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Asymptotic Properties of MLE's for Distributions Generated from an Exponential Distribution by a Generalized Log-Logistic Transformation

James U. Gleaton
University of North Florida
Ping Sa
University of North Florida
Dr. Sami Hamid
University of North Florida

ABSTRACT

A generalized log-logistic (GLL) family of lifetime distributions is one in which any pair of distributions are related through a GLL transformation, for some (non-negative) value of the transformation parameter κ (the odds function of the second distribution is the κ-th power of the odds function of the first distribution). We consider GLL families generated from an exponential distribution. It is shown that the Maximum Likelihood Estimators (MLE's) for the parameters of the generated, or composite, distribution have the properties of strong consistency and asymptotic normality and efficiency. Data simulation is also found to support the condition of asymptotic efficiency.

Keywords Generalized log-logistic exponential distribution; asymptotic properties of MLE's; simulation

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□ James U. Gleaton, Ping Sa, Dr. Sami Hamid are affiliated with the Department of Mathematics and Statistics at University of North Florida, email: jug_head@earthlink.net.

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1. Introduction.

Gleaton and Rahman (2010) examined the asymptotic properties of the maximum likelihood estimators for parameters of a distribution generated from a 2-parameter Weibull distribution through a ***generalized log-logistic (GLL) transformation***, defined below. It was shown that, under certain restrictions on the parameter space, the MLE's are strongly consistent and asymptotically normal and efficient.

In an earlier paper, Gleaton and Lynch (2006), discussed properties of lifetime distributions belonging to families generated by a GLL transformation:

$$G_\kappa(x) = \Lambda_\kappa \circ G(x) = \frac{\left(G(x)\right)^\kappa}{\left(G(x)\right)^\kappa + \left(\bar{G}(x)\right)^\kappa}, for\ x > 0,$$

relating two lifetime distribution functions $G(x)$ and $G_\kappa(x)$. Here the distribution $G(x)$ may also be a function of an *m*-dimensional non-negative parameter vector $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_m)$. The transformation is defined for each κ > 0 by

$$\Lambda_\kappa(u) = \left[1 + \left(\frac{\bar{u}}{u}\right)^\kappa\right]^{-1}, for\ 0 < u < 1,$$

where $\bar{u} = 1 - u$.

In introducing GLL families of lifetime distributions, Gleaton and Lynch (2006) showed that:

- The set of GLL transformations forms an abelian group with the binary operation of composition.
- The group partitions the set of all lifetime distributions, with any two members of an equivalence class being related to each other through a GLL transformation.
- Either every distribution in an equivalence class has a moment generating function (m.g.f.), or none does, and every distribution in an equivalence class has the same number of moments.
- Each equivalence class is linearly ordered according to the transformation parameter, with larger values of the parameter corresponding to smaller dispersion of a distribution about the common class median.
- The ***log-odds rates***,
$$\omega_\kappa = \frac{d}{dx}[ln(G_\kappa/\bar{G}_\kappa)] = \kappa\frac{d}{dx}[ln(G/\bar{G})] = \kappa\omega,$$
for any two distributions in an equivalence class are the same apart from a multiplicative constant, which is the transformation parameter. The above equation may be taken as the defining characteristic of GLL families.
- Within an equivalence class, the Kullback-Leibler information is an increasing function of the ratio of transformation parameters.

In each equivalence class, one arbitrarily chosen distribution (often taken to be the distribution whose c.d.f. is of simplest form) called the ***embedded distribution***, may be considered the generator of the class. The other distributions in the class are called ***composite distributions***.

Since the introduction of this family of distributions by Gleaton and Lynch (2006), other researchers have extended the family through composition of the GLL transformation with other monotone transformations and have applied the GLL family and the extended families to fitting data sets from various sources.

For example, Cordeiro, et. al. (2017) extended the family by composing the GLL transformation with a power transformation, leading to the generalized odd log-logistic (GOLL) family, which could also be called the GLL-Exponentiated family. They applied this composite transformation to normal, Weibull, and gamma distributions. In each case, they examined quantiles, moments, order statistics, hazard rates, and MLE's. They applied the distribution families to fitting a set of data consisting of lifetimes of a sample of $n = 50$ industrial devices. They found that the GOLL-Weibull(2) distribution provided a better fit than a Weibull(2) distribution.

In another example, Cordeiro, et. al. (2015) extended the family by composing the GLL transformation with a Zografos-Balakrishnan (2009) transformation, yielding what they termed a Zografos-Balakrishnan odd log-logistic (ZBOLL) family of distributions. They examined various generating distributions, including the Weibull(2), normal, and Gumbel distributions. They examined quantiles, entropies, order statistics, and MLE's for the resulting families of distributions. They then applied these distributions to two data sets. In the first case, they found that a ZBOLL-Weibull(2) distribution provided a good fit to as set of data consisting of the logarithm of the time to first calving for a sample of $n = 897$ female Brazilian Newlore breed cattle. In the second case, the found that ZBOLL-Normal distribution provided a good fit to a data set consisting of the daily temperatures for a city in Brazil over a one-year period.

Gleaton and Lynch followed their paper on GLL distributions (2006) with another paper extending the family by composing the GLL transformation with a proportional odds transformation (2010). In this paper, they also found that a GLL-Exponential distribution, the subject of the current paper, provided a better fit than a Weibull(2) distribution to a set of data consisting of the tensile breaking strengths of $n = 64$ ten-millimeter-long carbon fibers.

In this paper, we establish sufficient conditions for the asymptotic properties of the MLE's of the parameters of GLL-Exponential distributions, in which the generating distribution is exponential, with c.d.f. given by:

$$G(x|\lambda) = \left(1 - e^{-\lambda x}\right) I_{(0,\infty)}(x), \qquad \mathbf{(1.1)}$$

where $\lambda > 0$. The c.d.f. of the transformed distribution is

$$G_\kappa(x|\lambda) = \frac{\left(1 - e^{-\lambda}\ \right)^\kappa}{(1 - e^{-\lambda x})^\kappa + e^{-\kappa\lambda}} I_{(0,\infty)}(x), \qquad \mathbf{(1.2)}$$

In Section 2, we present the regularity conditions that must be satisfied for the desired asymptotic properties of the MLE's to hold. In Section 3, we prove a theorem that shows that, if the value of the transformation parameter exceeds 1, the likelihood equations have a sequence of solutions satisfying conditions of strong consistency and asymptotic normality and efficiency. In Section 4, simulation is used to verify that asymptotic efficiency of the MLE's is satisfied for various values of the parameters. Throughout the following sections, whenever there is no ambiguity, the argument x may be omitted as understood.

2. Regularity conditions for asymptotic properties of MLE's.

For a given probability p.d.f. $f(x;\boldsymbol{\theta})$ depending on an *m*-dimensional parameter vector $\boldsymbol{\theta}$, it is well-known that the MLE's of the parameters of the distribution are jointly strongly consistent and asymptotically normally distributed and efficient if certain regularity conditions are satisfied. The conditions are (Schervish, 1995; Serfling, 1980; Stuart, et. al., 1999; Wijsman, 1973; Wilks, 1962):

a) For almost all *x*, the derivatives

$$\frac{\partial ln(f)}{\partial\theta_i}, \frac{\partial^2 ln(f)}{\partial\theta_i\partial\theta_j}, \text{and} \frac{\partial^3 ln(f)}{\partial\theta_i\partial\theta_j\partial\theta_k}$$

exist for all values of $\boldsymbol{\theta}$ belonging to a nongenerate open parameter space, Ω.

b) For all $\theta \in \Omega$, where

$$\left|\frac{\partial f}{\partial\theta_i}\right| < F_{1i}(x), \left|\frac{\partial^2 f}{\partial\theta_i\partial\theta_j}\right| < F_{2ij}(x), \text{and} \left|\frac{\partial^3 f}{\partial\theta_i\partial\theta_j\partial\theta_k}\right| < H_{ijk}(x),$$

the functions $F_{1i}(x)$ and $F_{2ij}(x)$ are integrable with respect to x over $(-\infty, +\infty)$, for all $i, j \epsilon \{1, 2, \dots, m\}$, and

$$\int_{-\infty}^{\infty} H_{ijk}(x) f(x;\boldsymbol{\theta}) dx < M_{ijk},$$

where M_{ijk} is positive and independent of $\boldsymbol{\theta}$, for all $i, j, k \epsilon \{1,2, \dots, m\}$.

c) For all $\boldsymbol{\theta} \in \Omega$,

$$\int_{-\infty}^{\infty} \left(\frac{\partial ln(f)}{\partial\theta_i}\right)^2 f(x;\boldsymbol{\theta})\, dx$$

is positive and finite for $i \epsilon \{1, 2, \dots, m\}$.

3. Asymptotic properties of MLE's for distributions generated by a GLL transformation of an exponential distribution.

In this section, the regularity conditions will be verified for the class of continuous lifetime distributions generated from an exponential distribution by a GLL transformation. Throughout, the value of the transformation parameter κ is assumed to be greater than 1.

If the embedded, or generating, distribution for a GLL equivalence class has a p.d.f. of exponential-class form, then the other members of the equivalence class are not members of exponential families, since their p.d.f.'s have the form

$$g_\kappa(x) = \frac{\kappa\left(G(x)\bar{G}(x)\right)^{\kappa-1} g(x)}{\left[\left(G(x)\right)^\kappa + \left(\bar{G}(x)\right)^\kappa\right]^2} = \frac{\kappa\lambda}{G(x)} \Lambda_\kappa\left(G(x)\right)\Lambda_\kappa\left(\bar{G}(x)\right), for\ x \geq 0. \qquad \mathbf{(3.1)}$$

It is not possible to write this function in exponential-class form, unless κ =1.

The hazard rate for the GLLE distribution is then

$$h_\kappa(x) = \frac{g_\kappa(x)}{\bar{G}_\kappa(x)} = \kappa\Lambda_\kappa\left(G(x)\right)\frac{g(x)}{G(x)\bar{G}(x)} = \kappa\lambda\frac{G_\kappa(x)}{G(x)}.$$

It is clear that, for κ = 1, $h_1(x) = \lambda$, the exponential hazard rate.

LEMMA 3.1: For a GLLE(λ, κ) distribution, the first, second, and third partial derivatives of the natural logarithm of the density function with respect to the components of the parameter vector exist for all $x \in (0, \infty)$, for $\kappa > 1$.
PROOF: Gleaton and Rahman (2010) proved that, for a two-parameter Weibull generating distribution, the first-, second-, and third partial derivatives of the natural logarithm of the GLL-Weibull(2) p.d.f. with respect to the parameters exist for all $x \in (0, +\infty)$ for $\kappa > 1$. Since the GLLE distribution is a GLL-Weibull(2) with shape parameter 1, this shows that the first regularity condition holds for GLLE distributions.

Gleaton and Rahman (2010) also showed that the remaining regularity conditions hold for a GLL-Weibull(2) distribution, provided $\kappa > 3$. In this paper, the remaining regularity conditions will be addressed for the subset of GLLE distributions, with the restriction on κ relaxed.

It is straightforward to show that

$$\frac{\partial \Lambda_\kappa(u)}{\partial \kappa} = \Lambda_\kappa(u)\Lambda_\kappa(\bar{u}) \ln\left(\frac{u}{\bar{u}}\right), \qquad \mathbf{(3.2.a)}$$

and that

$$\frac{\partial \Lambda_\kappa(u)}{\partial \lambda} = \kappa\Lambda_\kappa(u)\Lambda_\kappa(\bar{u}) \frac{\frac{\partial u}{\partial \lambda}}{u\bar{u}}. \qquad \mathbf{(3.2.b)}$$

3.1 First partial derivatives of the density function with respect to the parameters

LEMMA 3.2: For a GLLE(λ, κ) distribution, the first partial derivatives of the density function with respect to the components of the parameter vector are bounded in absolute value by integrable functions for $\kappa > 1$.
PROOF: We have, using **(3.1)** and **(3.2.b)**,

$$\frac{\partial g_\kappa}{\partial \lambda} = \frac{1}{\lambda} g_\kappa(x) - x\frac{\bar{G}(x)}{G(x)} g_\kappa(x) + \kappa \frac{x}{G(x)}\left(\Lambda_\kappa\left(\bar{G}(x)\right) - \Lambda_\kappa\left(G(x)\right)\right) g_\kappa(x). \qquad \mathbf{(3.3)}$$

Then, for $\kappa > 1$, we may use Lemmas 1 and 3 from the Appendix to obtain

$$\left|\frac{\partial g_\kappa}{\partial \lambda}\right| \le \frac{1}{\lambda} g_\kappa(x) + M_\eta g_\kappa(x) + \kappa 2^{\kappa-1} M_\eta g_\kappa(x) + \kappa 2^{\kappa-1} x g_\kappa(x). \qquad \mathbf{(3.4)}$$

Since all moments of the composite distribution exist (Gleaton and Lynch, 2006), all terms on the RHS are integrable. Hence, the first partial derivative of the p.d.f. with respect to the scale parameter is bounded in absolute value by an integrable function.

Next, using **(3.1)** and **(3.2.a)**,

$$\frac{\partial g_\kappa}{\partial \kappa} = \frac{1}{\kappa} g_\kappa(x) + g_\kappa(x)\left(\Lambda_\kappa\left(\bar{G}(x)\right) - \Lambda_\kappa\left(G(x)\right)\right) ln\left(\frac{G(x)}{\bar{G}(x)}\right). \qquad \mathbf{(3.5)}$$

Using Lemma 1 in the Appendix, we find, for $\kappa > 1$,

$$\left|\frac{\partial g_\kappa}{\partial \kappa}\right| \le \frac{1}{\kappa} g_\kappa(x) + 2^\kappa g_\kappa(x)\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|$$
$$\le \frac{1}{\kappa} g_\kappa(x) + 2^\kappa g_\kappa(x)\left|ln\left(G(x)\right)\right| + 2^\kappa g_\kappa(x)\left|ln\left(\bar{G}(x)\right)\right|. \qquad \mathbf{(3.6)}$$

The first term is integrable. Each of the other two terms is integrable by Lemma 4 in the Appendix. Hence, the first partial derivative of the p.d.f. with respect to the transformation parameter is bounded in absolute value by an integrable function for $\kappa > 1$.

3.2 Second partial derivatives of the density function with respect to the parameters

LEMMA 3.3: For a GLLE(λ, κ) distribution, the second partial derivatives of the density function with respect to the components of the parameter vector are bounded in absolute value by integrable functions for $\kappa > 1$.

PROOF: We have, using **(3.3)** and **(3.2.b)**,

$$\frac{\partial^2 g_\kappa}{\partial \lambda^2} = -\frac{1}{\lambda^2} g_\kappa(x) + \frac{1}{\lambda}\frac{\partial g_\kappa}{\partial \lambda} + \frac{x^2 \bar{G}(x)}{\left(G(x)\right)^2} g_\kappa(x) - \frac{x\bar{G}(x)}{G(x)}\frac{\partial g_\kappa}{\partial \lambda}$$
$$-\kappa \frac{x^2 \bar{G}(x)}{\left(G(x)\right)^2}\left(1 - 2\Lambda_\kappa\left(G(x)\right)\right) g_\kappa(x) - \frac{2\kappa^2 x^2}{\left(G(x)\right)^2}\Lambda_\kappa\left(G(x)\right)\Lambda_\kappa\left(\bar{G}(x)\right) g_\kappa(x)$$
$$+\frac{\kappa x}{G(x)}\left(1 - 2\Lambda_\kappa\left(G(x)\right)\right)\frac{\partial g_\kappa}{\partial \lambda}. \qquad (\mathbf{3.7})$$

Thus, for $\kappa > 1$, using Lemmas 1 and 3 in the Appendix,

$$\left|\frac{\partial^2 g_\kappa}{\partial \lambda^2}\right| \le \frac{1}{\lambda^2} g_\kappa(x) + \frac{1}{\lambda}\left|\frac{\partial g_\kappa}{\partial \lambda}\right| + M_\nu g_\kappa(x) + M_\eta \left|\frac{\partial g_\kappa}{\partial \lambda}\right|$$
$$+\kappa M_\nu g_\kappa(x) + \kappa^2 2^{2\kappa-1} M_\nu g_\kappa(x) + \frac{\kappa x}{G(x)}\left|\frac{\partial g_\kappa}{\partial \lambda}\right|. \qquad (\mathbf{3.8})$$

The first six terms on the RHS are obviously integrable. We expand the seventh term on the RHS using **(3.4)**. Using **(3.1)** and Lemmas 1 and 3 in the Appendix and assuming $\kappa > 1$, we have

$$\frac{\kappa x}{G(x)}\left|\frac{\partial g_\kappa}{\partial \lambda}\right| \le \kappa^2 2^{2\kappa-2}\left[M_\eta + \lambda M_\eta^2 + \kappa\lambda 2^{\kappa-1} M_\eta^2 + \kappa\lambda 2^{\kappa-1} M_\eta x\right] e^{-(\kappa-1)\lambda x}. \qquad (\mathbf{3.9})$$

The RHS of this inequality is integrable. Thus, for $\kappa > 1$, the second partial derivative of the composite p.d.f. with respect to λ is bounded in absolute value by an integrable function.

Using **(3.5)** and **(3.2.a)**, we find that the second partial derivative of the p.d.f. with respect to the transformation parameter is

$$\frac{\partial^2 g_\kappa}{\partial \kappa^2} = -\frac{1}{\kappa^2} g_\kappa(x) + \frac{1}{\kappa}\frac{\partial g_\kappa}{\partial \kappa} + \frac{\partial g_\kappa}{\partial \kappa}\left(\Lambda_\kappa\left(\bar{G}(x)\right) - \Lambda_\kappa\left(G(x)\right)\right) ln\left(\frac{G(x)}{\bar{G}(x)}\right)$$
$$-2 g_\kappa(x)\Lambda_\kappa\left(G(x)\right)\Lambda_\kappa\left(\bar{G}(x)\right)\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2. \qquad (\mathbf{3.10})$$

Using Lemma 1 in the Appendix, we have

$$\left|\frac{\partial^2 g_\kappa}{\partial \kappa^2}\right| \le \frac{1}{\kappa^2} g_\kappa(x) + \frac{1}{\kappa}\left|\frac{\partial g_\kappa}{\partial \kappa}\right| + \left|\frac{\partial g_\kappa}{\partial \kappa}\right|\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|$$

$$+2^{2\kappa-1}g_\kappa(x)\sum_{j=0}^{2}\binom{2}{j}\left(ln\big(G(x)\big)\right)^j\left(ln\big(\bar{G}(x)\big)\right)^{2-j}. \qquad \mathbf{(3.11)}$$

The first two terms on the RHS are obviously integrable. Lemma 4 implies that the fourth term on the RHS of **(3.11)** is integrable.
We expand the derivative in the third term, using **(3.6)**, and also use Lemma 1, obtaining

$$\left|\frac{\partial g_\kappa}{\partial\kappa}\right|\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| \le \frac{1}{\kappa}g_\kappa(x)\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| + 2^\kappa g_\kappa(x)\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|^2. \qquad \mathbf{(3.12)}$$

By Lemma 4 in the Appendix, both terms on the RHS of **(3.12)** are integrable. Hence, the second partial derivative of the p.d.f. with respect to the transformation parameter is bounded in absolute value by an integrable function.

Using **(3.5)**, we find the mixed second partial derivative of the p.d.f.:

$$\frac{\partial^2 g_\kappa}{\partial\lambda\partial\kappa} = \frac{1}{\kappa}\frac{\partial g_\kappa}{\partial\lambda} + \frac{\partial g_\kappa}{\partial\lambda}\left[1-2\Lambda_\kappa\big(G(x)\big)\right]\ln\left(\frac{G(x)}{\bar{G}(x)}\right) - 2g_\kappa(x)\frac{\partial\Lambda_\kappa\big(G(x)\big)}{\partial\lambda}\ln\left(\frac{G(x)}{\bar{G}(x)}\right)$$
$$+g_\kappa(x)\left[1-2\Lambda_\kappa\big(G(x)\big)\right]\left(\frac{x\bar{G}(x)}{G(x)}+x\right). \qquad \mathbf{(3.13)}$$

Then, using **(3.2.b)**, we have

$$\left|\frac{\partial^2 g_\kappa}{\partial\lambda\partial\kappa}\right| \le \frac{1}{\kappa}\left|\frac{\partial g_\kappa(x)}{\partial\lambda}\right| + \left|\frac{\partial g_\kappa(x)}{\partial\lambda}\right|\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| + \frac{2}{\lambda}x\big(g_\kappa(x)\big)^2\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|$$
$$+g_\kappa(x)\left(\frac{x\bar{G}(x)}{G(x)}+x\right). \qquad \mathbf{(3.14)}$$

The first term on the RHS is integrable; using Lemma 3 in the Appendix and the fact that all moments of the distribution exist, we find that the fourth term on the RHS is also integrable. For the second and third terms, if we use **(3.4)** and Lemma 1 in the Appendix, we have

$$\left|\frac{\partial g_\kappa}{\partial\lambda}\right|\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| + \frac{2}{\lambda}x\big(g_\kappa(x)\big)^2\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|$$
$$\le\left[\frac{1}{\lambda}+M_\eta+\kappa 2^{\kappa-1}M_\eta+\kappa 2^{\kappa-1}x\right]g_\kappa(x)\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|$$
$$+\kappa 2^{2\kappa-1}x g_\kappa(x)\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|. \qquad \mathbf{(3.15)}$$

Assuming $\kappa > 1$ and using Lemma 4 in the Appendix and **(3.1)**, we find that both terms on the RHS of **(3.15)** are integrable.

Hence, all second partial derivatives of $g_\kappa(x)$ with respect to the parameter components are bounded in absolute value by integrable functions, so long as $\kappa > 1$. ■

3.3 Third partial derivatives of the p.d.f. with respect to the parameter components.

LEMMA 3.4: For a GLLE(λ, κ) distribution, the third partial derivatives of the density function with respect to the components of the parameter vector are bounded in absolute value by functions with finite expectations for $\kappa > 1$.
PROOF: The third partial derivative of the p.d.f. with respect to the transformation parameter is:

$$\frac{\partial^3 g_\kappa}{\partial \kappa^3} = \frac{2}{\kappa^3} g_\kappa(x) - \frac{2}{\kappa^2}\frac{\partial g_\kappa}{\partial \kappa} + \frac{1}{\kappa}\frac{\partial^2 g_\kappa}{\partial \kappa^2} + \frac{\partial^2 g_\kappa}{\partial \kappa^2}\left[1 - 2\Lambda_\kappa\big(G(x)\big)\right] \ln\left(\frac{G(x)}{\bar{G}(x)}\right)$$
$$-4\frac{\partial g_\kappa}{\partial \kappa}\Lambda_\kappa\big(G(x)\big)\Lambda_\kappa\big(\bar{G}(x)\big)\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2$$
$$-2g_\kappa(x)\Lambda_\kappa\big(G(x)\big)\Lambda_\kappa\big(\bar{G}(x)\big)\left[1 - 2\Lambda_\kappa\big(G(x)\big)\right]\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^3. \qquad \mathbf{(3.16)}$$

Hence, for $\kappa > 1$, using Lemma 1 in the Appendix and the boundedness of the c.d.f,

$$\left|\frac{\partial^3 g_\kappa}{\partial \kappa^3}\right| \le \frac{2}{\kappa^3} g_\kappa(x) + \frac{2}{\kappa^2}\left|\frac{\partial g_\kappa}{\partial \kappa}\right| + \frac{1}{\kappa}\left|\frac{\partial^2 g_\kappa}{\partial \kappa^2}\right| + \left|\frac{\partial^2 g_\kappa}{\partial \kappa^2}\right|\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|$$
$$+2^{2\kappa}\left|\frac{\partial g_\kappa}{\partial \kappa}\right|\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2 + 2^{2\kappa-1} g_\kappa(x)\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|^3. \qquad \mathbf{(3.17)}$$

The first three terms on the RHS of **(3.17)** are obviously integrable. Lemma 4 in the Appendix implies that the last term on the RHS of **(3.17)** is integrable.

Using **(3.5)**, **(3.6)** and Lemma 1 from the Appendix, we may write the fifth term on the RHS of **(3.17)** as

$$2^{2\kappa}\left|\frac{\partial g_\kappa}{\partial \kappa}\right|\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2$$
$$\le 2^{2\kappa}\left[\frac{1}{\kappa} g_\kappa(x) + 2^\kappa g_\kappa(x)\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\right]\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2. \qquad \mathbf{(3.18)}$$

It is clear from Lemma 4 in the Appendix that the RHS of **(3.18)** is integrable.

If we expand the fourth term on the RHS of **(3.17)** using **(3.11)**, we find

$$\left|\frac{\partial^2 g_\kappa}{\partial \kappa^2}\right|\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| \le \frac{1}{\kappa^2} g_\kappa(x)\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| + \frac{1}{\kappa}\left|\frac{\partial g_\kappa}{\partial \kappa}\right|\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|$$
$$+\left|\frac{\partial g_\kappa}{\partial \kappa}\right|\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|^2 + 2^{2\kappa-1} g_\kappa(x)\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|^3. \qquad \mathbf{(3.19)}$$

From Lemma 4 in the Appendix, the first and last terms on the RHS of **(3.19)** are integrable. It was shown above that the second term on the RHS of **(3.19)** is integrable.
If we expand the third term on the RHS of **(3.19)** using **(3.5)** and **(3.6)**, we find

$$\left|\frac{\partial g_\kappa}{\partial \kappa}\right|\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|^2 \le \frac{1}{\kappa} g_\kappa(x)\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|^2 + 2^\kappa g_\kappa(x)\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|^3. \qquad \mathbf{(3.20)}$$

From Lemma 4 in the Appendix, we find that the RHS of **(3.20)** is integrable.

Hence the third partial derivative of $g_\kappa(x)$ with respect to the transformation parameter is bounded in absolute value by an integrable function. Since $g_\kappa(x)$ is also bounded by Lemma 1, it follows that $\frac{\partial^3 g_\kappa}{\partial \kappa^3}$ is bounded in absolute value by a function with finite expectation.

The third partial derivative of the p.d.f. with respect to the scale parameter is:

$$\frac{\partial^3 g_\kappa}{\partial\lambda^3}=\frac{2}{\lambda^3}g_\kappa(x)-\frac{2}{\lambda^2}\frac{\partial g_\kappa}{\partial\lambda}+\frac{1}{\lambda}\frac{\partial^2 g_\kappa}{\partial\lambda^2}-\frac{x^3\bar{G}(x)}{\left(G(x)\right)^3}\left(1+\bar{G}(x)\right)g_\kappa(x)+2\frac{x^2\bar{G}(x)}{\left(G(x)\right)^2}\frac{\partial g_\kappa}{\partial\lambda}$$
$$-\frac{x\bar{G}(x)}{G(x)}\frac{\partial^2 g_\kappa}{\partial\lambda^2}+\kappa\frac{x^3\bar{G}(x)}{\left(G(x)\right)^3}\left(1+\bar{G}(x)\right)\left(1-2\Lambda_\kappa\left(G(x)\right)\right)g_\kappa(x)$$
$$+\frac{4\kappa}{\lambda}\frac{x^3\bar{G}(x)}{\left(G(x)\right)^2}\left(g_\kappa(x)\right)^2-2\kappa\frac{x^2\bar{G}(x)}{\left(G(x)\right)^2}\left(1-2\Lambda_\kappa\left(G(x)\right)\right)\frac{\partial g_\kappa}{\partial\lambda}$$
$$+\frac{2\kappa}{\lambda^2}\frac{x^2}{G(x)}\left(g_\kappa(x)\right)^2-\frac{6\kappa}{\lambda}\frac{x^2}{G(x)}g_\kappa(x)\frac{\partial g_\kappa}{\partial\lambda}+\frac{\kappa x}{G(x)}\left(1-2\Lambda_\kappa\left(G(x)\right)\right)\frac{\partial^2 g_\kappa}{\partial\lambda^2}.\quad(\mathbf{3.21})$$

Hence, using Lemma 3 in the Appendix and the boundedness of the c.d.f., we find

$$\left|\frac{\partial^3 g_\kappa}{\partial\lambda^3}\right|\le\frac{2}{\lambda^3}g_\kappa(x)+\frac{2}{\lambda^2}\left|\frac{\partial g_\kappa}{\partial\lambda}\right|+\frac{1}{\lambda}\left|\frac{\partial^2 g_\kappa}{\partial\lambda^2}\right|+2(\kappa+1)\frac{x^3\bar{G}(x)}{\left(G(x)\right)^3}g_\kappa(x)+M_\eta\left|\frac{\partial^2 g_\kappa}{\partial\lambda^2}\right|$$
$$+\frac{4\kappa}{\lambda}xM_\upsilon\left(g_\kappa(x)\right)^2+2(\kappa+1)M_\upsilon\left|\frac{\partial g_\kappa}{\partial\lambda}\right|+\frac{2\kappa}{\lambda^2}\frac{x^2}{G(x)}\left(g_\kappa(x)\right)^2$$
$$+\frac{6\kappa}{\lambda}\frac{x^2}{G(x)}g_\kappa(x)\left|\frac{\partial g_\kappa}{\partial\lambda}\right|+\frac{\kappa x}{G(x)}\left|\frac{\partial^2 g_\kappa}{\partial\lambda^2}\right|.\quad(\mathbf{3.22})$$

It is clear that terms 1, 2 ,3, 5, and 7 on the RHS of **(3.22)** are bounded in absolute value by integrable functions, and thus have finite expectations. The other terms will be considered individually.

Assuming $\kappa>1$, we use Lemmas 1 and 3 in the Appendix, together with **(3.1)** and **(3.2.b)** to find that the expectation of the absolute value of term 4 on the RHS of **(3.22)** is

$$2\kappa^2(\kappa+1)\lambda^2\int_0^\infty\frac{x^3\left(\bar{G}(x)\right)^3}{\left(G(x)\right)^3}\frac{\left(G(x)\right)^{2\kappa-2}\left(\bar{G}(x)\right)^{2\kappa-2}}{\left[\left(G(x)\right)^\kappa+\left(\bar{G}(x)\right)^\kappa\right]^4}dx$$
$$\le\kappa^2(\kappa+1)\lambda^2 2^{4\kappa-3}M_\eta^3\int_0^\infty e^{-2(\kappa-1)\lambda x}dx$$
$$=\frac{\kappa^2(\kappa+1)\lambda}{\kappa-1}2^{4(\kappa-1)}M_\eta^3.$$

Since the p.d.f. is bounded and all moments of the distribution exist, the expectation of term 6 on the RHS of **(3.22)** is finite.

Assuming $\kappa>1$, we use Lemmas 1 and 3 in the Appendix, together with **(3.1)** and **(3.2.b)** to find that the expectation of the absolute value of term 8 on the RHS of **(3.22)** is

$$\frac{2\kappa}{\lambda^2}\int_0^\infty\frac{x^2}{G(x)}\left(g_\kappa(x)\right)^3dx=2\kappa^4\lambda\int_0^\infty\frac{x^2\bar{G}(x)}{G(x)}\frac{\left(G(x)\right)^{3(\kappa-1)}\left(\bar{G}(x)\right)^{3\kappa-1}}{\left[\left(G(x)\right)^\kappa+\left(\bar{G}(x)\right)^\kappa\right]^6}dx$$
$$\le 2\kappa^4\lambda M_\rho 2^{6(\kappa-1)}\int_0^\infty e^{-(3\kappa-1)\lambda x}dx=\frac{2\kappa^4}{3\kappa-1}M_\rho 2^{6(\kappa-1)}.$$

Let $\alpha\left(\kappa,\lambda,M_\eta\right)=\kappa^2 2^{2(\kappa-1)}M_\eta\left(1+\lambda M_\eta+\kappa\lambda 2^{\kappa-1}M_\eta\right)$, and $\beta\left(\kappa,\lambda,M_\eta\right)=\kappa^3\lambda 2^{3(\kappa-1)}M_\eta$. Then, using Lemma 1 in the Appendix and **(3.9)**, we find that the expectation of term 9 on the RHS of **(3.22)** is

$$E\left[\frac{6\kappa}{\lambda}\frac{X^2}{G(X)}g_\kappa(X)\left|\frac{\partial g_\kappa}{\partial\lambda}\right|\right]=\frac{6\kappa}{\lambda}\int_0^\infty\frac{x^2}{G(x)}\big(g_\kappa(x)\big)^2\left|\frac{\partial g_\kappa}{\partial\lambda}\right|dx$$

$$\leq\frac{6}{\lambda}\alpha\big(\kappa,\lambda,M_\eta\big)\int_0^\infty x\,e^{-(\kappa-1)\lambda x}\big(g_\kappa(x)\big)^3dx+\frac{6}{\lambda}\beta\big(\kappa,\lambda,M_\eta\big)\int_0^\infty x^2e^{-(\kappa-1)\lambda x}\big(g_\kappa(x)\big)^3dx$$

$$\leq(3)2^{6\kappa-5}\kappa^3\lambda^2\left\{\alpha\big(\kappa,\lambda,M_\eta\big)\int_0^\infty xe^{-(\kappa-1)\lambda x}dx+\beta\big(\kappa,\lambda,M_\eta\big)\int_0^\infty x^2e^{-(\kappa-1)\lambda x}dx\right\}.\quad\mathbf{(3.23)}$$

Both integrals on the RHS of **(3.23)** exist.

Let $C_1=\frac{1}{\lambda^2}+M_\nu(\kappa+1+\kappa^2 2^{2\kappa-1})$, and $C_2=\frac{1}{\lambda}+M_\eta$. Then, using **(3.8)**, we find that the expectation of the tenth term on the RHS of **(3.22)** is:

$$E\left[\frac{\kappa X}{G(X)}\left|\frac{\partial^2 g_\kappa}{\partial\lambda^2}\right|\right]=\kappa\int_0^\infty\frac{x}{G(x)}g_\kappa(x)\left|\frac{\partial^2 g_\kappa}{\partial\lambda^2}\right|dx\leq\kappa C_1\int_0^\infty\frac{x}{G(x)}\big(g_\kappa(x)\big)^2dx$$

$$+\kappa C_2\int_0^\infty\frac{x}{G(x)}g_\kappa(x)\left|\frac{\partial g_\kappa}{\partial\lambda}\right|dx+\kappa\int_0^\infty\frac{x^2}{\big(G(x)\big)^2}g_\kappa(x)\left|\frac{\partial g_\kappa}{\partial\lambda}\right|dx.\qquad\mathbf{(3.24)}$$

Using Lemma 5 in the Appendix, we find that the first integral in **(3.24)** is

$$\int_0^\infty\frac{x}{G(x)}\big(g_\kappa(x)\big)^2dx\leq\kappa^2\lambda^2 2^{4(\kappa-1)}\int_0^\infty\frac{x\big(\bar{G}(x)\big)^{2\kappa}}{G(x)}dx,$$

which is finite. If we let $C_3=\frac{1}{\lambda}+M_\eta+\kappa 2^{\kappa-1}$, and $C_4=\kappa 2^{\kappa-1}$, and use **(3.4)**, we find that the second integral in **(3.24)** is

$$\int_0^\infty\frac{x}{G(x)}g_\kappa(x)\left|\frac{\partial g_\kappa}{\partial\lambda}\right|dx\leq C_3\int_0^\infty\frac{x}{G(x)}\big(g_\kappa(x)\big)^2dx+C_4\int_0^\infty\frac{x^2}{G(x)}\big(g_\kappa(x)\big)^2dx.$$

Lemma 5 in the Appendix implies that both integrals on the RHS of the above equation are finite. Likewise, the last integral on the RHS of **(3.24)** was shown above to be finite.

Hence, the third partial derivative of the p.d.f. with respect to the scale parameter is bounded in absolute value by a function with finite expectation.

Third mixed partial derivatives. We have

$$\frac{\partial^3 g_\kappa}{\partial\kappa\partial\lambda^2}=-\frac{1}{\lambda^2}\frac{\partial g_\kappa}{\partial\kappa}+\frac{1}{\lambda}\frac{\partial^2 g_\kappa}{\partial\kappa\partial\lambda}+\frac{x^2\bar{G}(x)}{\big(G(x)\big)^2}\frac{\partial g_\kappa}{\partial\kappa}-\frac{x\bar{G}(x)}{G(x)}\frac{\partial^2 g_\kappa}{\partial\kappa\partial\lambda}$$

$$-\frac{x^2\bar{G}(x)}{\big(G(x)\big)^2}\Big(1-2\Lambda_\kappa\big(G(x)\big)\Big)g_\kappa(x)+2\kappa\frac{x^2\bar{G}(x)}{\big(G(x)\big)^2}\Lambda_\kappa\big(G(x)\big)\Lambda_\kappa\big(\bar{G}(x)\big)\ln\left(\frac{G(x)}{\bar{G}(x)}\right)g_\kappa(x)$$

$$-\frac{\kappa x^2\bar{G}(x)}{\big(G(x)\big)^2}\Big(1-2\Lambda_\kappa\big(G(x)\big)\Big)\frac{\partial g_\kappa}{\partial\kappa}-\frac{4\kappa x^2}{\big(G(x)\big)^2}\Lambda_\kappa\big(G(x)\big)\Lambda_\kappa\big(\bar{G}(x)\big)g_\kappa(x)$$

$$-\frac{2\kappa^2x^2}{\big(G(x)\big)^2}\Lambda_\kappa\big(G(x)\big)\Big(\Lambda_\kappa\big(\bar{G}(x)\big)\Big)^2 ln\left(\frac{G(x)}{\bar{G}(x)}\right)g_\kappa(x)$$

$$+\frac{2\kappa^2x^2}{\left(G(x)\right)^2}\left(\Lambda_\kappa\left(G(x)\right)\right)^2\Lambda_\kappa\left(\bar{G}(x)\right)ln\left(\frac{G(x)}{\bar{G}(x)}\right)g_\kappa(x)-\frac{2\kappa^2x^2}{\left(G(x)\right)^2}\Lambda_\kappa\left(G(x)\right)\Lambda_\kappa\left(\bar{G}(x)\right)\frac{\partial g_\kappa}{\partial\kappa}$$
$$+\frac{x}{G(x)}\left(1-2\Lambda_\kappa\left(G(x)\right)\right)\frac{\partial g_\kappa}{\partial\lambda}-2\frac{\kappa x}{G(x)}\Lambda_\kappa\left(G(x)\right)\Lambda_\kappa\left(\bar{G}(x)\right)\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\frac{\partial g_\kappa}{\partial\lambda}$$
$$+\frac{\kappa x}{G(x)}\left(1-2\Lambda_\kappa\left(G(x)\right)\right)\frac{\partial^2 g_\kappa}{\partial\kappa\partial\lambda}. \qquad (\mathbf{3.25})$$

Then, using Lemmas 1 and 3 in the Appendix, we find

$$\left|\frac{\partial^3 g_\kappa}{\partial\kappa\partial\lambda^2}\right|\le\frac{1}{\lambda^2}\left|\frac{\partial g_\kappa}{\partial\kappa}\right|+\frac{1}{\lambda}\left|\frac{\partial^2 g_\kappa}{\partial\kappa\partial\lambda}\right|+M_\nu\left|\frac{\partial g_\kappa}{\partial\kappa}\right|+M_\eta\left|\frac{\partial^2 g_\kappa}{\partial\kappa\partial\lambda}\right|$$
$$+M_\nu g_\kappa(x)+2^{2\kappa-1}\kappa M_\nu\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|g_\kappa(x)+\kappa M_\nu\left|\frac{\partial g_\kappa}{\partial\kappa}\right|+2^{2\kappa}\kappa M_\nu g_\kappa(x)$$
$$+2^{3\kappa-1}\kappa^2M_\nu\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|g_\kappa(x)+2^{2\kappa-1}\kappa^2M_\nu\left|\frac{\partial g_\kappa}{\partial\kappa}\right|+\frac{x}{G(x)}\left|\frac{\partial g_\kappa}{\partial\lambda}\right|$$
$$+2^{2\kappa-1}\kappa\frac{x}{G(x)}\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\left|\frac{\partial g_\kappa}{\partial\lambda}\right|+\kappa\frac{x}{G(x)}\left|\frac{\partial^2 g_\kappa}{\partial\kappa\partial\lambda}\right|. \qquad (\mathbf{3.26})$$

The first, second, third, fourth, fifth, seventh, eighth, and tenth terms on the RHS of **(3.26)** are clearly integrable. Lemma 4 in the Appendix implies that the sixth and ninth terms are also integrable. Hence, all of these terms have finite expectations.

Using **(3.4)**, we find that the expectation of the eleventh term is

$$E\left[\frac{X}{G(X)}\left|\frac{\partial g_\kappa}{\partial\lambda}\right|\right]\le\left(\frac{1}{\lambda}+M_\eta+\kappa 2^{\kappa-1}M_\eta\right)\int_0^\infty\frac{x}{G(x)}\left(g_\kappa(x)\right)^2dx$$
$$+\kappa 2^{\kappa-1}\int_0^\infty\frac{x^2}{G(x)}\left(g_\kappa(x)\right)^2dx$$

Applying **(3.1)** and Lemmas 1 and 3 in the Appendix, we find

$$E\left[\frac{X}{G(X)}\left|\frac{\partial g_\kappa}{\partial\lambda}\right|\right]\le\kappa\lambda 2^{2(\kappa-1)}M_\eta\left(\frac{1}{\lambda}+M_\eta+\kappa 2^{\kappa-1}M_\eta\right)+\kappa^2\lambda 2^{3(\kappa-1)}M_\eta E[X].$$

The expectation of the twelfth term is

$$2^{2\kappa-1}\kappa E\left[\frac{X}{G(X)}\left|\ln\left(\frac{G(X)}{\bar{G}(X)}\right)\right|\left|\frac{\partial g_\kappa}{\partial\lambda}\right|\right]=2^{2\kappa-1}\kappa\int_0^\infty\frac{x}{G(x)}\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\left|\frac{\partial g_\kappa}{\partial\lambda}\right|g_\kappa(x)dx.$$

Using **(3.4)**, we find

$$E\left[\frac{X}{G(X)}\left|\ln\left(\frac{G(X)}{\bar{G}(X)}\right)\right|\left|\frac{\partial g_\kappa}{\partial\lambda}\right|\right]\le\left(\frac{1}{\lambda}+M_\eta+\kappa 2^{\kappa-1}M_\eta\right)\int_0^\infty\frac{x}{G(x)}\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\left(g_\kappa(x)\right)^2dx$$
$$+\kappa 2^{\kappa-1}\int_0^\infty\frac{x^2}{G(x)}\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\left(g_\kappa(x)\right)^2dx. \qquad (\mathbf{3.27})$$

If we apply **(3.1)** and Lemma 1 in the Appendix, we find the first integral on the RHS of **(3.27)** is

$$\int_0^\infty \frac{x}{G(x)}\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\left(g_\kappa(x)\right)^2 dx \le \kappa\lambda 2^{2(\kappa-1)}M_\eta \int_0^\infty \left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| g_\kappa(x)dx.$$

The integral on the RHS above exists, by Lemma 4 in the Appendix. Similarly, the second integral on the RHS of **(3.27)** is

$$\int_0^\infty \frac{x^2}{G(x)}\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\left(g_\kappa(x)\right)^2 dx \le \kappa\lambda 2^{2(\kappa-1)}M_\eta \int_0^\infty x\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| g_\kappa(x)dx.$$

Again, Lemma 4 in the Appendix implies that the integral on the RHS above exists.

The expectation of the thirteenth term on the RHS of **(3.26)** is

$$E\left[\kappa\frac{X}{G(X)}\left|\frac{\partial^2 g_\kappa}{\partial\kappa\partial\lambda}\right|\right] = \kappa\int_0^\infty \frac{x}{G(x)}\left|\frac{\partial^2 g_\kappa}{\partial\kappa\partial\lambda}\right| g_\kappa(x)dx.$$

Using **(3.14)**, we find

$$E\left[\kappa\frac{X}{G(X)}\left|\frac{\partial^2 g_\kappa}{\partial\kappa\partial\lambda}\right|\right] \le E\left[\frac{X}{G(X)}\left|\frac{\partial g_\kappa}{\partial\lambda}\right|\right] + \kappa E\left[\frac{X}{G(X)}\left|\ln\left(\frac{G(X)}{\bar{G}(X)}\right)\right|\left|\frac{\partial g_\kappa}{\partial\lambda}\right|\right]$$
$$+\frac{2\kappa}{\lambda}\int_0^\infty \frac{x^2}{G(x)}\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\left(g_\kappa(x)\right)^3 dx$$
$$+\kappa\int_0^\infty \frac{x}{G(x)}\left(x+\frac{x\bar{G}(x)}{G(x)}\right)\left(g_\kappa(x)\right)^2 dx. \qquad \mathbf{(3.28)}$$

It was shown above that the first two terms on the RHS of **(3.28)** are finite. For the third term on the RHS of **(3.28)**, if we use **(3.1)** and Lemmas 1 and 3 in the Appendix, we have

$$\frac{2\kappa}{\lambda}\int_0^\infty \frac{x^2}{G(x)}\left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\left(g_\kappa(x)\right)^3 dx \le \kappa^3\lambda 2^{4\kappa-3}M_\nu \int_0^\infty \left|\ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| g_\kappa(x)dx \qquad \mathbf{(3.29)}$$

The fourth term on the RHS of **(3.28)** may be written, using **(3.1)** and Lemmas 1 and 3 in the Appendix, as

$$\kappa\int_0^\infty \frac{x^2}{\left(G(x)\right)^2}\left(g_\kappa(x)\right)^2 dx = \kappa^2\lambda\int_0^\infty \frac{x^2}{\left(G(x)\right)^3}\Lambda_\kappa\left(G(x)\right)\Lambda_\kappa\left(\bar{G}(x)\right)g_\kappa(x)dx$$
$$\le \kappa^2\lambda 2^{2(\kappa-1)}M_\nu.$$

Hence, $\left|\frac{\partial^3 g_\kappa}{\partial\kappa\partial\lambda^2}\right|$ is bounded by a function with finite expectation.

Finally, using **(3.10)**, we find

$$\frac{\partial^3 g_\kappa}{\partial\lambda\partial\kappa^2} = -\frac{1}{\kappa^2}\frac{\partial g_\kappa}{\partial\lambda} + \frac{1}{\kappa}\frac{\partial^2 g_\kappa}{\partial\lambda\partial\kappa} - \frac{2}{\lambda}x g_\kappa(x)\frac{\partial g_\kappa}{\partial\kappa} ln\left(\frac{G(x)}{\bar{G}(x)}\right)$$
$$+\left(1-2\Lambda_\kappa\left(G(x)\right)\right)\frac{\partial^2 g_\kappa}{\partial\lambda\partial\kappa} ln\left(\frac{G(x)}{\bar{G}(x)}\right) + \frac{x}{G(x)}\left(1-2\Lambda_\kappa\left(G(x)\right)\right)\frac{\partial g_\kappa}{\partial\kappa}$$
$$-\frac{2}{\kappa\lambda}x\bar{G}(x)\left(g_\kappa(x)\right)^2\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2 - \frac{4}{\kappa\lambda}G(x)g_\kappa(x)\frac{\partial g_\kappa}{\partial\lambda}\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2$$

$$-\frac{4}{\kappa\lambda}x\big(g_\kappa(x)\big)^2 ln\left(\frac{G(x)}{\bar{G}(x)}\right). \qquad \mathbf{(3.30)}$$

Hence,

$$\left|\frac{\partial^3 g_\kappa}{\partial\lambda\partial\kappa^2}\right| \le \frac{1}{\kappa^2}\left|\frac{\partial g_\kappa}{\partial\lambda}\right| + \frac{1}{\kappa}\left|\frac{\partial^2 g_\kappa}{\partial\lambda\partial\kappa}\right| + \frac{2}{\lambda}xg_\kappa(x)\left|\frac{\partial g_\kappa}{\partial\kappa}\right|\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|$$
$$+\left|\frac{\partial^2 g_\kappa}{\partial\lambda\partial\kappa}\right|\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| + \frac{x}{G(x)}\left|\frac{\partial g_\kappa}{\partial\kappa}\right| + \frac{2}{\kappa\lambda}x\big(g_\kappa(x)\big)^2\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2$$
$$+\frac{4}{\kappa\lambda}g_\kappa(x)\left|\frac{\partial g_\kappa}{\partial\lambda}\right|\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2 + \frac{4}{\kappa\lambda}x\big(g_\kappa(x)\big)^2\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|. \qquad \mathbf{(3.31)}$$

The first two terms on the RHS of **(3.31)** are bounded by integrable functions, and thus have finite expectations. The boundedness of the p.d.f., together with Lemma 4 in the Appendix, imply that the sixth and eighth terms on the RHS of **(3.31)** are bounded by integrable functions, and thus have finite expectations.

Using the fact that the p.d.f. is bounded, we find that the expectation of the third term on the RHS of **(3.31)** is

$$\frac{2}{\lambda}E\left[Xg_\kappa(X)\left|\frac{\partial g_\kappa}{\partial\kappa}\right|\left|ln\left(\frac{G(X)}{\bar{G}(X)}\right)\right|\right] \le \kappa\lambda 2^{4\kappa-3}\int_0^\infty xg_\kappa(x)\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|dx$$
$$+\kappa^2\lambda 2^{5\kappa-3}\int_0^\infty xg_\kappa(x)\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|^2 dx,$$

which is finite by Lemma 4 in the Appendix.

Using **(3.14)**, we find that the expectation of the fourth term on the RHS of **(3.31)** is

$$\int_0^\infty\left|\frac{\partial^2 g_\kappa}{\partial\lambda\partial\kappa}\right|\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|g_\kappa(x)dx \le \frac{1}{\kappa}\int_0^\infty\left|\frac{\partial g_\kappa}{\partial\kappa}\right|\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|g_\kappa(x)dx$$
$$+\int_0^\infty\left|\frac{\partial g_\kappa}{\partial\kappa}\right|\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|^2 g_\kappa(x)dx + \frac{2}{\lambda}\int_0^\infty x\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|^2\big(g_\kappa(x)\big)^3dx$$
$$+\int_0^\infty x\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\big(g_\kappa(x)\big)^2dx + \int_0^\infty\frac{x\bar{G}(x)}{G(x)}\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\big(g_\kappa(x)\big)^2dx. \qquad \mathbf{(3.32)}$$

It was shown above that the first term – see **(3.12)** – and the second term – see **(3.20)** – on the RHS of **(3.31)** are finite.

Using **(3.6)**, we find that the expectation of the fifth term on the RHS of **(3.31)** is

$$E\left[\frac{X}{G(X)}\left|\frac{\partial g_\kappa}{\partial\kappa}\right|\right] = \int_0^\infty\frac{x}{G(x)}\left|\frac{\partial g_\kappa}{\partial\kappa}\right|g_\kappa(x)dx \le \frac{1}{\kappa}\int_0^\infty\frac{x}{G(x)}\big(g_\kappa(x)\big)^2dx$$
$$+2^\kappa\int_0^\infty\frac{x}{G(x)}\left|ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right|\big(g_\kappa(x)\big)^2dx.$$

Using **(3.6)** and Lemmas 1 and 3 in the Appendix, we find

$$E\left[\frac{X}{G(X)}\left|\frac{\partial g_\kappa}{\partial \kappa}\right|\right] \leq \lambda 2^{2(\kappa-1)} M_\eta + \kappa\lambda 2^{3\kappa-2} M_\eta \int_0^\infty \left| ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right| g_\kappa(x)dx.$$

The integral on the RHS above exists, by Lemma 4 in the Appendix. Hence, the fifth term on the RHS of **(3.31)** is bounded by an integrable function.

Using **(3.1)**, **(3.4)**, and Lemmas 1 and 3 in the Appendix, we find that the expectation of the seventh term on the RHS of **(3.31)** is

$$\frac{4}{\kappa\lambda}\int_0^\infty \left|\frac{\partial g_\kappa}{\partial \lambda}\right| \left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2 \left(g_\kappa(x)\right)^2 dx$$

$$\leq \kappa\lambda 2^{4\kappa-2}\left(\frac{1}{\lambda} + M_\eta + 2^{\kappa-1}M_\eta\right)\int_0^\infty \left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2 g_\kappa(x)dx$$

$$+\kappa^2\lambda 2^{5\kappa-3}\int_0^\infty x\left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2 g_\kappa(x)dx.$$

Then, Lemma 4 in the Appendix implies that the integrals on the RHS above are finite. Thus, the expectation of the seventh term is bounded by an integrable function.

Hence, the eighth term on the RHS of **(3.31)** is bounded by a function with finite expectation.

Thus, $\frac{\partial^3 g_\kappa}{\partial\lambda\partial\kappa^2}$ is bounded in absolute value by a function with finite expectation. ■

3.4. Expectation of the squares of the first partial derivatives of the logarithm of the p.d.f. with respect to the two parameters

LEMMA 3.5: For a GLLE(λ, κ) distribution, the expectation of the square of the natural logarithm of the density function with respect to each component of the parameter vector is positive and finite for $\kappa > 1$.

PROOF: From **(3.1)**, we have

$$\frac{\partial ln\left(g_\kappa(x)\right)}{\partial\kappa} = \frac{1}{\kappa} + \left(1 - 2\Lambda_\kappa\left(G(x)\right)\right) ln\left(\frac{G(x)}{\bar{G}(x)}\right).$$

Then, using Lemma 4 in the Appendix, we find that

$$E\left[\left(\frac{\partial ln\left(g_\kappa(X)\right)}{\partial\kappa}\right)^2\right] \leq \frac{1}{\kappa^2} + \frac{2}{\kappa}\int_0^\infty ln\left(\frac{G(x)}{\bar{G}(x)}\right) g_\kappa(x)dx + \int_0^\infty \left(ln\left(\frac{G(x)}{\bar{G}(x)}\right)\right)^2 g_\kappa(x)dx$$

is finite.

Also from **(3.1)**, we have

$$\frac{\partial ln\left(g_\kappa(x)\right)}{\partial\lambda} = \frac{1}{\lambda} - \frac{x\bar{G}(x)}{G(x)} + \frac{\kappa x}{G(x)}\left(1 - 2\Lambda_\kappa\left(G(x)\right)\right).$$

Thus, by Lemma 3 in the Appendix, we have

$$\left|\frac{\partial ln\left(g_\kappa(x)\right)}{\partial\lambda}\right| \leq \frac{1}{\lambda} + M_\eta + \frac{\kappa x}{G(x)},$$

and

$$E\left[\left(\frac{\partial ln\left(g_\kappa(X)\right)}{\partial\lambda}\right)^2\right] \le \left(\frac{1}{\lambda}+M_\eta\right)^2 + 2\kappa\left(\frac{1}{\lambda}+M_\eta\right)\int_0^\infty \frac{x}{G(x)}g_\kappa(x)dx$$
$$+\kappa^2\int_0^\infty \frac{x^2}{\left(G(x)\right)^2}g_\kappa(x)dx$$

Using **(3.1)** and Lemmas 1 and 3 in the Appendix, we find that the first integral on the RHS above is

$$\int_0^\infty \frac{x}{G(x)}g_\kappa(x)dx \le \kappa\lambda M_\eta 2^{2(\kappa-1)}\int_0^\infty e^{-(\kappa-1)\lambda x}dx = \frac{\kappa M_\eta 2^{2(\kappa-1)}}{\kappa-1}.$$

Also using **(3.1)** and Lemmas 1 and 3 in the Appendix, we find that the second integral on the RHS above is

$$\int_0^\infty \frac{x^2}{\left(G(x)\right)^2}g_\kappa(x)dx \le \kappa\lambda M_\nu 2^{2(\kappa-1)}\int_0^\infty e^{-(\kappa-1)\lambda x}dx = \frac{\kappa M_\nu 2^{2(\kappa-1)}}{\kappa-1}.$$

Thus the last regularity condition is satisfied for $\kappa > 1$. ■

3.5. Asymptotic properties theorem

As a result of the five preceding lemmas in this section, we may state (Schervish, 1995; Serfling, 1980; Stuart, et. al., 1999; Wijsman, 1973; Wilks, 1962) the following theorem.

THEOREM: If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution which is Generalized Log-Logistic Exponential, with scale parameter $\lambda > 0$ and transformation parameter $\kappa > 1$, then with probability 1, the likelihood equations admit a sequence of solutions $\{\widehat{\boldsymbol{\theta}}_n\}$ satisfying: a) strong consistency: $\widehat{\boldsymbol{\theta}}_n \to \boldsymbol{\theta}$ as $n \to \infty$, and b) asymptotic normality and efficiency: $\widehat{\boldsymbol{\theta}}_n$ is $AN\left(\boldsymbol{\theta}, \boldsymbol{n^{-1}I_\theta^{-1}}\right)$, where the information matrix

$$\boldsymbol{I_\theta} = \left[E_{G_\kappa}\left\{\left((lng_\kappa)_{(\theta_i)}\right)\left((lng_\kappa)_{(\theta_j)}\right)\right\}\right].$$

4. Simulation Results

A simulation study was conducted to evaluate the joint asymptotic efficiency of the MLE's of the parameters of the GLLE distribution. Without loss of generality, we chose the scale parameter to be 1. The chosen values of the transformation parameter were 3.0, 2.0, 1.5, 1.0, 0.5, and 0.25. The simulation study contained two parts. In the first part, the MLE's and their variances were simulated. In the second part, we evaluated the Cramer-Rao Lower Bound (CRLB) by simulating the information matrix. The results were compared to decide whether the variances of the MLE's tended to approach the CRLB.

Let $U \sim Uniform(0,1)$. To generate a random variate from a GLLE(λ, κ) distribution, with chosen values for λ and κ, we set

$$u = \frac{\left(e^{\lambda x} - 1\right)^{\kappa}}{1 + \left(e^{\lambda x} - 1\right)^{\kappa}},$$

and invert, obtaining $x = \frac{1}{\lambda}\ln\left(1 + \left(\frac{u}{1-u}\right)^{1/\kappa}\right)$. We substitute n generated values of U into this equation to obtain a simulated random sample of size n from a GLLE(λ, κ) distribution.

In terms of the baseline and composite distributions, the likelihood equations are:

$$\frac{\partial l}{\partial \kappa} = \frac{n}{\kappa} + \sum_{i=1}^{n} \ln\left(G(x_i)\right) - \lambda \sum_{i=1}^{n} x_i - 2\sum_{i=1}^{n} \ln\left(G(x_i)\right) G_{\kappa}(x_i) + 2\lambda \sum_{i=1}^{n} x_i \bar{G}_{\kappa}(x_i),$$

and

$$\frac{\partial l}{\partial \lambda} = \frac{n}{\lambda} + (\kappa - 1)\sum_{i=1}^{n} \frac{x_i \bar{G}(x_i)}{G(x_i)} - \kappa \sum_{i=1}^{n} x_i - 2\kappa \sum_{i=1}^{n} \frac{x_i \bar{G}(x_i)}{G(x_i)} G_{\kappa}(x_i) + 2\kappa \sum_{i=1}^{n} x_i \bar{G}_{\kappa}(x_i).$$

These were set equal to 0 and solved numerically to find the MLE's.

The algorithm for simulating the MLE's and estimated covariance matrix:

1. Generate a random sample of size n from a GLLE distribution.
2. Calculate the MLE's.
3. Repeat the first two steps 1000 times.
4. Calculate the estimated covariance matrix.

The algorithm for evaluating the CRLB:

1. Generate a random variate from a GLLE distribution with given scale parameter and transformation parameter.
2. Evaluate the squares of the first partial derivatives of the logarithm of the density function with respect to the two respective parameters, and the product of the two first partial derivatives.
3. Repeat the first two steps 5000 times.
4. Find the averages of the above three quantities to estimate the four elements of the Fisher information matrix.
5. Invert the matrix and multiply the diagonal elements by 1/n to obtain the estimated CRLB.

The results of the simulation are presented in the tables below. Tables 1 to 6 present the simulation results for the MLE's and estimated variances for both parameters. Tables 7 to 12 present the differences between the estimated variances and the corresponding CRLB's.

Table 1: κ = 3 and λ = 1

n	$\hat{E}(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$\hat{E}(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	1.001125749	0.001768968	3.013491483	0.065186515
110	1.000785286	0.001456501	3.026518598	0.06318737
120	0.999340124	0.001389415	3.0250789	0.053447569
130	1.000480552	0.001241102	3.028950807	0.049939951
140	1.001798459	0.001221275	3.016643976	0.042107998
150	0.998726951	0.001071179	3.019782171	0.04155098
160	1.000839726	0.001097054	3.013302705	0.040841092
170	1.001673574	0.001030968	3.013898422	0.036684858
180	1.001255294	0.00093284	3.005612011	0.03606937
190	0.999299428	0.000808745	3.006299152	0.034189518
200	1.000179172	0.000871689	3.016910752	0.031295812

Table 2: κ = 2 and λ = 1

n	$\hat{E}(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$\hat{E}(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	1.002454292	0.003407492	2.005670216	0.027603
110	1.004599604	0.003119688	2.01955518	0.027636
120	1.001600936	0.003200932	2.016340406	0.023595
130	1.001174895	0.002631337	2.00964898	0.022263
140	1.003221361	0.002791891	2.009530121	0.020723
150	1.001494831	0.002336678	2.007397904	0.017988
160	1.002517937	0.002440983	2.00182019	0.01919
170	0.999816102	0.002109983	2.010170053	0.017719
180	1.002440618	0.002005028	2.013965088	0.015169
190	1.0018852	0.001816137	2.013453912	0.014212
200	1.002927173	0.001895421	2.014119136	0.014125

Table 3: κ = 1.5 and λ = 1

n	$\hat{E}(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$\hat{E}(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	1.004913748	0.006099656	1.517384639	0.016007
110	1.006263712	0.005472738	1.514604842	0.015646
120	1.004553852	0.004468002	1.508908329	0.014388
130	1.006066782	0.004892317	1.50626504	0.011974
140	1.004739657	0.004056058	1.510753485	0.01136
150	1.000800482	0.003649807	1.508005461	0.010293
160	1.003654916	0.00371035	1.50338242	0.010151
170	1.004894546	0.003607423	1.5075619	0.009785
180	1.003217785	0.003473121	1.511898558	0.00975
190	1.002131601	0.003104311	1.508210397	0.008309
200	1.000240384	0.002893984	1.50458012	0.007536

Table 4: κ = 1 and λ = 1

n	$\hat{E}(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$\hat{E}(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	1.011348156	0.0109347	1.00296116	0.006757
110	1.009089758	0.0096083	1.00565177	0.006626
120	1.011014586	0.0094702	1.00480053	0.005552
130	1.007037837	0.0083208	1.00284849	0.005512
140	1.008081372	0.0075836	1.00753959	0.00503
150	1.007591632	0.0072112	1.00991224	0.004845
160	1.015388201	0.0067138	1.01033131	0.004435
170	1.004809568	0.006134	1.00213605	0.004225
180	1.008245478	0.0056417	1.00506997	0.003941
190	1.01074676	0.0057851	1.00420861	0.004128
200	1.002714724	0.0050746	1.00217774	0.003706

Table 5: κ = 0.5 and λ = 1

n	$\hat{E}(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$\hat{E}(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	1.020816584	0.0173823	0.50495578	0.001771
110	1.019769348	0.0161177	0.50395102	0.001558
120	1.013707054	0.0138546	0.5020536	0.001503
130	1.015732208	0.0131783	0.50238774	0.001395
140	1.012712389	0.0130471	0.50467292	0.001401
150	1.01657043	0.0105934	0.50265586	0.001205
160	1.012332079	0.0108858	0.50260116	0.00112
170	1.008269904	0.0095322	0.50120098	0.000924
180	1.011543402	0.0090271	0.50109543	0.001026
190	1.00863372	0.0083533	0.50291172	0.000886
200	1.010423929	0.0081023	0.50164918	0.000907

Table 6: κ = 0.25 and λ = 1

n	$\hat{E}(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$\hat{E}(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	1.023429598	0.016222	0.25241718	0.000438
110	1.018713738	0.0166837	0.25180928	0.000451
120	1.018160677	0.0157249	0.25189155	0.0004
130	1.015540994	0.0143469	0.25167838	0.000352
140	1.013978264	0.0111233	0.25179948	0.000336
150	1.013925343	0.0111865	0.25144254	0.000301
160	1.011180729	0.0108696	0.2513737	0.000283
170	1.015184211	0.0105859	0.2514993	0.000282
180	1.016492297	0.0096427	0.25162673	0.000237
190	1.00933421	0.0087129	0.25136223	0.000224
200	1.013720433	0.0085688	0.25238844	0.000223

Table 7: κ = 3 and λ = 1

n	$CRLB(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$CRLB(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	1.63513E-05	0.001768968	0.000631778	0.065186515
110	1.39208E-05	0.001456501	0.000524663	0.06318737
120	1.16737E-05	0.001389415	0.000462971	0.053447569
130	9.79998E-06	0.001241102	0.000379027	0.049939951
140	8.81020E-06	0.001221275	0.000323560	0.042107998
150	7.71129E-06	0.001071179	0.000291942	0.04155098
160	6.45611E-06	0.001097054	0.000254583	0.040841092
170	5.78695E-06	0.001030968	0.000226281	0.036684858
180	5.27834E-06	0.00093284	0.000197507	0.03606937
190	4.74250E-06	0.000808745	0.000179838	0.034189518
200	4.25858E-06	0.000871689	0.000156712	0.031295812

Table 8: κ = 2 and λ = 1

n	$CRLB(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$CRLB(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	3.54035E-05	0.003407492	0.000292617	0.027603469
110	3.12867E-05	0.003119688	0.000245548	0.027636468
120	2.56908E-05	0.003200932	0.000198794	0.023594641
130	2.20802E-05	0.002631337	0.000172064	0.022262681
140	1.83566E-05	0.002791891	0.000153608	0.020723469
150	1.67602E-05	0.002336678	0.000130659	0.017988025
160	1.46984E-05	0.002440983	0.000115759	0.019189793
170	1.27992E-05	0.002109983	0.000100777	0.017719104
180	1.14493E-05	0.002005028	9.39030E-05	0.015169107
190	1.00901E-05	0.001816137	8.24004E-05	0.014211697
200	8.88526E-06	0.001895421	7.30947E-05	0.014125095

Table 9: κ = 1.5 and λ = 1

n	$CRLB(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$CRLB(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	6.20146E-05	0.006099656	0.00018024	0.016006769
110	5.18892E-05	0.005472738	0.00013768	0.015646155
120	4.30093E-05	0.004468002	0.000118736	0.014387556
130	3.83645E-05	0.004892317	0.000102972	0.011973794
140	3.30375E-05	0.004056058	9.03979E-05	0.011360234
150	2.75008E-05	0.003649807	7.65130E-05	0.010293458
160	2.45281E-05	0.00371035	6.73736E-05	0.010151349
170	2.20230E-05	0.003607423	5.95262E-05	0.009785136
180	1.92334E-05	0.003473121	5.34973E-05	0.009750133
190	1.72763E-05	0.003104311	4.76184E-05	0.008308865
200	1.62548E-05	0.002893984	4.29802E-05	0.007536144

Table 10: κ = 1 and λ = 1

n	$CRLB(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$CRLB(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	0.000117654	0.010934718	8.43314E-05	0.006756552
110	9.75270E-05	0.009608251	7.30740E-05	0.006626176
120	8.32848E-05	0.009470224	5.82261E-05	0.005551558
130	6.98807E-05	0.008320847	5.01450E-05	0.005511549
140	6.07769E-05	0.00758364	4.30533E-05	0.005030476
150	5.33687E-05	0.007211223	3.75767E-05	0.004844793
160	4.87387E-05	0.0067138	3.28210E-05	0.004435137
170	4.24524E-05	0.006134016	2.93007E-05	0.004224744
180	3.75354E-05	0.005641733	2.62918E-05	0.003941071
190	3.33194E-05	0.005785119	2.35158E-05	0.004127568
200	3.03357E-05	0.005074613	2.12919E-05	0.003706077

Table 11: κ = 0.5 and λ = 1

n	$CRLB(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$CRLB(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	0.000239788	0.017382284	2.64371E-05	0.001770712
110	0.000204437	0.016117651	2.17209E-05	0.001558213
120	0.00017332	0.013854568	1.85515E-05	0.001503341
130	0.000143753	0.013178294	1.6404E-05	0.001395243
140	0.000126519	0.013047066	1.38956E-05	0.001400904
150	0.000111575	0.010593407	1.21155E-05	0.001204757
160	9.54672E-05	0.010885846	1.05242E-05	0.001120218
170	8.48157E-05	0.009532231	9.29180E-06	0.000924
180	7.56372E-05	0.009027064	8.24801E-06	0.001026008
190	6.85013E-05	0.008353267	7.43450E-06	0.000886202
200	6.14205E-05	0.008102274	6.64519E-06	0.000906536

Table 12: κ = 0.25 and λ = 1

n	$CRLB(\hat{\lambda})$	$\hat{V}(\hat{\lambda})$	$CRLB(\hat{\kappa})$	$\hat{V}(\hat{\kappa})$
100	Indeterminate	0.016222038	Indeterminate	0.000438493
110	Indeterminate	0.016683656	Indeterminate	0.000450572
120	Indeterminate	0.015724927	Indeterminate	0.000399894
130	Indeterminate	0.014346928	Indeterminate	0.000352489
140	Indeterminate	0.0111233	Indeterminate	0.000336
150	Indeterminate	0.01118653	Indeterminate	0.000300584
160	Indeterminate	0.010869561	Indeterminate	0.000282979
170	Indeterminate	0.010585914	Indeterminate	0.000282421
180	Indeterminate	0.009642659	Indeterminate	0.000237439
190	Indeterminate	0.008712918	Indeterminate	0.000223855
200	Indeterminate	0.008568814	Indeterminate	0.000222519

From examination of the tabulated results, it is apparent that both estimators tend to overestimate the parameters slightly. There appears to be a somewhat stronger tendency for $\hat{\kappa}$ to overestimate.

It is clear that the estimated variances approach the CRLB as the sample size increases, albeit somewhat slowly. We were unable to assess the efficiency for $\kappa = 0.25, \lambda = 1.0$, since we were unable to compute the CRLB for that case.

6. Conclusion.

We have proved that, for any positive value of the scale parameter and for any value of the transformation parameter exceeding 1, the Generalized Log-Logistic Exponential distribution family satisfies the standard regularity conditions. Hence, the MLE's of the parameters, in this parameter space, are strongly consistent and jointly asymptotically normally distributed and efficient.

In addition, the simulation results imply that the MLE's are jointly efficient, with the variances of the estimators approaching the Cramer-Rao Lower Bounds, not only for the parameter values in the space mentioned in the preceding paragraph, but also for some values of the transformation parameter less than 1. Only for $\kappa = 0.25$ were we unable to assess the asymptotic efficiency.

Some practical implications of these results are that standard normally-based statistical inferential procedures may be used in situations in which a GLLE distribution appears to provide a good fit to a data set. For example, in Section 1, it was mentioned that Gleaton and Lynch (2010) found that a GLLE distribution appeared to provide a better fit than a 2-parameter Weibull distribution to a data set consisting of measured tensile breaking strengths of a sample of $n = 64$ ten-millimeter-long carbon fibers.

Appendix I: Lemmas

LEMMA 1: Let $G(x) = \left(1 - e^{-\lambda x}\right) I_{(0,+\infty)}(x)$, for $\lambda \geq 0$, and let $G_\kappa(x) = \Lambda_\kappa\left(G(x)\right)$. Then, for $\kappa \geq 1$, we have

a) $\left[\left(G(x)\right)^\kappa + \left(\bar{G}(x)\right)^\kappa\right]^{-1} \leq 2^{\kappa-1}$, and

b) $0 \leq g_\kappa(x) = G_\kappa{}'(x) \leq 2^{2(\kappa-1)}\kappa\lambda$.

PROOF: Assume that $\kappa \geq 1$. Let $u = \left(1 - e^{-\lambda x}\right) I_{(0,+\infty)}(x)$, $\bar{u} = 1 - u$, and let

$$h_\kappa(u) = u^\kappa + \bar{u}^\kappa.$$

We have $\lim_{u \downarrow 0} h_\kappa(u) = \lim_{u \to 1} h_\kappa(u) = 1$, and $h_\kappa(0.5) = 2^{-(\kappa-1)} \leq 1$, with $h_\kappa(0.5) = 1$ if and only if $\kappa = 1$.

Differentiating, we have

$$h'_\kappa(u) = \kappa(u^{\kappa-1} - \bar{u}^{\kappa-1}).$$

Now, $h'_\kappa(0.5) = 0$. In addition, it is clear that, for $0 < u < 0.5$, $h'_\kappa(u) < 0$, and that for $0.5 < u < 1$, $h'_\kappa(u) > 0$. Since $h_\kappa(u)$ is differentiable on the unit interval, it follows that $h_\kappa(u)$ has a minimum at $u = 0.5$. Thus, $h_\kappa(u) \geq 2^{-(\kappa-1)}$, or $\left(h_\kappa(u)\right)^{-1} \leq 2^{(\kappa-1)}$.

For $\kappa \geq 1$, it is clear that $u^{\kappa-1} \leq 1$ and that $\bar{u}^\kappa \leq 1$ Then, using the result above, we have

$$g_\kappa(u) = \frac{\kappa\lambda u^{\kappa-1}\bar{u}^\kappa}{(u^\kappa + \bar{u}^\kappa)^2} \leq 2^{2(\kappa-1)}\kappa\lambda. \quad \blacksquare$$

LEMMA 2: Let κ ≥ 1, let $\Lambda_\kappa(u) = \frac{u^\kappa}{u^\kappa+\bar{u}^\kappa} I_{(0,1)}(u)$, and $\bar{\Lambda}_\kappa(u) = 1 - \Lambda_\kappa(u) = \Lambda_\kappa(\bar{u})$. Then, for any positive integers n, m, there exists positive constants A_{nm} and β_m such that

$$\Lambda_\kappa^n\big(G(x)\big)\,\Lambda_\kappa^m\big(\bar{G}(x)\big) \leq A_{nm}e^{-\beta_m x},$$

for all x.

PROOF: Let κ ≥ 1, and let n, m be positive integers. We have

$$\Lambda_\kappa^n\big(G(x)\big)\,\Lambda_\kappa^m\big(\bar{G}(x)\big) = \frac{\big(G(x)\big)^{n\kappa}\big(\bar{G}(x)\big)^{m\kappa}}{\Big(\big(G(x)\big)^\kappa + \big(\bar{G}(x)\big)^\kappa\Big)^{n+m}}.$$

Now, $0 \leq G(x) \leq 1$, so that $0 \leq \big(G(x)\big)^{n\kappa} \leq 1$, and

$$\Lambda_\kappa^n\big(G(x)\big)\Lambda_\kappa^m\big(\bar{G}(x)\big) \leq \frac{\big(\bar{G}(x)\big)^{m\kappa}}{\Big(\big(G(x)\big)^\kappa + \big(\bar{G}(x)\big)^\kappa\Big)^{n+m}}.$$

Using part (a) of Lemma 1, we have

$$\Lambda_\kappa^n\big(G(x)\big)\Lambda_\kappa^m\big(\bar{G}(x)\big) \leq 2^{(n+m)(\kappa-1)}\big(\bar{G}(x)\big)^{m\kappa} = 2^{(n+m)(\kappa-1)}e^{-m\kappa} \quad ,$$

for all x. Thus, $A_{nm} = 2^{(n+m)(\kappa-1)}$, and $\beta_m = m\kappa\lambda$. ■

LEMMA 3: The functions $\eta(x) = x\frac{\bar{G}(x)}{G(x)}$, $\nu(x) = x^2\frac{\bar{G}(x)}{\big(G(x)\big)^2}$, $\rho(x) = x^2\frac{\bar{G}(x)}{G(x)}$, and $\tau(x) = x^3\frac{\bar{G}(x)}{\big(G(x)\big)^2}$ are bounded.

PROOF: It is straightforward to show, using L'Hopital's Rule, that

$$\lim_{x\to\infty}\eta(x) = 0, \lim_{x\to\infty}\nu(x) = 0, \lim_{x\to\infty}\rho(x) = 0, \lim_{x\to\infty}\tau(x) = 0,$$

and that

$$\lim_{x\downarrow 0}\eta(x) = \frac{1}{\lambda}, \lim_{x\downarrow 0}\nu(x) = \frac{1}{\lambda^2}, \lim_{x\downarrow 0}\rho(x) = 0, \lim_{x\downarrow 0}\tau(x) = 0.$$

Each function is non-negative, bounded, and continuously differentiable on $(0, +\infty)$. Thus, they have maxima, M_η, M_ν, M_ρ, and M_τ, respectively. ■

LEMMA 4: If $\kappa \geq 1$, then for any positive integers n and m, the functions
$i)\ \varphi_{1n}(x) = g_\kappa(x)\big|ln\big(G(x)\big)\big|^n$, $\quad ii)\ \varphi_{2n}(x) = g_\kappa(x)\big|ln\big(\bar{G}(x)\big)\big|^n$,
$iii)\ \varphi_{3nm}(x) = x^m g_\kappa(x)\big|ln\big(G(x)\big)\big|^n$, and $iv)\ \varphi_{4nm}(x) = x^m g_\kappa(x)\big|ln\big(\bar{G}(x)\big)\big|^n$
are integrable for $\kappa \geq 1$.

PROOF:

i) Using the change of variable $u = G(x) = 1 - e^{-\lambda x}$, it is straightforward to show that, for $\kappa \geq 1$ and using Lemma 1 and Gradshteyn (2015), we have

$$\int_0^\infty \varphi_{1n}(x)dx = \kappa\int_0^1 \frac{1}{u\bar{u}}\Lambda_\kappa(u)\Lambda_\kappa(\bar{u})|ln(u)|^n du \leq \kappa 2^{2\kappa-2}\int_0^1 |ln(u)|^n du = \kappa 2^{2\kappa-2}n!.$$

ii) Similarly, letting $u = \bar{G}(x) = e^{-\lambda x}$, we find

$$\int_0^\infty \varphi_{2n}(x)dx \leq \kappa 2^{2\kappa-2} n!.$$

iii) The common class median of the distribution is $m = \frac{ln(2)}{\lambda}$. If we let $u = G(x) = 1 - e^{-\lambda x}$, then, using Lemma 1 and Gradshteyn (2015), we have

$$\int_0^\infty \varphi_{3nm}(x)dx = \kappa 2^{2\kappa-2} \int_0^1 |ln(u)|^n |ln(\bar{u})|^m du.$$

For $0 < u < 0.5$, we have $|ln(\bar{u})| \leq ln(2)$, so that, using Gradshteyn (2015), we have

$$\int_0^m \varphi_{3nm}(x)dx \leq \kappa 2^{2\kappa-2} \big(ln(2)\big)^m \int_0^{0.5} |\ln(u)|^n du \leq \kappa 2^{2\kappa-2} \big(ln(2)\big)^m n!.$$

For $0.5 < u < 1$, we have $|ln(u)| \leq ln(2)$, so that, using Gradshteyn (2015), we have

$$\int_m^\infty \varphi_{3nm}(x)dx \leq \kappa 2^{2\kappa-2} \big(ln(2)\big)^n \int_{0.5}^1 |\ln(\bar{u})|^m du \leq \kappa 2^{2\kappa-2} \big(ln(2)\big)^n m!.$$

Hence,

$$\int_0^\infty \varphi_{3nm}(x)dx \leq \kappa 2^{2\kappa-2} \left[\big(ln(2)\big)^m n! + \big(ln(2)\big)^n m!\right],$$

iv) Similarly, if we let $u = \bar{G}(x) = e^{-\lambda}$ and use Lemma 1 and Gradshteyn (2015), we have

$$\int_0^\infty \varphi_{4n}\ (x)dx \leq \kappa 2^{2\kappa-2} \left[\big(ln(2)\big)^m n! + \big(ln(2)\big)^n m!\right]. \qquad \blacksquare$$

LEMMA 5: Let *n* be a positive integer. Let *a* and *b* be real numbers with $a > b$. Then the following function is integrable:

$$f_n(x) = \frac{x^n e^{-ax}}{1 - e^{-bx}} I_{(0,\infty)}(x).$$

PROOF: First, it will be shown that the (strictly positive) function is bounded above. Then, it will be shown that the tail of the function is bounded by an integrable function.

First, we have

$$\lim_{x\to\infty} f_n(x) = 0.$$

Then, using L'Hopital's Rule, we find that

$$\lim_{x\downarrow 0} f_n(x) = \begin{cases} \frac{1}{\lambda}, for\ n = 1, or \\ 0, for\ n > 1. \end{cases}$$

Then, since the function is strictly positive and continuously differentiable, it is bounded above.

Next, let $x_0 > 0$. Then, we have

$$\int_{x_0}^{\infty} f_n(x)dx \le \frac{1}{1-e^{-bx_0}} \int_0^{\infty} x^n e^{-ax} dx = \frac{\Gamma(n+1)}{a^{n+1}(1-e^{-bx_0})}. \quad \blacksquare$$

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