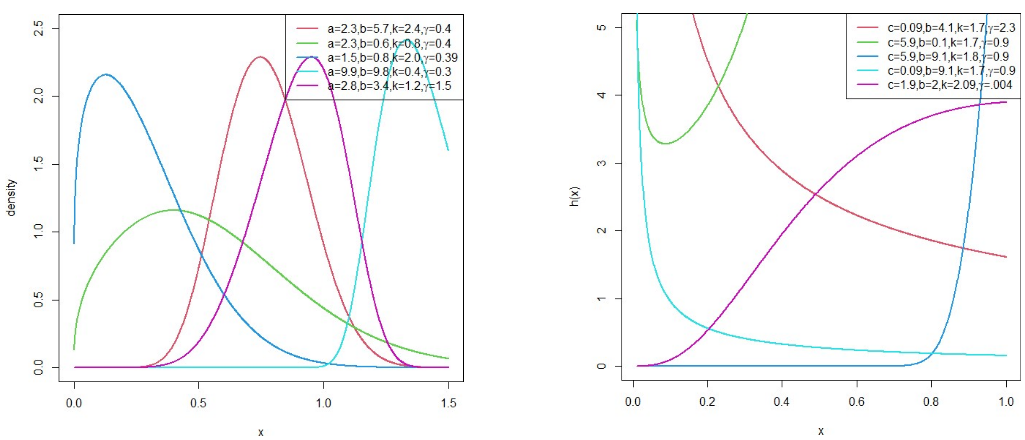


Figure 1: Plots of the pdf and hrf for the Gom-TL-BXII distribution

2.2 Gompertz-Topp-Leone-Weibull (Gom-TL-W) Distribution

Consider the weibull distribution with cdf and pdf given by $G(x;a) = 1-e^{-\lambda x^a}$ and $g(x;a) = \lambda a x^{a-1}e^{-\lambda x^a}$, for $a,\lambda>0$ respectively, as the baseline distribution. The cdf and pdf of the Gom-TL-W distribution are given by

$$F(x;\gamma,b,\lambda,a) \quad = \quad 1-\exp\left\{\frac{1}{\gamma}[1-(1-[1-e^{-2\lambda x^a}]^b)^{-\gamma}]\right\}$$

and

$$\begin{aligned} f(x;\gamma,b,\lambda,a) &= 2b\lambda a x^{a-1} e^{-2\lambda x^{a}} [1-e^{-2\lambda x^{a}}]^{b-1} (1-[1-e^{-2\lambda x^{a}}]^{b})^{-\gamma-1} \\ &\times \exp\left\{\tfrac{1}{\gamma}[1-(1-[1-e^{-2\lambda x^{a}}]^{b})^{-\gamma}]\right\}, \end{aligned}$$

respectively, for $\gamma, b, \lambda, a > 0$. The pdf of the Gom-TL-W distribution can handle data that has decreasing, left or right-skewed and almost symmetric shapes. Also, the hrf of the distribution exhibit increasing, decreasing, bathtub, J and reverse-J shapes.

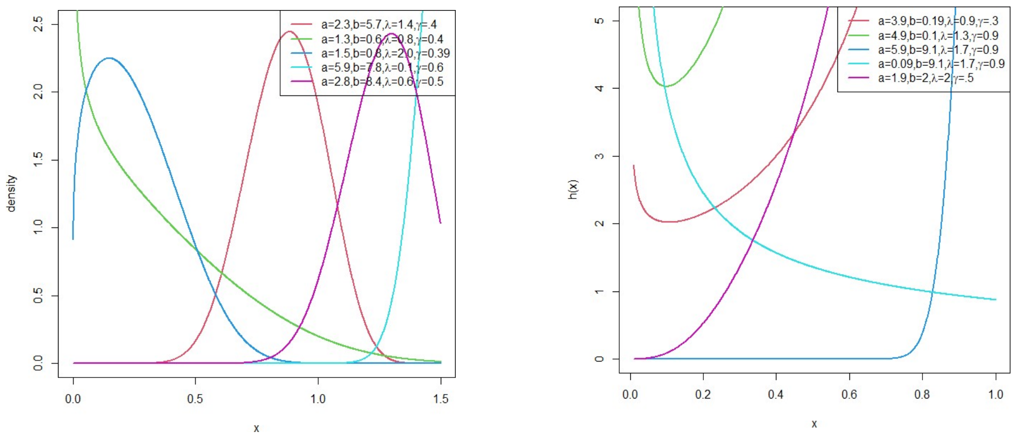


Figure 2: Plots of the pdf and hrf for the Gom-TL-W distribution

2.3 Gompertz-Topp-Leone-Lindley (Gom-TL-L) Distribution

If we take the baseline distribution to be the Lindley distribution with cdf and pdf $G(x;\lambda) = 1 - e^{-\lambda}\ (1+\frac{\lambda x}{1+\lambda})$ and $g(x;\lambda) = \lambda^2 \frac{e^{-\lambda}}{1+\lambda}(1+x)$, for $\lambda > 0$, respectively, we obtain the Gom-TL-L distribution with cdf and pdf

$$F(x;\gamma,b,\lambda) = 1 - \exp\left\{\tfrac{1}{\gamma}\left[\left(1-\left[1-\left(e^{-\lambda x}\left(1+\tfrac{\lambda x}{1+\lambda}\right)\right)^{2}\right]^{b}\right)^{-\gamma}\right]\right\}$$

and

$$\begin{aligned} f(x;\gamma,b,\lambda) &= 2b\lambda^{2}\tfrac{e^{-2\lambda x}}{1+\lambda}(1+x)(1+\tfrac{\lambda x}{1+\lambda})[1-(e^{-\lambda x}(1+\tfrac{\lambda x}{1+\lambda}))^{2}]^{b-1} \\ &\times (1-[1-\left(e^{-\lambda x}\left(1+\tfrac{\lambda x}{1+\lambda}\right)\right)^{2}\Big])^{-\gamma-1} \\ &\times \exp\left\{\tfrac{1}{\gamma}\left[\left(1-\left[1-\left(e^{-\lambda}\ \left(1+\tfrac{\lambda x}{1+\lambda}\right)\right)^{2}\right]^{b}\right)^{-\gamma}\right]\right\}, \end{aligned}$$

respectively, for $\gamma, b, \lambda > 0$. The pdf of the Gom-TL-L distribution exhibits symmetric, left or right-skewed, J and reverse-J shapes. The hrf can handle data that is increasing, decreasing, bathtub and J-shaped.

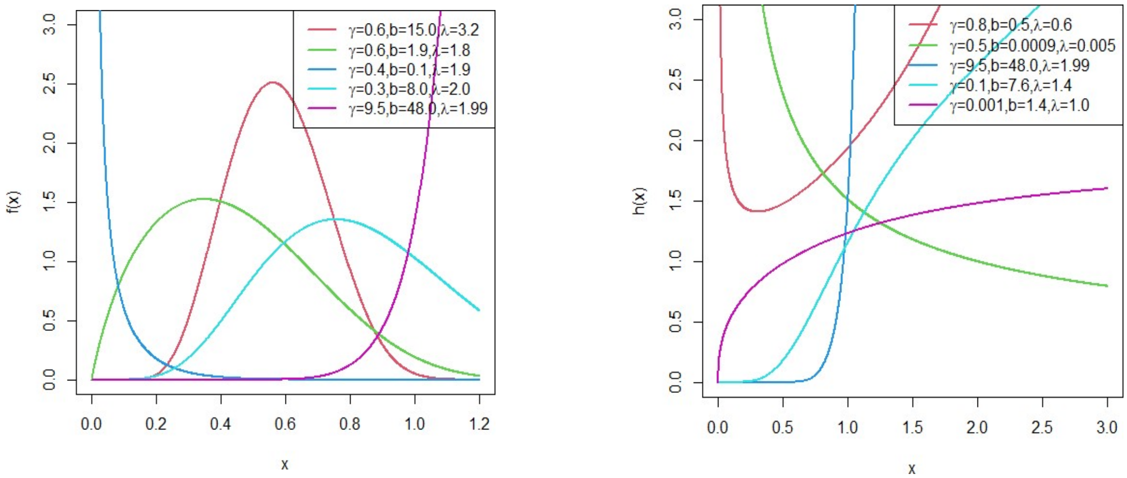


Figure 3: Plots of the pdf and hrf for the Gom-TL-W distribution

3. Statistical Properties

In this section, we provide some statistical properties of the Gom-TL-G family of distributions including the distribution of the i^{th} order statistics, Rényi entropy, moments and the quantile function.

4.1 Distribution of Order Statistics

Let $X_1, X_2, \ldots, X_n$ be random variables from the Gom-TL-G family of distributions and suppose that $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$ denotes the corresponding order statistics. The pdf of the i^{th} order statistic is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!\,(n-i)!} \sum_{k,l,p,q,r,s,z=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{j+k+q+r+s+z} 2b(k+1)^p}{\gamma^p p!\,(z+1)} \binom{n-i}{j} \\ &\times \binom{j+i-1}{k} \binom{p}{q} \binom{-\gamma(q+1)-1}{r} \binom{b(r+1)-1}{s} \\ &\times \binom{2s+1}{z} (z+1) g(x;\zeta) G^z(x;\zeta) \\ &= \sum_{z=0}^{\infty} \eta_{z+1}\, g_{z+1}(x;\zeta), \end{aligned}$$

where $g_{z+1}(x;\zeta) = (z+1)g(x;\zeta)G^z(x;\zeta)$ is an Exp-G distribution with power parameter $(z+1)$ and linear component

$$
\begin{aligned}
\eta_{z+1} \quad = & \quad \frac{i!}{(i-1)!\,(n-i)!} \sum_{k,l,p,q,r,s=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{j+k+q+r+s+z} 2b}{\gamma^p p!\,(z+1)} \binom{n-i}{j} \\
\times & \quad \binom{j+i-1}{k} \binom{p}{q} \binom{-\gamma(q+1)-1}{r} \binom{b(r+1)-1}{s} \binom{2s+1}{z}.
\end{aligned}
$$

Visit the appendix for derivations.

4.2 Rényi Entropy

Rényi entropy (Rényi[16]) is an extension of Shannon entropy (Shannon[19]). Rényi entropy of the Gom-TL-G family of distributions is given by

$$
I_R(v) = \frac{1}{1-v} \log\left(\sum_{p=0}^{\infty} \phi_{p+1}\, e^{(1-v) I_{REG}} \right),
$$

where

$$
I_{REG} = \frac{1}{1-v} \log\left(\int_0^\infty \left(\left[1+\frac{p}{v}\right] g(x;\zeta) G^{\frac{p}{v}}(x;\zeta) \right)^v dx \right),
$$

is the Rényi entropy of the Exp-G distribution with power parameter $\left[1+\frac{p}{v}\right]$ and

$$
\begin{aligned}
\phi_{p+1} \quad = & \quad \sum_{i,j,k,l=0}^{\infty} \frac{(-1)^{j+k+l+m} (2b)^v v^i}{\gamma^i i!} \binom{i}{j} \binom{-\gamma(v+j)-v}{k} \\
\times & \quad \binom{b(k+v)-v}{l} \binom{2l+v}{p}.
\end{aligned}
$$

Therefore, Rényi entropy of the Gom-TL-G family of distributions can be readily derived from Rényi entropy of the Exp-G distribution. See the appendix for derivations.

4.3 Moments and Generating Function

Let $X \sim$ Gom-TL-G(γ, b, ζ) and Y follow an Exp-G distribution with power parameter $(q+1)$. The r^{th} ordinary moment of the Gom-TL-G family of distributions is given as

$$
E(X^r) \sum_{q=0}^{\infty} \nu_{q+1}\, E(Y^r),
$$

where ν_{q+1} is as given in equation (3) and $E(Y^r)$ is the r^{th} moment of the Exp-G distribution. The r^{th} incomplete moment of X is given by

$$
\phi_r(z) = \int_{-\infty}^{z} x^r f(x; b, \gamma, \zeta) dx = \sum_{q=0}^{\infty} \nu_{q+1} \int_{-\infty}^{z} x^r g_q(x;\zeta) dx.
$$

The moment generating function (mgf) of X is given as

$$M_X(t) = E(e^{tX}) = \sum_{q=0}^{\infty} \nu_{p+1}\, M_Y(t),$$

where $M_Y(t)$ is the mgf of the Exp-G distribution.

Figures 4 and 5 shows the 3D plots of skewness and kurtosis for the Gompertz-Topp-Leone-Log-Logistic (Gom-TL-LLoG) and Gompertz-Topp-Leone-Lindley (Gom-TL-L) distributions. As observed from the figures, the Gom-TL-LLoG distribution can handle various levels of skewness and kurtosis when we fix different parameters.

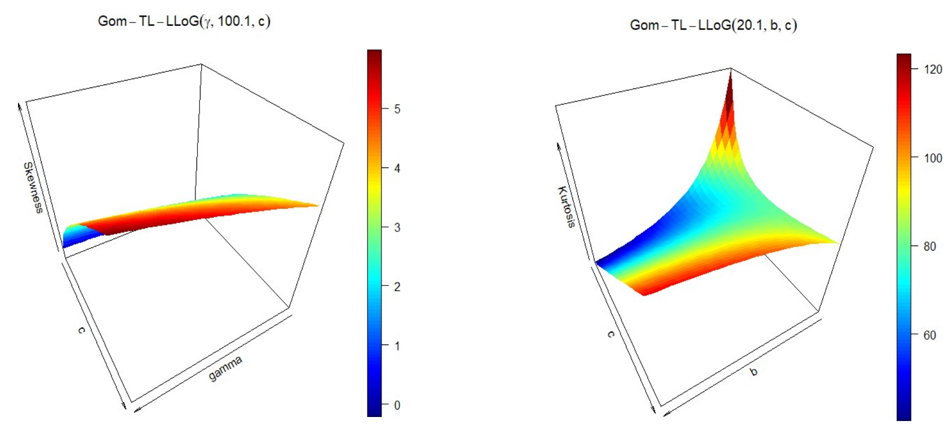


Figure 4: Plots of skewness and kurtosis for the Gom-TL-LLoG distribution

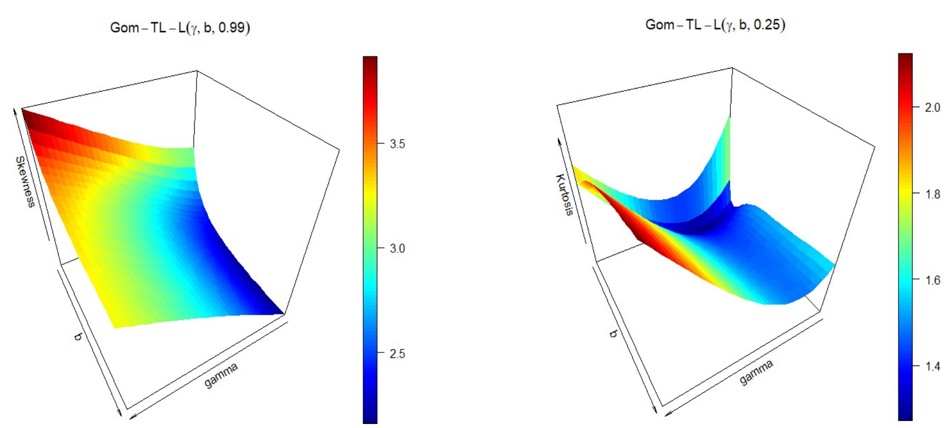


Figure 5: Plots of skewness and kurtosis for the Gom-TL-L distribution

4.3.1 Probability Weighted Moments

The probability weighted moments (PWMs) of the GOM-TL-G family of distributions is defined as

$$\vartheta_{j,i} = E(X^j F(X)^i) = \int_{-\infty}^{\infty} x^j\, f(x)[F(x)]^i dx.$$

Using similar expansions from Section 4.1, we can write

$$\begin{aligned} f(x)[F(x)]^i &= \sum_{k,l,p,q,r,s,z=0}^{\infty} \frac{(-1)^{k+q+r+s+z} 2b(k+1)^p}{\gamma^p p!} \binom{i}{k}\binom{p}{q} \\ &\times \binom{-\gamma(q+1)-1}{r}\binom{b(r+1)-1}{s}\binom{2s+1}{z} g(x;\zeta)G^z(x;\zeta), \end{aligned}$$

which can be written as

$$f(x)F(x)^i = \sum_{z=0}^{\infty} \rho_{z+1}\, g_{z+1}(x;\zeta),$$

where

$$\begin{aligned} \rho_{z+1} &= \sum_{k,l,p,q,r,s=0}^{\infty} \frac{(-1)^{j+k+q+r+s+z} 2b(k+1)^p}{\gamma^p p!\,(z+1)} \\ &\times \binom{i}{k}\binom{p}{q}\binom{-\gamma(q+1)-1}{r}\binom{b(r+1)-1}{s}\binom{2s+1}{z}. \end{aligned}$$

As such, the PWM of the Gom-TL-G family of distributions is given by

$$\vartheta_{j,i} = \sum_{z=0}^{\infty} \rho_{z+1} \int_{-\infty}^{\infty} x^i\, g_{z+1}(x;\zeta)dx = \sum_{z=0}^{\infty} \rho_{z+1}\, E(Y^j),$$

where $E(Y^j)$ is the j^{th} power of an Exp-G distribution with power parameter $(z+1)$.

4.4 Quantile Function

We invert the cdf of the Gom-TL-G family of distributions to obtain the quantile function. The quantile function for the Gom-TL-G family of distributions is given by

$$Q_X(u) = G^{-1}\left[1-(1-[1-(1-\gamma\ln(1-u))^{1/\gamma}]^{1/b})^{1/2}\right].$$

Visit the appendix for the derivations.

4. Maximum Likelihood Estimation

Let $X_i \sim$ Gom-TL-G(γ, b, ζ) and $\Delta = (\gamma, b, \zeta)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\ell(\Delta) = n\log(2b) + \sum_{i=1}^{n} \ln[g(x_i;\zeta)] + \sum_{i=1}^{n} \ln[\bar{G}(x_i;\zeta)] + (b-1)\sum_{i=1}^{n} \ln[1-\bar{G}^2(x_i;\zeta)]$$
$$- (\gamma+1)\sum_{i=1}^{n} \ln[1-[1-\bar{G}^2(x_i;\zeta)]^b] + \sum_{i=1}^{n} \frac{1}{\gamma}[1-(1-[1-\bar{G}^2(x_i;\zeta)]^b)^{-\gamma}].$$

The elements of the score vector are given in the appendix. The maximum likelihood estimates of the parameters, are determined by solving the non-linear equation $\left(\frac{\partial\ell}{\partial\gamma}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\zeta_k}\right) = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution $N_{q+2}(\mathbf{0}, J(\widehat{\boldsymbol{\Delta}})^{-1})$, where the mean vector $\mathbf{0} = (0,0,\underline{0})^T$ and $J(\widehat{\boldsymbol{\Delta}})^{-1}$is the observed Fisher information matrix evaluated at $\widehat{\boldsymbol{\Delta}}$ can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

5. Simulation Study

The consistency of the maximum likelihood estimates is examined is this section by conducting a simulation study for the Gom-TL-LLoG model. We simulated for the sample sizes n=25, 50, 100, 200, 400, 800 and 1000 for N=1000 for each sample. If the model performs better, we expect the mean to approximate the true parameter values, the Root Mean Square Error (RMSE) to decrease and the average bias to decay toward zero for an increase in sample size. From the results shown in Table 1, the mean values approximate the true parameter values, RMSE decreases and bias decay towards zero for all the parameter values with increasing sample size.

Table 1: Monte Carlo Simulation Results for Gom-TL-LLoG Distribution:Mean, RMSE and Average Bias

		$\gamma = 1.0, b = 1.0, c = 1.5$			$\gamma = 0.1, b = 0.9, c = 0.9$		
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
γ	25	2.4420	4.7691	1.4420	2.0594	6.1418	1.9594
	50	1.7943	2.2077	0.7943	0.8825	1.6980	0.7825
	100	1.6588	2.2381	0.6588	0.5146	0.8118	0.4146
	200	1.5650	1.4388	0.5650	0.3340	0.4574	0.2340
	400	1.3386	1.0098	0.3386	0.2027	0.1857	0.1027
	800	1.2483	0.8553	0.2483	0.1534	0.1248	0.0534
	1000	1.2043	0.7306	0.2043	0.1386	0.0959	0.0386
b	25	1.5296	1.1846	0.5296	1.8725	1.5882	0.9725
	50	1.3477	0.8758	0.3477	1.4851	0.9772	0.5851
	100	1.2773	0.8461	0.2773	1.2833	0.6381	0.3833
	200	1.2752	0.7667	0.2752	1.1479	0.4272	0.2479
	400	1.1696	0.5981	0.1696	1.0268	0.2244	0.1268
	800	1.1224	0.5232	0.1224	0.9686	0.1567	0.0686
	1000	1.1031	0.4668	0.1031	0.9527	0.1228	0.0527

c	25	1.6006	1.4186	0.1006	0.6161	0.4148	-0.2839
	50	1.5554	0.9855	0.0554	0.6860	0.3349	-0.2140
	100	1.6294	0.9612	0.1294	0.7340	0.2696	-0.1660
	200	1.5859	0.8821	0.0859	0.7757	0.2136	-0.1243
	400	1.5895	0.7252	0.0895	0.8253	0.1449	-0.0747
	800	1.5792	0.6155	0.0792	0.8582	0.1111	-0.0418
	1000	1.5694	0.5682	0.0694	0.8675	0.0929	-0.0325
		$\gamma = 1.5, b = 1.1, c = 1.5$			$\gamma = 0.3, b = 0.7, c = 0.7$		
Parameter	n	Mean	RMSE	Bias	Mean	RMSE	Bias
γ	25	2.6994	5.1845	1.1994	1.5159	2.3441	1.2159
	50	2.1332	2.1682	0.6332	1.1071	1.3022	0.8071
	100	2.0676	2.1983	0.5676	0.9407	1.0320	0.6407
	200	1.8996	1.2653	0.3996	0.7658	0.7301	0.4658
	400	1.7683	1.0095	0.2683	0.6678	0.5420	0.3678
	800	1.6793	0.8579	0.1793	0.5341	0.2598	0.2341
	1000	1.6684	0.7913	0.1684	0.5182	0.2238	0.2182
b	25	1.4570	1.0134	0.3570	1.4538	1.1349	0.7538
	50	1.3171	0.7757	0.2171	1.2776	0.8524	0.5776
	100	1.2936	0.7707	0.1936	1.1932	0.7232	0.4932
	200	1.2537	0.6535	0.1537	1.0977	0.5773	0.3977
	400	1.2050	0.5442	0.1050	1.0293	0.4630	0.3293
	800	1.1638	0.4836	0.0638	0.9390	0.2531	0.2390
	1000	1.1634	0.4482	0.0634	0.8984	0.2081	0.1984
c	25	1.8362	1.8039	0.3362	0.5133	0.3976	-0.1867
	50	1.7942	1.4149	0.2942	0.5252	0.3088	-0.1748
	100	1.7491	1.1559	0.2491	0.5307	0.2611	-0.1693
	200	1.6920	0.9223	0.1920	0.5572	0.2491	-0.1428
	400	1.6356	0.7130	0.1356	0.5682	0.2234	-0.1318
	800	1.6276	0.6087	0.1276	0.5674	0.1435	-0.1326
	1000	1.5950	0.5372	0.0950	0.5832	0.1224	-0.1168

6. Applications

In this section, we present real data examples to demonstrate the usefulness of the Gom-TL-LLoG distribution. The proposed model is compared to various competing non-nested models. Different goodness-of-fit statistics are used to examine the performance of the model and these are: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramér-von Mises (W^*), Andersen-Darling (A^*) (Chen and Balakrishnan [4]), Kolmogorov-Smirnov (K-S) and its p-value. The model that has the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic is deemed as the best fitting model.

The maximum likelihood estimation technique using the nlm package in R software was used to estimate model parameters. Tables 2, 3, 4 and 5 contains parameter estimates of the model alongside their standard errors which are shown in parenthesis, as well as the goodness-of-fit-statistics for the two data sets. Additionally, fitted Empirical Probability Distribution Function (EPDF), Probability plots, Kaplan-Meier, Empirical Cumulativ Distribution function (ECDF), Total Time on Test (TTT) and Hazard Rate Funtion (HRF) plots are presented to show how the Gom-TL-LLoG model fits the selected data sets. The measure of closeness of the probability plots to the diagonal line was determined and is given by the sum of squares

$$SS = \sum_{i=1}^{n} \left[F\left(x_{(i)}; \hat{\gamma}, \hat{b}, \hat{\zeta}\right) - \left(\frac{i - 0.375}{n + 0.25}\right) \right]^2.$$

The non-nested models considered in this paper are the Kumaraswamy inverse Gompertz (KuIG) by El-Morshedy et al. [9], generalized Gompertz (GG) by El-Gohary et al. [8], Topp-Leone Lomax (TLLo) by Oguntunde et al. [15], Topp-Leone generalized exponential (TL-GE) by Sangsanit and Bodhisuwan [18], Topp-Leone-Marshall-Olkin-log logistic (TL-MO-LLo) by Chipepa et al. [5] and alpha power Weibull (APW) (Nassar et al. [14]) distributions. The pdfs of the non-nested models are:

$$f_{KuIG}(x; \alpha, \beta, \gamma) = \frac{\alpha\gamma}{x^2} e^{\frac{\beta}{x}} e^{-\frac{\alpha}{\beta} e^{\frac{\beta}{x}-1}} (1 - e^{-\frac{\alpha}{\beta} e^{\frac{\beta}{x}-1}})^{\gamma-1},$$

for $\alpha, \beta, \gamma > 0$,

$$f_{GG}(x; \lambda, c, \theta) = \theta\lambda e^{-\frac{\lambda}{c}(e^{cx}-1)}$$

for $\lambda, c, \theta > 0$,

$$f_{TLLo}(x; a, b, c) = 2abc(1 + cx)^{-(2b+1)}[1 - (1 + cx)^{-2b}]^{a-1},$$

for $a, b, c > 0$,

$$f_{TL-MO-LLo}(x; b, \delta, c) = \frac{2b\delta^2 cx^{c-1}(1 + x^c)^{-3}}{[1 - \bar{\delta}(1 + x^c)^{-1}]^3} [1 - \frac{\delta^2(1 + x^c)^{-2}}{[1 - \bar{\delta}(1 + x^c)^{-1}]^2}]^{b-1},$$

for $b, \delta, c > 0$,

$$f_{TL-GE}(x; \alpha, \beta, \lambda) = 2\alpha\beta\lambda e^{-\lambda x}(1 - (1 - e^{-\lambda x})^{\beta}(1 - e^{-\lambda x})^{\beta\alpha-1}(2 - (1 - e^{-\lambda x})^{\beta}))^{\alpha-1},$$

for $\alpha, \beta, \lambda > 0$, and

$$f_{APW}(x; \alpha, \beta, \theta) = \frac{\log(\alpha)}{(\alpha - 1)} \beta\theta x^{\beta-1} e^{-\theta x^{\beta}} \alpha^{1-e^{-\theta x^{\beta}}},$$

for $\alpha, \beta, \theta > 0$.

7.1 Survival Times Data

The first data represents survival times in weeks for male rats that were exposed to a high level of radiation. The data was reported by Lawless [13] and the observations are: 40, 62, 69, 77, 83, 88, 94, 101, 109, 115, 123, 125, 128, 136, 137, 152, 152, 153, 160, 165.

Table 2 Parameter estimates for various models fitted for survival times data set

Model	γ	b	c
Gom-TL-LLoG	10.7072 (5.1081)	182.8135 (0.1988)	0.4695 (0.0227)
	α	β	γ
KulG	348.6700 (75.6200)	5.1950×10^{-6} (0.0868)	17.9060 (11.3840)
	λ	c	θ
GG	0.0275 (5.6764×10^{-3})	2.2161×10^{-6} (3.2946×10^{-3})	12.9460 (9.7046×10^{-6})
	a	b	c
TLLo	13.0860 (2.1272×10^{-11})	142.3300 (5.7970×10^{-12})	9.7644×10^{-5} (8.3970×10^{-6})
	α	β	λ
TL-GE	0.2424 (0.0916)	43.0850 (4.3233×10^{-4})	0.0234 (3.0587×10^{-3})
	b	δ	c
TL-MO-LLo	1.1568 (0.3632)	7.2778×10^{7} (3.6089×10^{-10})	3.7108 (0.0847)
	α	β	λ
APW	6.4700×10^{9} (5.1975×10^{-14})	0.6110 (0.0778)	0.1951 (0.7085)

Table 3 Goodness-of-fit statistics for various models fitted for survival times data set

Model	-2 log L	AIC	AICC	BIC	W*	A*	KS	P-value	SS
Gom-TL-LLoG	197.1	203.1	204.6	206.1	0.0255	0.1971	0.1240	0.9184	0.0334
KulG	202.6	208.6	210.1	211.6	0.0756	0.5224	0.1617	0.6723	0.0888
GG	203.1	209.1	210.6	212.1	0.0805	0.5496	0.1401	0.8274	0.0698
TLLo	203.1	209.1	210.6	212.1	0.0811	0.5532	0.1409	0.8220	0.0712
TL-GE	201.1	207.1	208.6	210.1	0.0601	0.4223	0.1393	0.8328	0.0594
TL-MO-LLo	201.4	207.4	208.9	210.4	0.0604	0.4106	0.1457	0.7899	0.0655
APW	208.4	214.4	215.9	217.4	0.0825	0.5615	0.2384	0.2057	0.1484

The estimated variance-covariance matrix from the survival times data is given by

$$\begin{bmatrix} 26.0923 & 1.0157 & -0.1131 \\ 1.0157 & 0.0395 & -0.0044 \\ -0.1131 & -0.0044 & 0.0005 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\gamma \in [10.7072 \pm 10.0118]$, $b \in [182.8135 \pm 0.3897]$ and $c \in [0.4695 \pm 0.0444]$.

Results in Tables 2 and 3 indicate that the Gom-TL-LLoG model performs better than the non-nested models considered on survival times data because it has the lowest values of the goodness-of-fit statistics, K-S (and the largest p-value for the K-S statistic) and the SS values. The EPDF and probability plots in Figure 6 shows how well the new model fits the data. Furthermore, we conclude that our model is performing well because the observed and fitted Kaplan-Meier and ECDF curves are close to each other as illustrated in Figure 7. The TTT plot is above the diagonal line indicating an increasing HRF as shown in Figure 8.

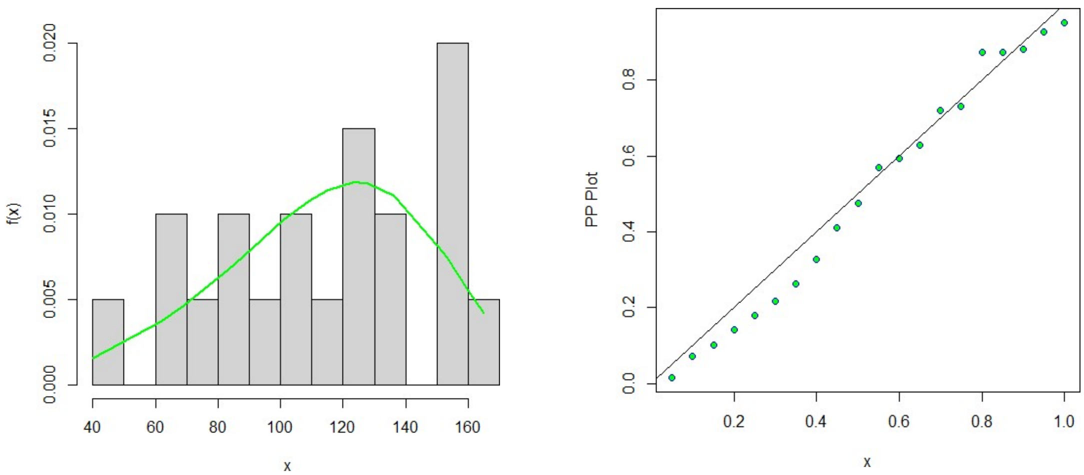


Figure 6: Fitted Density and Probability plots for survival times data

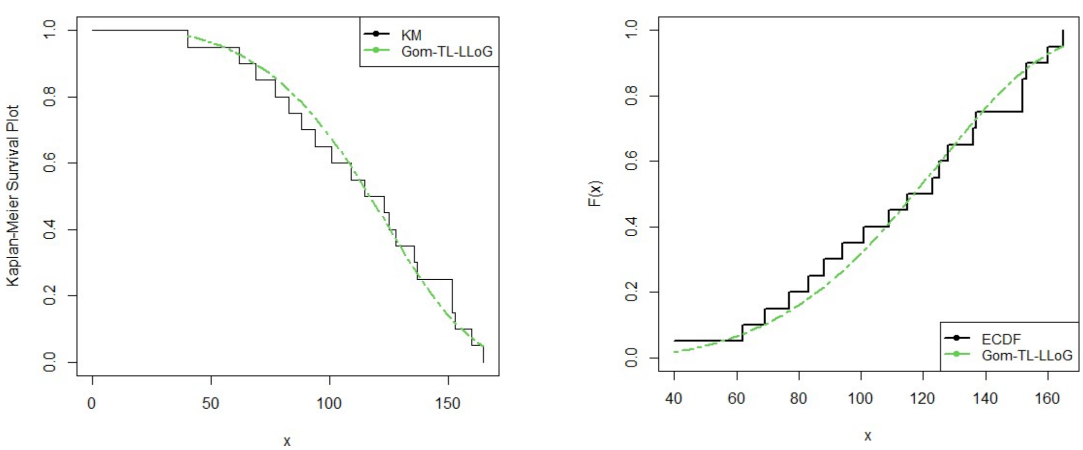


Figure 7: Fitted Kaplan-Meier and ECDF plots for survival times data

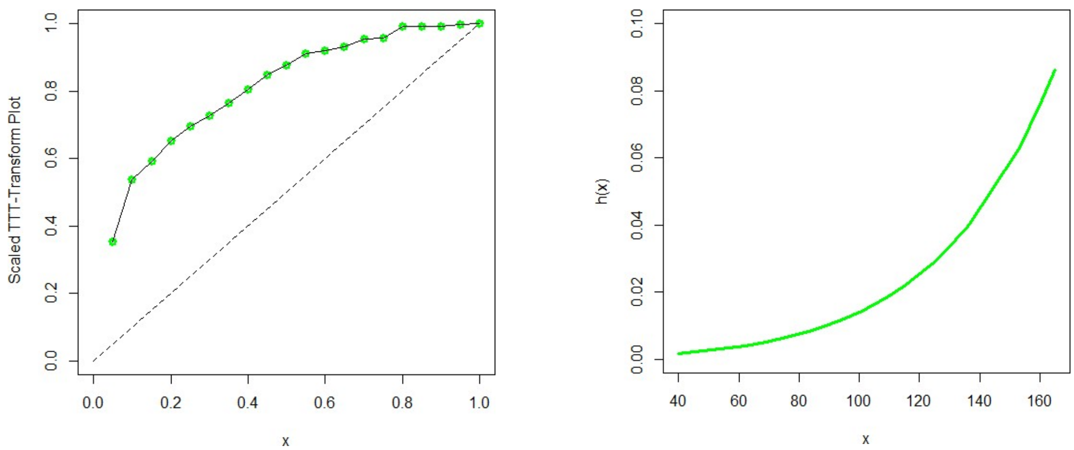


Figure 8: Fitted TTT and HRF plots for survival times data

7.2 Maximum Rainfall Data

The second data set considered by Bakouch et al. [3] describes the maximum rainfall in mm of the whole year of Jiwani town located along the Gulf of Oman in the Gwadar district of the Balochistan province in Pakistan from 1981 to 2010. The observations are: 21.7, 172.9, 69.5, 96.5, 12.6, 265.5, 154, 28, 142.8, 14.2, 74.8, 32.5, 25, 28.5, 113.8, 25.7, 116.3, 28, 16.9, 6, 9, 17.6, 47.3, 55, 129, 72, 92, 28, 113, 194.

Table 4 Parameter estimates for various models fitted for maximum rainfall data set

Model	γ	b	c
Gom-TL-LLoG	1.2426 (1.0485)	35.7013 (17.7949)	0.4178 (0.1056)
	α	β	γ
KulG	1.0000 (0.5711)	28.7642 (5.3004)	0.2942 (0.0675)
	λ	c	θ
GG	0.0065 (0.0035)	0.0070 (0.0038)	0.8564 (0.2831)
	a	b	c
TLLo	1.3575 (1.5971×10^{-9})	56.3250 (6.2835×10^{-11})	1.4747×10^{-4} (2.3798×10^{-5})
	α	β	λ
TL-GE	0.0180 (4.2848×10^{-3})	72.2050 (1.0063×10^{-6})	0.0154 (3.0172×10^{-3})
	b	δ	c
TL-MO-LLo	26.4112 (6.4742)	0.5899 (0.1067)	0.3355 (0.0469)
	α	β	λ
APW	1.0647×10^{11} (3.2013×10^{-14})	0.3420 (0.0404)	0.9397 (0.1514)

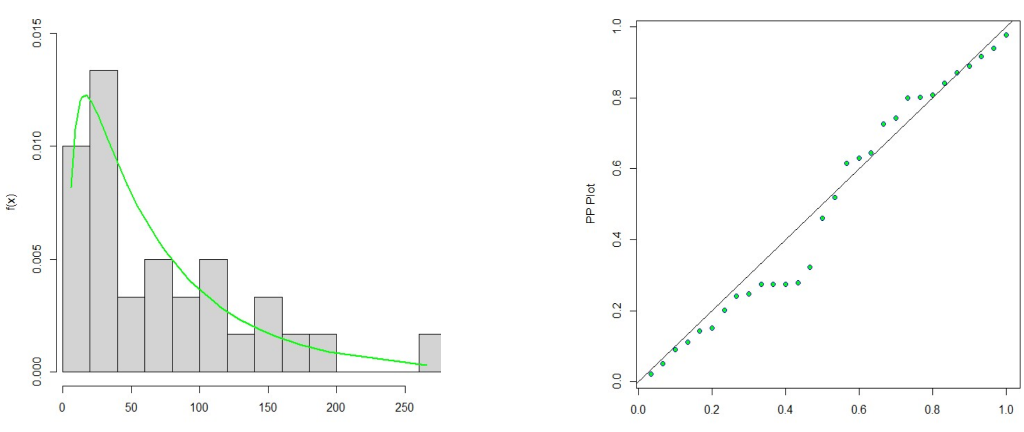
Table 5 Goodness-of-fit statistics for various models fitted for maximum rainfall data set

Model	-2 log L	AIC	AICC	BIC	W*	A*	KS	P-value	SS
Gom-TL-LLoG	314.7	320.7	321.6	324.9	0.0760	0.3935	0.1541	0.4746	0.0885
KulG	349.1	355.1	356.1	359.3	0.3381	2.1410	0.3050	0.0075	0.8894
GG	319.0	325.0	325.9	329.2	0.1041	0.6241	0.2021	0.1725	0.1745
TLLo	316.5	322.5	323.4	326.7	0.0948	0.5081	0.1678	0.3671	0.1002
TL-GE	316.2	322.2	323.1	326.4	0.0910	0.4920	0.1690	0.3580	0.0977
TL-MO-LLo	341.3	347.3	348.3	351.5	0.0906	0.5163	0.2685	0.0265	0.6527
APW	317.0	323.0	324.0	327.2	0.0916	0.4979	0.1671	0.3724	0.1187

The estimated variance-covariance matrix from the maximum rainfall data is given by

$$\begin{bmatrix} 1.0994 & -14.6116 & -0.1025 \\ -14.6116 & 316.6586 & 1.7727 \\ -0.1025 & 1.7727 & 0.0112 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\gamma \in [1.2426 \pm 2.0551]$, $b \in [35.7013 \pm 34.8780]$ and $c \in [0.4178 \pm 0.2070]$. The Gom-TL-LLoG model performs better than the non-nested models considered in this paper on maximum rainfall data as shown by Table 3. The fitted density plot and probability plot in Figure 9 shows how well the new model fits the data. The EPDF plot also indicates that our model can be applied to heavy tailed data. The observed and fitted Kaplan-Meier and ECDF curves are close to each other as illustrated by Figure 10, a clear indication that the model is performing well. The HRF plot which is an updside-down bathtub is reflecting what is suggested by the TTT plot in Figure 11.



Fitted 9: Fitted Density and Probability plots for maximum rainfall data

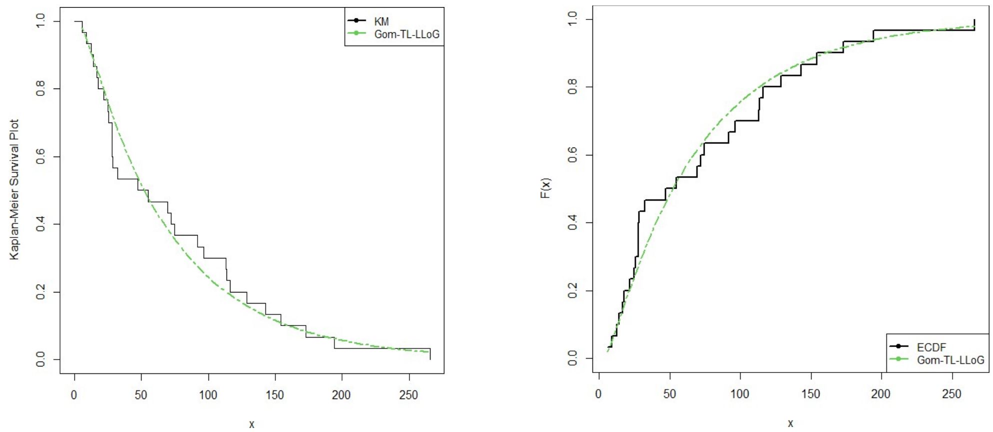


Figure 10: Fitted Kaplan-Meier and ECDF plots for maximum rainfall data

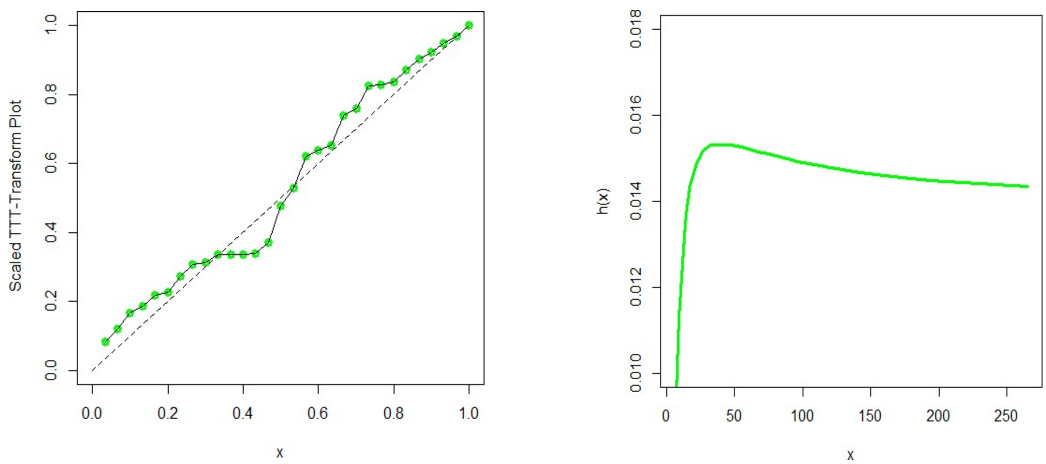


Figure 11: Fitted TTT and HRF plots for maximum rainfall data

8 Concluding Remarks

A new family of distributions called the Gompertz-Topp-Leone-G distribution have been developed. Distributional properties of this model are derived. Estimation of the model parameters via the method of maximum likelihood is presented. A simulation study to assess the performance of the maximum likelihood estimates was conducted. Application of the Gom-TL-LLoG model to real data sets is presented to illustrate its applicability and usefulness.

Appendix

The following URL contains derivations of statistical properties and elements of the score vector.
https://drive.google.com/file/d/1276L6LZn2_Qz_hrOf1YOga6qeds3Kg1E/view?usp=sharing

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