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A Note on Early Termination in the 2^{k-p} Designs

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ABSTRACT

A complete factorial experiment can resolve problems that occur in experiments that to determine suitable tolerances for the factors in a manufacturing process. Two-level fractional factorial designs have been widely used to investigate the effect of factors in which several factors are involved. In many circumstances, the experiment may be stopped before all the points have been run. This article argues that one should not arrange the points of the experiment in random order. Instead, one should consider adjusting the run order to protect against the risk of early termination, that is, a systematic run order should be carefully applied to the case. In this article, we will use semi-folding design as examples to illustrate the idea of how to take advantage of the sequential run order.

Keywords: Defining contrast subgroup; Minimum aberration; Independent defining words; Semi-folding

1. Introduction

The 2^{k-p} fractional factorial designs have been widely used in the experiments in which several factors are involved. These designs consist of $k-p$ basic factors and p independent generated factors. We may also say that there are p independent defining words, where a "word" consists of letters which are the names of the factors denoted by A, B, ... The group generated by these p defining words is called the defining contrast subgroup. The defining contrast subgroup for the design consists of the p generators initially chosen and their $2^p - p - 1$ generalized interactions.

A regular 2^{k-p} fractional factorial design is orthogonal, so the construction of the design and the analysis of the data from the experiments are reasonably straightforward. Usually, the optimal fractional factorial designs are chosen according the resolution, proposed by Box and Hunter (1961). A reasonable criterion is to select the best generators such that the resulting design has the highest resolution. Although it is the criterion to choose a good design, sometime resolution alone is insufficient to distinguish among designs with the same resolution. Fries and Hunter (1980) proposed the criterion for the minimum aberration (MA) to discriminate designs with the same resolution. Fold over designs, in which the levels

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of the factors are completely changed, are commonly used in the follow-up experiments. Box and Wilson (1951) were the first to discuss the properties for the fold-over strategy. How to use the fold-over technique to break the alias chains were also described by authors, such as Box and Hunter (1961), Daniel (1962), John (1971), Montgomery and Runger (1996), Li and Mee (2002), and Montgomery (2013). Li and Lin (2003) used the minimum aberration criterion and utilized a computer to find the optimal foldover design for giving k and p in the 2^{k-p} fractional factorial designs. Note that the augmented design after foldover is still orthogonal. However, the technique of foldover requires the same run size as the original experiment. This can sometimes be quite wasteful for breaking the alias chains.

Using the technique of foldover and running the half-run size of the first experiment, called semi-folding design, was considered by some authors such as John (1962), John (2000) and Mee and Peralta (2000). In general, we describe a semi-folding plan in the following way: foldover on ___; subset on ___ (Mee and Peralta, 2000). In fact, the combined design produced by the original 2^{k-p} design plus the 2^{k-p-1} design by semi-folding technique is an irregular design which three fractions are formed in this case. Hence the criterion of MA may not be used as a rule for choosing the optimal design. In this situation, we will apply the criterion of "clear" for design selection. We shall call a main effect or two-factor interaction "clear to estimate" if it can be estimated when other main effects or two-factor interactions appear in the same model. Huang et al. (2008) defined an optimal semi-folding design and utilized a computer to search the corresponding designs for 16 and 32 runs.

In agricultural experimentation, the varieties are shown simultaneously and the experiment points are assigned to the plots at random. Experiments in industry are different from experiments in agriculture, it is usual to consider a planned experimental design to be run in sequence. In practice, due to budget of time or money may run out before the whole experiment as originally planned is finished, or vital piece of equipment may fail with no replacement available. Furthermore, due to the epidemic, for example COVID-19, experimenters must be quarantined and the laboratory was forced to temporarily suspend work. If this happened, it would be preferable that the part of the experiment could offer the better information. Some examples of designs for factorial experiments with three, four, six, seven, and eight factors will be illustrated for the early termination in this article.

2. The experimental points make in sequence

The followings we will use some examples to illustration how to run the experiment in sequence to robust against the early termination.

Example 1. Three Factors in Six Points

Suppose three factors, *A*, *B*, and *C*, each at two levels, are of interest and one would take six points for the experiment. The six points could be divided into three fractions, each having two points, defined by:

(i) *I* = *A* = *ABC* = *BC*, (ii) *I* = -*A* = *ABC* = -*BC*, (iii) *I* = *A* = -*ABC* = -*BC*.

The third fraction (iii) is obtained by the way "$f_0 = C;\ ss = A^+$" from the resolution III plan *I* = *ABC*. Consequently, these three fractions will form three combined designs *I* = *ABC*, *I* = *A*, and *I* = -*BC*. We can estimate all the main effects and two-factor interactions *BC* and *AB* if the effects *AC* and *ABC* are negligible. Suppose, for example, the experiment stops after only four of the six points have been run. In this situation, the experimenter has to select a sequence such that the better information can be obtained. If the four points, *a*, *abc*, *b*, and *c*, are performed first, then all main effects can be estimated from these points. It means that we combine the first two fractions, (i) and (ii), and obtain the resolution III design defined by *I* = *ABC*. Suppose that four points *b*, *c*, *ab*, *ac* (from (ii) and (iii)) are taken. Then main effects *B* and *C* are aliased with each other in the combined design defined by *I* = -*BC*. On the other hand, a random choice of order for the points may produce the significant different results.

Example 2. Four Factors in Twelve Points

Usually, we may take a MA 2^{4-1} design with defining contrast subgroup *I* = *ABCD*. The optimal new semi-folding fraction could be obtained by the way "$f_0 = D;\ ss = A^+$". Hence, it forms three fractions:

(i) *I* = *A* = *ABCD* = *BCD*, (ii) *I* = -*A* = *ABCD* = -*BCD*, (iii) *I* = *A* = -*ABCD* = -*BCD*.

After performing these twelve points, we will obtain three combined designs *I* = *ABCD*, *I* = *A*, and *I* = -*BCD*. Obviously, main effects *A*, *B*, *C*, and *D* can be "clear" to estimate from the fraction *I* = *ABCD*; two-factor interactions *BC*, *BD*, and *CD* are "clear" to estimate from the fraction *I* = *A*; and main effect *A* and three two-factor interactions *AB*, *AC*, and *AD* are "clear" to estimate from the fraction *I* = -*BCD*. This is a resolution V plan, that is, all main effects and two-factor interactions are clear to estimate. The variance of the average of the estimates of *A* from the two half replicates is $3\sigma^2/8$, and the other effects have variance $\sigma^2/2$. Hence, the effect *A* has higher efficiency than the other effects. Suppose, for some reasons, that the experiment has to stop after running eight points. If the experimenter takes fractions (i) and (ii) to be run, in which the combined design is defined by *I* = *ABCD*, it means that all main effects are clear to estimate but two-factor interactions are aliased with each other. On the other hand, for instance, if the eight points are taken from fractions (ii) and (iii), then the resulting design is a resolution III design with defining contrast subgroup *I* = -*BCD*. Note that main effect *A* and all two-factor interactions involving *A* are clear to estimate, but no other main effects are clear to estimate.

Example 3. Six Factors in Twenty-four Points

Consider a MA 2^{6-2} design with defining contrast subgroup *I* = *ABCE* = *ABDF* = *CDEF*, and its corresponding optimal semi-folding design is defined by "$f_0 = E;\ ss = A^{+}$". It is divided into three fractions:

(i) *I* = *A* = *ABCE* = *BCE* = *ABDF* = *BDF* = *CDEF* = *ACDEF*,
(ii) *I* = -*A* = *ABCE* = -*BCE* = *ABDF* = -*BDF* = *CDEF* = -*ACDEF*,
(iii) *I* = *A* = -*ABCE* = -*BCE* = *ABDF* = *BDF* = -*CDEF* = -*ACDEF*.

Then all six main effects plus nine two-factor interactions, *AC*, *AE*, *BC*, *BE*, *CD*, *CE*, *CF*, *DE*, and *EF*, are clear to estimate in the combined design with twenty-four points. If we have prior information that factors *C* and *E* are more important than the other factors, then all two-factor interactions involving factors *C* and *E* are clear to estimate in this plan. Furthermore, it can be shown that the variance of effects *A*, *D*, *F*, *CD*, *CF*, *DE*, and *EF* equals $3\sigma^2/16$; the variance of effects *B*, *C*, *E*, *AC*, *AE*, *BC*, *BE*, and *CE* equals $\sigma^2/4$. That is, the effects *A*, *D*, *F*, *CD*, *CF*, *DE*, and *EF* have higher efficiency than the other effects. If the fractions (i) and (ii) are considered by the experimenter, in this circumstance, then the combined design is a resolution IV design in which six main effects are clear to estimate and two-factor interactions are aliased with each other. Suppose that the experimenter decides to run the fractions (i) and (iii) first. The combined design with sixteen points just allows us to clear estimate two-factor interactions *BC*, *BE*, *CD*, *CE*, *CF*, *DE*, and *EF*, and one thing we need to notice is that no main effects are clear to estimate.

Considering another optimal semi-folding design, we take the eight points from the technique "$f_0 = EF;\ ss = A^{+}$". Then the defining contrast subgroup of new fraction is

(iv) *I* = *A* = -*ABCE* = -*BCE* =- *ABDF* = -*BDF* = *CDEF* = *ACDEF*.

The resulting design, from fractions (i), (ii), and (iv), then provides all six main effects and nine two-factor interactions *AB*, *AC*, *AD*, *AE*, *AF*, *BC*, *BD*, *BE*, and *BF*, that are clear of the other effects. Just as above design, all main effects and nine two-factor interactions are clear to estimate in the combined design. But the variance of the factor *A* is $3\sigma^2/16$, and the variance of the other fourteen effects is $\sigma^2/4$. We may pay attention to one thing that all two-factor interactions can be clear to estimate for factors *A* and *B*. If one performs the sixteen points, from (i) and (iv), then only four effects, *BC*, *BD*, *BE*, and *BF*, are clear to estimate.

Another selection, we may take the design defined by *I* = *ABE* = *ACDF* = *BCDEF* and its corresponding optimal semi-folding design is defined by "$f_0 = EF$; $ss = AC^+$". The twenty-four points could be divided into three fractions:

(i) *I* = *AC* = *ABE* = *BCE* = *ACDF* = *DF* = *BCDEF* = *ABDEF*,
(ii) *I* = -*AC* = *ABE* = -*BCE* = *ACDF* = -*DF* = *BCDEF* = -*ABDEF*,
(iii) *I* = *AC* = -*ABE* = -*BCE* = -*ACDF* = -*DF* = *BCDEF* = *ABDEF*.

Then all main effects and eleven effects, *AB*, *AC*, *AE*, *BC*, *BD*, *BE*, *BF*, *CE*, *DE*, *DF*, and *EF*, are clear to estimate in these twenty-four points. Note that the effects *D*, *F*, *BD*, *BF*, *DE*, and *EF* have higher efficiency than the other effects and all two-factor interactions involving *B* and *E* are clear to estimate. Suppose that the fractions (i) and (iii) are performed in the experiment. The effects *B*, *D*, *E*, *F*, *BD*, *BE*, *BF*, *DE*, *DF*, and *EF* are clear to estimate in these sixteen points, that is, this is resolution V design for factors *B*, *D*, *E*, and *F*. If the fractions (ii) and (iii) are considered to be run, in this case, only four effects *A*, *AB*, *AC*, and *AE* are clear to estimate.

Example 4. Seven Factors in Forty-eight Points

We may start a MA 2^{7-2} fractional factorial design that has the two independent defining words *I* = *ABCF* and *I* = *ABDEG*. The optimal semi-folding design of the design is taken from "$f_0 = F$; $ss = A^+$", and it forms three fractions:

(i) *I* = *A*, *I* = *BCF*, and *I* = *BDEG*, (ii) *I* = -*A*, *I* = -*BCF*, and *I* = -*BDEG*,
(iii) *I* = *A*, *I* = -*BCF*, and *I* = *BDEG*.

All main effects and two-factor interactions are clear to estimate in these forty-eight points. It can be showed that the variance of effects *CD*, *CE*, *CG*, *DF*, and *FG* is $\sigma^2/12$; the variance of effects *A*, *D*, *E*, *G*, *AD*, *AE*, *AH*, *AG*, *BD*, *BE*, *BG*, *DE*, *DG*, *EF*, and *EG* is $3\sigma^2/32$; the other effects have the variance $\sigma^2/8$. That is, the effects *CD*, *CE*, *CG*, *DF*, and *FG* have the highest efficiency among the twenty-eight effects. If the fractions (i) and (ii) are considered to do the experiment at the beginning, then there are twenty-two clear effects including all main effects and all two-factor interactions for factors *D*, *E*, and *G* in these thirty-two points. If the thirty-two points, from fractions (ii) and (iii), are run first, then the combined design provides main effects *A*, *D*, *E*, *G* and all their two-factor interactions and the other twelve two-factor interactions are clear to estimate.

Suppose that we use another 2^{7-2} fractional factorial design defined by $I = ABF$ and $I = ACDG$. Its corresponding optimal semi-folding design is obtained by the way "$f_0 = FG;\ ss = E^+$". The points will be formed three combined designs:

(i) *I* = *E*, *I* = *ABF*, and *I* = *CDG*, (ii) *I* = -*E*, *I* = *ABF*, and *I* = *CDG*,
(iii) *I* = *E*, *I* = -*ABF*, and *I* = -*CDG*.

We may obtain that all main effects and two-factor interactions are clear to estimate. In this circumstance, the variance of *AC*, *AD*, *AG*, *BC*, *BD*, *BG*, *CF*, *DF*, and *FG* is $\sigma^2/12$; the variance of effects *E* and *CE* is $3\sigma^2/32$; the other effects have the variance $\sigma^2/8$. If we consider doing the fraction (i) and (ii) first, then only one main effect *E* and fifteen two-factor interactions are clear to estimate in these thirty-four points. On the other hand, if fractions (ii) and (iii) are considered by the experimenter, then all main effects and two-factor interactions, *AC*, *AD*, *AG*, *BC*, *BD*, *BG*, *CF*, *DF*, and *FG*, are clear to estimate.

Example 5. Eight Factors in Forty-eight Points

Consider an experiment in which eight factors *A*, *B*, *C*, *D*, *E*, *F*, *G*, and *H* are being studied. The 2^{8-3} MA design has three independent defining words *I* = *ABCF*, *I* = *ABDG*, and *I* = *ACDEH*. This is a resolution IV design that consists of thirty-two points. The corresponding optimal semi-folding is obtained by the way "$f_0 = F;\ ss = A^+$". The points can be divided into three fractions:

(i) *I* = *A*, *I* = *BCF*, *I* = *BDG*, and *I* = *CDEH*, (ii) *I* = -*A*, *I* = -*BCF*, *I* = -*BDG*, and *I* = -*CDEH*,
(iii) *I* = *A*, *I* = -*BCF*, *I* = *BDG*, and *I* = *CDEH*.

Consequently, the forty-eight points provide that all main effects and twenty-two two-factor interactions, *AC*, *AE*, *AF*, *AH*, *BC*, *BE*, *BF*, *BH*, *CD*, *CE*, *CF*, *CG*, *CH*, *DE*, *DF*, *DH*, *EF*, *EG*, *EH*, *FG*, *FH*, and *GH*, are clear to estimate. The variance of effects, *BE* and *BH*, is $\sigma^2/12$; the variance of effects, *A*, *D*, *E*, *G*, *H*, *AE*, *AH*, *CE*, *CG*, *CH*, *DE*, *DF*, *DH*, *EF*, *EG*, *FH*, and *GH*, is $3\sigma^2/32$; the other effects have the variance $\sigma^2/8$. Hence the effects *BE* and *BH* have the highest efficiency among these thirty effects. One thing worth mentioning is that all two-factor interactions involving factors *C*, *E*, *F*, and *H* are clear to estimate in these forty-eight points. If the thirty-two points, from (i) and (ii), are considered by the experimenter in the first experiment, then there are twenty-one clear effects including all main effects. Note that all two-factor interactions with factor *E* are clear to estimate. Suppose that we take the fractions (i) and (iii) in the first experiment. Consequently, these thirty-two points provide estimate of only twelve two-factor interactions.

Suppose that we consider another experiment that the three independent defining words are *I* = *ABF*, *I* = *ACDG*, and *I* = *BCEH*. Its corresponding optimal semi-folding design is taken by the way "$f_0 = FGH$; $ss = ADE^+$", in this case, all main effects and twenty-eight two-factor interactions are clear to estimate. Note that there are only twenty-two two-factor interactions could be clear to estimate in the previous design. Notice again that effects *C*, *D*, *F*, *AB*, *AC*, *AE*, *AH*, *BC*, *BD*, *BG*, *CF*, *DG*, *DH*, *EF*, *EG*, *EH*, and *FG* have the variance $3\sigma^2/32$ and the variance of the other effects is $\sigma^2/8$. In fact, these forty-eight points form three fractions defined by:

(i) *I* = *ADE*, *I* = *ABF*, *I* = *ACDG*, and *I* = *ABCDH*, (ii) *I* = -*ADE*, *I* = *ABF*, *I* = *ACDG*, and *I* = -*ABCDH*, (iii) *I* = *ADE*, *I* = -*ABF*, *I* = -*ACDG*, and *I* = -*ABCDH*.

If we consider running thirty-two points first, there are three selections, (i) and (ii), (i) and (iii), or (ii) and (iii). Each selection allows us to estimate eighteen clear effects. Note that if we choose (i) and (iii), then only two main effects *C* and *F* plus sixteen two-factor interactions are clear to estimate.

Remark 1: In the worst situation in which only four points can be run for four factors design as the discussion of example 2, we may first take the fraction *I* = -*A* = *ABCD* = -*BCD* with the points (1), *bc*, *bd*, and *cd*, then we have half replicate of the factors, *B*, *C*, and *D*, that has resolution III design. The addition of two points, *ad* and *abcd*, is considered next that defined by *I* = *D* = *A* = *AD* = *ABCD* = *ABC* = *BCD* = *BC*, in this circumstance, it makes a plan with six points for all four factors with resolution III. Furthermore, adding two points, *ac* and *ab*, will form a resolution IV design, and then, four points, *abc*, *a*, *acd*, and *bcd* are added in the experiment. Finally a resolution V design is produced.

Remark 2: Reconsidering Example 3, the sixteen points for the quarter replicate is defined by *I* = *ABCE* = *ABDF* = *CDEF* with the experiment points *df*, *aef*, *bde*, *ab*, *ce*, *acd*, *bcf*, *abcdef*, (1), *abdf*, *cdef*, *abce*, *bef*, *acf*, *ade*, and *bc*. If only eight points can be run for under certain circumstances, we may choose the first eight points which the three independent defining words are *I* = *ACD*, *I* = *ABCE*, and *I* = *BCF*. This is a resolution III design. We can then estimate all main effects and one of the two-factor interactions *AB* (*CE* or *DF*). Adding the next four points (1), *abdf*, *cdef*, and *abce* to the experiment, which the defining words are *I* = *AB*, *I* = -*ACD*, *I* = *CE*, and *I* = -*ACF*, produces a plan that all main effects and four two-factor interactions can be estimated. Since the linear relations between these points are: *D* – *F* – *AC* (= *BE*) + *AE* (= *BC*) = 0, *C* – *E* – *AD* (= *BF*) + *AF* (= *BD*) = 0, *A* – *B* – *CD* (= *EF*) + *CF* (= *DE*) = 0, and *AB* = *CE* = *DF*, it means that two-factor interactions form four groups which are (*AC*, *AE*, *BC*, *BE*), (*AD*, *AF*, *BD*, *BF*), (*CD*, *CF*, *DE*, *EF*), and (*AB*, *CE*, *DF*). And we may take one of the effects from each of the four groups to be a potential candidate to be estimated with all main effects. Consequently, if the sixteen points are performed, then this is a MA design with resolution IV. After its optimal semi-folding design is added, the design allows us to estimate all main effects plus eleven two-factor interactions that are clear of the other effects, as we have already mentioned in example 3.

3. Conclusion

For the regular 2^{k-p} fractional factorial designs, the MA criterion is commonly used for choosing optimal plans. However, the criterion of MA does not seem to work in all situations, especially in irregular designs. In this article, we apply the criterion of "clear" to judge the competitive plans. And we have used five examples and two remarks to illustrate how to make a plan that is robust against early termination, that is, the points in a design are run in sequence. In general, if the experimenter has some prior information about certain specific factors that may more important than the others factors, then she/he may arrange the experimental points in advance to obtain the better information in order to against the early termination. Just as we had mentioned in above, different order of the experiment may be conducted depend on different purposes. On the other hand, a random selection of order for the experiment is likely to produce a design that is far away from being an efficient sub-design.

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