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The Exponentiated Half Logistic-Generalized-G Power Series Class of Distributions: Properties and Applications

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ABSTRACT
A new generalized class of distributions called the Exponentiated Half Logistic-Generalized-G Power Series (EHL-GGPS) distribution is proposed. We present some special cases of the proposed distribution. Several mathematical properties of the EHL-GGPS distribution were also derived including order statistics, moments and maximum likelihood estimates. A simulation study for selected parameter values is presented to examine the consistency of the maximum likelihood estimates. Finally, some real data applications of the EHL-GGPS distribution are presented to illustrate the usefulness of the proposed class of distributions.

Keywords: Half Logistic, Exponentiated-G Distribution, Maximum Likelihood Estimation, Power Series.

1 Introduction

The half logistic distribution has been widely used to model real lifetime data in various areas such as physics and hydrology. The distribution was proposed by Balakrishnan [2] and has several applications in scenarios with monotonic hazard rate functions. The half logistic distribution has some disadvantages when modeling data with non-monotonic hazard rate functions. Extensions of

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the half logistic distribution has produced distributions that can handle extreme tailed data and model non-monotonic hazard function.

The generalized-G (G-G) class of distributions has cdf and pdf given by

$$F_{G-G}(x;\beta,\xi) = 1 - \bar{G}^{\beta}(x;\xi) \tag{1.1}$$

and

$$f_{G-G}(x;\beta,\xi) = \beta g(x;\xi)[\bar{G}(x;\xi)]^{\beta-1}, \tag{1.2}$$

where $G(x;\xi)$ is the baseline cdf with parameter vector ξ, $\bar{G}(x;\xi) = 1 - G(x;\xi)$ and $\beta > 0$ is shape parameter. These generalized distributions give more flexibility by adding one or more parameter(s) to the baseline distribution. There are various generators in the literature, which includes the exponentiated-G by Gupta et al. [17], half logistic-G by Cordeiro et al. [11] and Weibull-G by Bourguignon et al. [6], to mention a few.

Cordeiro et al. [12], developed the exponentiated half logistic-generalized-G (EHL-GG) class of distributions with the cumulative distribution function (cdf) and probability density function (pdf) given by

$$F(x;\beta,\delta,\xi) = \left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta} \tag{1.3}$$

and

$$f(x;\beta,\delta,\xi) = \frac{2\beta\delta g(x;\xi)\bar{G}^{\beta-1}(x;\xi)[1-\bar{G}^{\beta}(x;\xi)]^{\delta-1}}{(1+\bar{G}^{\beta}(x;\xi))^{\delta+1}}, \tag{1.4}$$

where $G(x;\xi)$ is the baseline cdf with parameter vector ξ, $\bar{G}(x;\xi) = 1 - G(x;\xi)$ and $\beta,\delta > 0$ are shape parameters.

The primary motivation for developing the Exponentiated Half Logistic-Generalized-G Power Series (EHL-GGPS) class of distributions is the versatility and flexibility derived from compounding continuous distributions. The EHL-GGPS class of distributions was proposed in order to obtain a new class of distributions with desirable properties that include the rate function that exhibits increasing, decreasing, bathtub, bathtub followed by upside down bathtub, J and reverse-J shapes. Moreover, we were motivated by several applications of power series distributions in medicine, exponential tilting in finance and actuarial sciences, as well as economics.

This paper is organized as follows: The model and structural properties of the EHL-GGPS class of distributions are presented in section 2. Section 3 contains some special cases of the EHL-GGPS class of distributions. Monte Carlo simulation study is presented in section 4. Applications of the proposed model to real data are given in section 5, followed by summary remarks in section 6.

2 The Model and Properties

Now, let N be a discrete random variable following a power series distribution assumed to be truncated at zero, whose probability mass function (pmf) is given by

$$P(N = n) = \frac{a_n\theta^n}{C(\theta)}, \quad n = 1,2,..., \tag{2.1}$$

where $C(\theta) = \sum_{n=1}^{\infty} a_n\theta^n$ is finite, $\theta > 0$, and $\{a_n\}_{n\geq 1}$ a sequence of positive real numbers. The power series family of distributions includes binomial, Poisson, geometric and logarithmic distributions. Table 1 shows some useful quantities including $a_n, C(\theta)$ and cdf for the exponentiated half logistic-generalized-G Poisson (EHL-GGP), exponentiated half logistic-generalized-G geometric (EHL-GGG), exponentiated half logistic-generalized G binomial (EHL-GGB) and exponentiated half logistic-generalized-G logarithmic (EHL-GGL) distributions.

We combine the EHL-GG distribution and power series distribution to obtain a new distribution, namely, EHL-GGPS distribution. Given N, let $Y_1, Y_2, \ldots, Y_N$ be identically and independently distributed (iid) random variable following EHL-GG distribution. Let $X = Y_{(1)} = \min(Y_1, \ldots, Y_N)$. The conditional distribution of X given $N = n$ is given by

$$G_{X|N=n}(x) = 1 - \prod_{i=1}^{n} (1 - G(x)) = 1 - S^n(x)$$
$$= 1 - \left[1 - \left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]^n.$$

Thus, the cdf of the life length of the whole system, X, say, F_θ, is given by

$$F_X(x) = 1 - \frac{C(\theta S(x))}{C(\theta)} = 1 - \frac{C\left(\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]\right)}{C(\theta)},$$

that is, the cdf of the EHL-GGPS distribution is given by the marginal distribution of X, that is,

$$F_X(x;\beta,\delta,\theta,\xi) = \sum_{n=1}^{\infty} \frac{a_n\theta^n}{C(\theta)}\left(1 - \left[1 - \left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]^n\right)$$
$$= 1 - \frac{C\left(\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]\right)}{C(\theta)}. \quad (2.2)$$

The corresponding pdf and hrf are given by

$$f_X(x;\beta,\delta,\theta,\xi) = \frac{2\delta\beta\theta g(x,\xi)\bar{G}^{\beta-1}(x;\xi)(1-\bar{G}^{\beta}(x;\xi))^{\delta-1}}{(1+\bar{G}^{\beta}(x;\xi))^{\delta+1}}$$
$$\times \frac{C'\left(\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]\right)}{C(\theta)} \quad (2.3)$$

and

$$h_X(x;\beta,\delta,\theta,\xi) = \frac{2\delta\beta\theta g(x,\xi)\bar{G}^{\beta-1}(x;\xi)(1-\bar{G}^{\beta}(x;\xi))^{\delta-1}}{(1+\bar{G}^{\beta}(x;\xi))^{\delta+1}}$$
$$\times \frac{C'\left(\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]\right)}{C\left(\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]\right)},$$

respectively, where $\delta, \beta, \theta > 0$ and ξ is the parameter vector from the baseline distribution. Table 1 below show special families of EHL-GGPS distribution when $C(\theta)$ is specified in equation (2.2).

Table 1: Special Cases of the EHL-GGPS Distribution

Distribution	a_n	$C(\theta)$	cdf
EHL-GG Poisson	$(n!)^{-1}$	$e^{\theta}-1$	$1-\frac{\exp\left(\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]\right)-1}{e^{\theta}-1}$
EHL-GG Geometric	1	$\theta(1-\theta)^{-1}$	$1-\frac{(1-\theta)\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]}{1-\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]}$
EHL-GG Logarithmic	n^{-1}	$-\log(1-\theta)$	$1-\frac{\log\left(1-\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]\right)}{\log(1-\theta)}$
EHL-GG Binomial	$\binom{m}{n}$	$(1+\theta)^{m}-1$	$1-\frac{\left(1+\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]\right)^{m}-1}{(1+\theta)^{m}-1}$

2.1 Some Sub-models of EHL-GGPS Distribution

In this sub-section, we discuss some sub-models of EHL-GGPS distribution.

- If $\delta = 1$, we obtain the half logistic-generalized-G power series (HL-GGPS) class of distributions with cdf given by

$$F_X(x;\beta,\theta,\xi)=1-\frac{C\left(\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)\right]\right)}{C(\theta)},$$

for $\beta,\theta>0$ and parameter vector ξ.

- If $\beta = 1$, we obtain the exponentiated half logistic-G power series (EHL-GPS) class of distributions with cdf given by

$$F_X(x;\delta,\theta,\xi)=1-\frac{C\left(\theta\left[1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\delta}\right]\right)}{C(\theta)},$$

for $\delta,\theta>0$ and parameter vector ξ.

- If $\beta = \delta = 1$, we obtain the half logistic-G power series (HL-GPS) class of distributions with cdf given by

$$F_X(x;\theta,\xi)=1-\frac{C\left(\theta\left[1-\left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)\right]\right)}{C(\theta)},$$

for $\theta>0$ and parameter vector ξ.

- If $\theta\to 0^{+}$, we obtain the exponentiated half logistic-generalized G (EHL-GG) class of distributions with cdf given by

$$F_X(x;\beta,\delta,\xi)=\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta},$$

for $\beta,\delta>0$ and parameter vector ξ.

• If δ = 1 and $\theta \to 0^+$, then we get the half logistic-generalized G (HL-GG) class of distributions with cdf given by

$$F_X(x;\beta,\xi) = \left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right),$$

for $\beta > 0$ and parameter vector ξ.

• If β = 1 and $\theta \to 0^+$, we obtain the exponentiated half logistic-G (EHL-G) class of distributions with cdf given by

$$F_X(x;\delta,\xi) = \left(\frac{G(x;\xi)}{1+\bar{G}(x;\xi)}\right)^{\delta},$$

for $\delta > 0$ and parameter vector ξ.

• If $\delta = \beta$ = 1 and $\theta \to 0^+$, we obtain the half logistic-G (HL-G) class of distributions with cdf given by

$$F_X(x;\xi) = \frac{G(x;\xi)}{1+\bar{G}(x;\xi)},$$

where ξ is the parameter vector.

2.2 Quantile Function

The quantile function of the EHL-GGPS class of distributions is obtained by inverting $F_\theta(x) = u$, $0 \le u \le 1$. This is equivalent to solving the equation

$$1-u = \frac{C\left(\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)\right]\right)}{C(\theta)}, \tag{2.4}$$

which can be expressed as

$$Q_{X_{(1)}}(u) = G^{-1}\left(\frac{[1-\frac{C^{-1}(C(\theta)(1-u))}{\theta}]^{\frac{1}{\delta}}}{[1-\frac{C^{-1}(C(\theta)(1-u))}{\theta}]^{\frac{1}{\delta}}}\right)^{\frac{1}{\beta}}. \tag{2.5}$$

The solution of the non-linear equation (2.4) gives the quantiles of the EHL-GGPS class of distributions.

2.3 Expansion of Density

In this sub-section, we present the series expansion of the EHL-GGPS class of distributions. The pdf in equation (2.3) can be written as

$$f_X(x;\beta,\delta,\theta,\xi) = \sum_{q,k,m,p=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{q+k+m+p}na_n\theta^n}{C(\theta)}\binom{\delta(k+1)-1}{k}\binom{\delta(k+1)-1}{m}$$
$$\times\binom{\beta(k+m+1)-1}{p}\frac{p+1}{p+1}g(x;\xi)G^p(x;\xi)$$
$$= \sum_{p=0}^{\infty} v_{p+1}g_{p+1}(x;\xi), \tag{2.6}$$

where $g_{p+1}(x;\xi) = (p+1)(G(x;\xi))^p g(x;\xi)$ is the exponentiated-G (Exp-G) distribution with power parameter $(p+1)$ and

$$v_{p+1} = \sum_{q,k,m,p=0}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{q+k+m+p}na_n\theta^n}{C(\theta)(p+1)}\binom{\delta(k+1)-1}{k}\binom{\delta(k+1)-1}{m}$$
$$\times\binom{\beta(k+m+1)-1}{p}. \tag{2.7}$$

Thus, the pdf of the EHL-GGPS class of distributions can be expressed as an infinite linear combination of Exp-G distribution. See the appendix for derivations.

2.4 Moments and Generating Function

If X follows the EHL-GGPS distribution and $Y \sim Exp - G\ (p+1)$, then the r^{th} moment, μ_r' of the EHL-GGPS class of distributions is obtained as

$$\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx = \sum_{p=0}^{\infty} v_{p+1} E(Y^r),$$

where v_{p+1} is given by equation (2.7). The moment generating function (MGF) $M(t) = E(e^{tX})$ is given by:

$$M_X(t) = \sum_{p=0}^{\infty} v_{p+1} M_Y(t),$$

where $M_Y(t)$ is the mgf of Y and v_{p+1} is given by equation (2.7).

2.5 Conditional Moments

The r^{th} conditional moment of the EHL-GGPS class of distributions is given by

$$E(X^r|X \geq w) = \frac{1}{\overline{F}(w;\beta,\delta,\theta,\xi)} \int_t^{\infty} x^r f(x;\beta,\delta,\theta,\xi)dx$$
$$= \frac{1}{\overline{F}(w;\beta,\delta,\theta,\xi)} \sum_{p=0}^{\infty} v_{p+1} E\left(Y^r I_{\{Y^r \geq t\}}\right),$$

where

$$E\left(Y^r I_{\{Y^r \geq t\}}\right) = \int_t^{\infty} y^r g_{\ p+1}(y;\xi)dy = p\int_{G(u;\xi)}^{1} [Q_G(u;\xi)]^r u^p du, \quad (2.8)$$

for $\beta, \delta, \theta > 0$, and parameter vector ξ.

2.6 Mean Deviation, Lorenz and Bonferroni Curves

The mean deviation about the mean D(μ), median $D(M)$, Lorenz $L(q)$ and Bonferroni $B(q)$ curves for the EHL-GGPS class of distributions are presented in this subsection.

2.6.1 Mean Deviations

By definition $D(\mu) = \int_0^{\infty} |x-\mu| f_X(x;\beta,\delta,\theta,\xi)dx$, $D(M) = \int_0^{\infty} |x-M| f_X(x;\beta,\delta,\theta,\xi)dx$, respectively, where $\mu = E(X)$ and $M = Median(X) = F^{-1}(\frac{1}{2})$ is the median of $F_{X_{(1)}}(x)$. Therefore,

$$D(\mu) = 2\mu F_{X_{(1)}}(x)(\mu) - 2\mu + 2\sum_{p=0}^{\infty} v_{p+1} E\left(Y I_{\{Y \geq \mu\}}\right),$$

and

$$D(M) = -\mu + 2\sum_{p=0}^{\infty} v_{p+1} E\left(Y_p I_{\{Y_p \geq M\}}\right),$$

where $E\left(Y I_{\{Y \geq M\}}\right)$ is given by (2.8) with $r = 1$ and M in place of t.

2.6.2 Lorenz and Bonferroni Curves

Lorenz and Bonferroni curves have various applications in medicine, reliability, and insurance.

$$L(p) = \frac{1}{\mu}\sum_{p=0}^{\infty} v_{p+1} \int_0^l x g_{p+1}(x;\xi)dx, \quad and \quad B(p) = \frac{1}{q\mu}\sum_{p=0}^{\infty} v_{p+1} \int_0^l x g_{p+1}(x;\xi)dx,$$

respectively, where $\int_0^l x g_{p+1}(x;\xi)dx$, is the first incomplete moment of the Exp-G distribution with $0 \leq l \leq 1$ and v_{p+1} is given in equation (2.7).

2.7 Order Statistics and Rényi Entropy

In this section, we present distribution of k^{th} order statistic and Rényi entropy for the EHL-GGPS class of distributions.

2.7.1 Order Statistics

Order statistics play an important role in probability and statistics. Let $X_1, X_2, \ldots, X_n$ be a random sample from EHL-GGPS class of distributions and suppose $X_{1:n} < X_{2:n}, \ldots < X_{n:n}$ denote the corresponding order statistics. The pdf of the k^{th} order statistic is given by

$$f_{k:n}(x) = \frac{n!(-1)^{p+s+m+j+q}}{(k-1)!(n-k)!} \sum_{p,z,s,m,j,q=0}^{\infty} \sum_{n=1}^{\infty} \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{2d_{z,p}\beta\delta\theta n a_n \theta^{n+z}}{C^{z+1}(\theta)}$$
$$\times \binom{k+i-1}{p} \binom{z+n-1}{s} \binom{\delta(s+1)-1}{m} \binom{-\delta(s+1)-1}{j} \binom{\beta(m+j+1)-1}{q} g(x;\xi) G^q(x;\xi)$$
$$= \sum_{q=0}^{\infty} v_{q+1}^* g_{q+1}(x;\xi), \tag{2.9}$$

where $g_{q+1}(x;\xi) = (q+1)g(x;\xi)G^q(x;\xi)$ is an Exp-G with power parameter $(q+1)$ and the linear component

$$v_{q+1}^* = \frac{n!(-1)^{p+s+m+j+q}}{(k-1)!(n-k)!} \sum_{p,z,s,m,j,q=0}^{\infty} \sum_{n=1}^{\infty} \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{2d_{z,p}\beta\delta\theta n a_n \theta^{n+z}}{C^{z+1}(\theta)(q+1)}$$
$$\times \binom{z+n-1}{s} \binom{\delta(s+1)-1}{m} \binom{-\delta(s+1)-1}{j} \binom{\beta(m+j+1)-1}{q} \left(\frac{1}{q+1}\right).$$

See the appendix for derivations. The t^{th} moment of the distribution of the k^{th} order statistic from EHL-GGPS class of distributions can be readily obtained from equation (2.9).

2.7.2 Rényi Entropy

In this subsection, Rényi entropy for EHL-GGPS class of distributions is derived. An entropy is a measure of uncertainty or variation of a random variable. Rényi entropy [24] is a generalization of Shannon entropy [26]. Rényi entropy for the EHL-GGPS distribution is given by

$$I_R(v) = \frac{1}{1-v} \log\left(\sum_{p=0}^{\infty} w_{p+1}^* e^{(1-v)I_{REG}}\right), \tag{2.10}$$

where $I_{REG} = \int_0^{\infty} [(1+p/v)g(x;\xi)G^{p/v}]^v dx$ is Rényi entropy for an Exp-G distribution with power parameter $(p/v+1)$ and

$$w_{p+1}^* = \sum_{z,m,q,s,p=0}^{\infty} d_{z,v} \theta^{z+v} (2\delta\beta)^v (-1)^{m+q+s+p} \binom{z}{m} \binom{\delta(m+v)-v}{q}$$
$$\times \binom{-\delta(m+v)-v}{s} \binom{-\beta(q+s+1)-1}{p} \frac{1}{(1+m/v)^v}. \tag{2.11}$$

Consequently, Rényi entropy of the EHL-GGPS class of distributions can be readily derived from Rényi entropy of the Exp-G. See the appendix for derivations.

2.8 Parameter Estimation

Let $X_i \sim EHL-GGPS(\beta,\delta,\theta;\xi)$ and $\Delta = (\beta,\delta,\theta;\xi)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ based on a random sample of size n is given by

$$\ell = \ell(\Delta) = n\ln[2\beta\delta\theta] + \sum_{i=1}^{n} \ln[g(x_i;\xi)] + (\beta-1)\sum_{i=1}^{n} \ln[\bar{G}(x_i;\xi)] - n\ln C(\theta)$$
$$+(\delta-1)\sum_{i=1}^{n} \ln[1-\bar{G}^{\beta}(x_i;\xi)] - (\delta+1)\sum_{i=1}^{n} \ln[1+\bar{G}^{\beta}(x_i;\xi)]$$
$$+\sum_{i=1}^{n} \ln\left(C'\left(\theta\left[1-\left(\frac{1-\bar{G}^{\beta}(x;\xi)}{1+\bar{G}^{\beta}(x;\xi)}\right)^{\delta}\right]\right)\right).$$

Elements of the score vector $U = \left(\frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \xi_k}\right)$ are given in the appendix. The maximum likelihood estimates of the parameters, denoted by $\widehat{\boldsymbol{\Delta}}$ is obtained by solving the nonlinear equation $\left(\frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \delta}, \frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \xi_k}\right)^T = \mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution $N_{q+3}(\underline{0}, J(\Delta)^{-1})$, where the mean vector $\underline{0} = (0,0,0,\underline{0})^T$ and $J(\Delta)^{-1}$ is the observed Fisher information matrix evaluated at Δ, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

3 Some Special Cases

In this section, we look at some special cases of the EHL-GGPS distribution. These special cases are the exponentiated half logistic-generalized Weibull Poisson (EHL-GWP), exponentiated half logistic-generalized Weibull Geometric (EHL-GWG), exponentiated half logistic-generalized log-logistic Poisson (EHL-GLLoGP), exponentiated half logistic-generalized log-logistic Geometric (EHL-GLLoGG), exponentiated half-generalized Burr III Poisson (EHL-GBP) and exponentiated half-generalized Burr III Geometric (EHL-GBG) distributions. The cdf and pdf of the Weibull distribution are given by $G(x;\alpha) = 1 - \exp(-x^\alpha)$ and $g(x;\alpha) = \alpha x^{\alpha-1}\exp(-x^\alpha)$, for $\alpha > 0$, for the log-logistic distribution are given by $G(x;\lambda) = 1 - (1 + x^\lambda)^{-1}$ and $g(x;\lambda) = \lambda x^{\lambda-1}(1 + x^\lambda)^{-2}$, for $\lambda > 0$ and for the Burr III distribution are given by $G(x;a,\gamma) = (1 + x^{-a})^{-\gamma}$ and $g(x;a,\gamma) = a\gamma x^{-a-1}(1 + x^{-a})^{-\gamma-1}$ for $a, \gamma > 0$.

3.1 Exponentiated Half Logistic-Generalized Weibull Poisson (EHL-GWP) Distribution

The cdf and pdf of the EHL-GWP distribution are given by

$$F_{EHL-G}\quad(x) = 1 - \frac{\exp\left(\theta\left[1-\left(\frac{1-e^{-\beta x^\alpha}}{1+e^{-\beta x^\alpha}}\right)^\delta\right]\right)-1}{exp(\theta)-1}$$

and

$$f_{EHL-GWP}(x) = \frac{2\alpha\delta\beta\theta\quad {}^{\alpha-1}e^{-x^\alpha}e^{(-\beta+1)x^\alpha}(1-e^{-\beta x^\alpha})^{\delta-1}}{(1+e^{-\beta x^\alpha})^{\delta+1}}\frac{\exp\left(\theta\left[1-\left(\frac{1-e^{-\beta x^\alpha}}{1+e^{-\beta x^\alpha}}\right)^\delta\right]\right)}{\exp(\theta)-1}$$

for θ, δ, β and $\alpha > 0$.

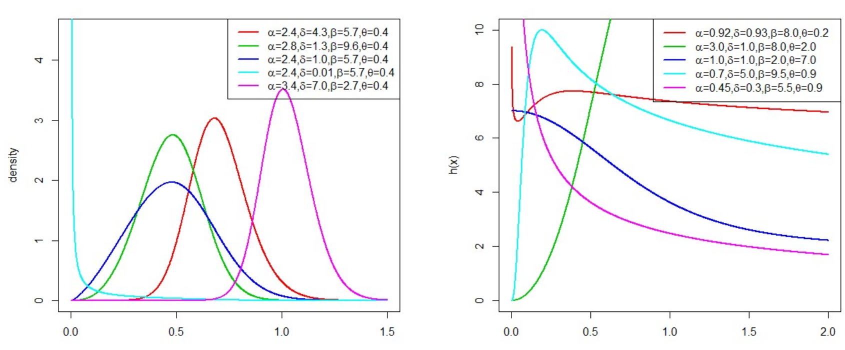

Figure 3: Plots of the pdf and hrf for the EHL-GWP distribution

The plots of pdf and hrf for the EHL-GWP distribution are shown in Figure 3. The pdf exhibit almost symmetric, right or left-skewed shapes. The hazard rate function exhibits both increasing, decreasing, upside bathtub, and bathtub followed by upside bathtub shapes.

3D plots of skewness and kurtosis for EHL-GGPS class of distributions are given in Figures 1 and 2. We consider two special cases from EHL-GGPS class of distributions. We observe that EHL-GLLoGG and EHL-GWP distributions can handle various levels of skewness and kurtosis, when we fix some parameters.

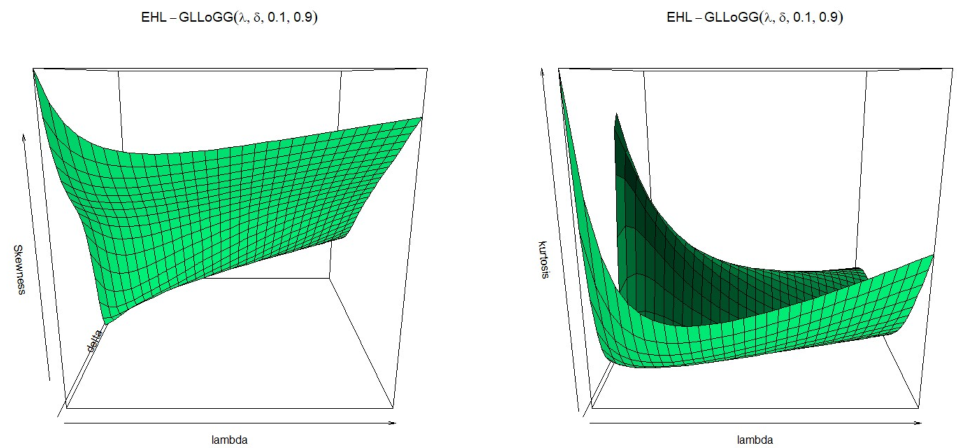

Figure 1: Plots of skewness and kurtosis for the EHL-GLLoGG distribution

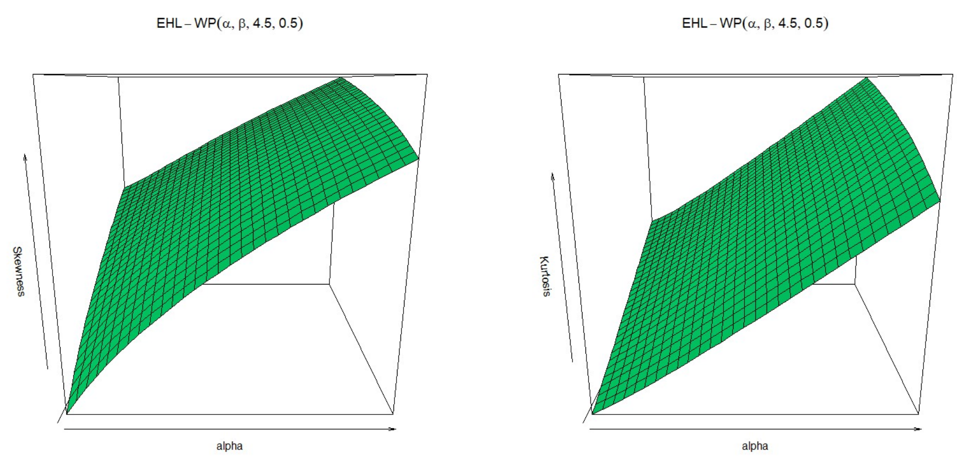

Figure 2: Plots of skewness and kurtosis for the EHL-GWP distribution

3.2 Exponentiated Half Logistic-Generalized Weibull Geometric (EHL-GWG) Distribution

The cdf and pdf of the EHL-GWG distribution are given by

$$F_{EHL-G} \quad (x) = 1 - \frac{(1-\theta)\left[1-\left(\frac{1-e^{-\beta x^{\alpha}}}{1+e^{-\beta x^{\alpha}}}\right)^{\delta}\right]}{1-\theta\left[1-\left(\frac{1-e^{-\beta x^{\alpha}}}{1+e^{-\beta x^{\alpha}}}\right)^{\delta}\right]}$$

and

$$f_{EHL-GWG}(x) = \frac{2\alpha\delta\beta\theta x^{\alpha-1}e^{-x^{\alpha}}e^{(-\beta+1)x^{\alpha}}(1-e^{-\beta x^{\alpha}})^{\delta-1}}{(1+e^{-\beta x^{\alpha}})^{\delta+1}} \times \frac{\left(1-\left(\theta\left[1-\left(\frac{1-e^{-\beta x^{\alpha}}}{1+e^{-\beta x^{\alpha}}}\right)^{\delta}\right]\right)\right)^{-2}}{(1-\theta)^{-1}}$$

for $\lambda, \gamma, \ a > 0$ and $0 < \theta < 1$.

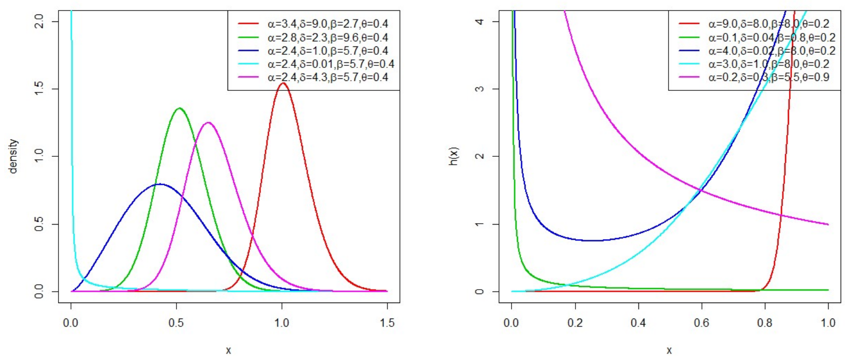

Figure 4: Plots of the pdf and hrf for the EHL-GWG distribution

Figure 4 shows the pdf and hrf plots for selected parameter values for the EHL-GWG distribution. The distribution is almost symmetric, right or left skewed. The hazard rate function exhibits both non-monotonic and monotonic shapes.

3.3 Exponentiated Half Logistic-Generalized Log-Logistic Poisson (EHL-GLLoGP) Distribution

The cdf and pdf of the EHL-GLLoGP distribution are given by

$$F_{EHL-GLLoG} \ (x) = 1 - \frac{\exp\left(\theta\left[1-\left(\frac{1-(1+x^{\lambda})^{-\beta}}{1+(1+x^{\lambda})^{-\beta}}\right)^{\delta}\right]\right)-1}{\exp(\theta)-1}$$

and

$$f_{EHL-GLLoG} \ (x) = \frac{2\lambda\delta\beta\theta \quad {}^{\lambda-1}(1+x^{\lambda})^{-2}(1+x^{\lambda})^{-\beta+1}(1-(1+x^{\lambda})^{-\beta})^{\delta-1}}{(1+(1+x^{\lambda})^{-\beta})^{\delta+1}}$$

$$\times \frac{\exp\left(\theta\left[1-\left(\frac{1-(1+x^{\lambda})^{-\beta}}{1+(1+x^{\lambda})^{-\beta}}\right)^{\delta}\right]\right)}{\exp(\theta)-1}$$

for θ, δ, β and $\lambda > 0$.

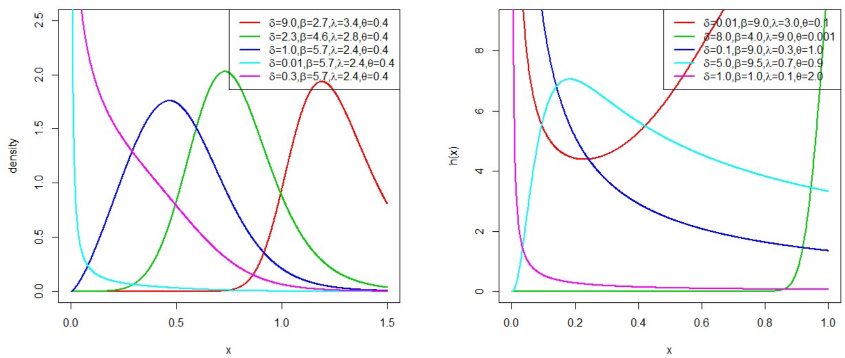

Figure 5: Plots of the pdf and hrf for the EHL-GLLoGP distribution

The EHL-GLLoGP distribution applies to data sets that are heavy tailed, left or right-skewed as shown in Figure 5. The hrf function exhibits both monotonic and non-monotonic shapes.

3.4 Exponentiated Half Logistic-Generalized Log-Logistic Geometric (EHL-GLLoGG) Distribution

The cdf and pdf of the EHL-GLLoGG distribution are given by

$$F_{EHL-GLLoGG}(x) = 1 - \frac{(1-\theta)\left(1-\left(\frac{1-(1+x^{\lambda})^{-\beta}}{1+(1+x^{\lambda})^{-\beta}}\right)^{\delta}\right)}{1-\theta\left(1-\left(\frac{1-(1+x^{\lambda})^{-\beta}}{1+(1+x^{\lambda})^{-\beta}}\right)^{\delta}\right)}$$

and

$$f_{EHL-GLLoGG}(x) = \frac{2\lambda\delta\beta\theta x^{\lambda-1}(1+x^{\lambda})^{-2}(1+x^{\lambda})^{-\beta+1}(1-(1+x^{\lambda})^{-\beta})^{\delta-1}}{(1+(1+x^{\lambda})^{-\beta})^{\delta+1}}$$

$$\times \frac{\left(1-\left(\theta\left[1-\left(\frac{1-(1+x^{\lambda})^{-\beta}}{1+(1+x^{\lambda})^{-\beta}}\right)^{\delta}\right]\right)\right)^{-2}}{(1-\theta)^{-1}},$$

for $\delta, \beta, \lambda > 0$ and $0 < \theta < 1$.

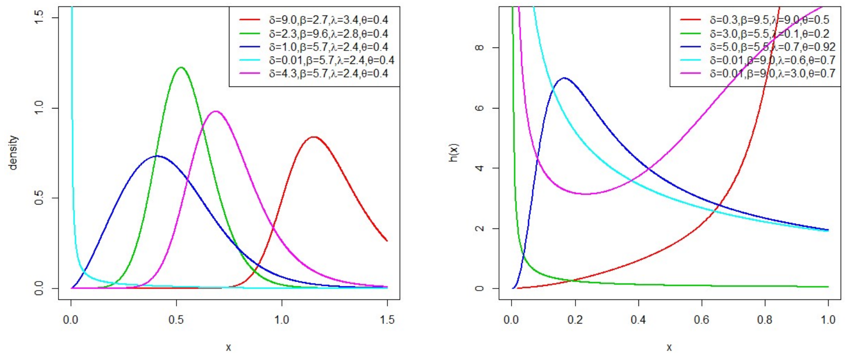

Figure 6: Plots of the pdf and hrf for the EHL-GLLoGG distribution

The EHL-GLLoGG distribution pdf exhibits extreme tails, almost symmetric and reverse-J shapes. The monotonic and non-monotonic hrf shapes are exhibited as shown in Figure 6.

3.5 Exponentiated Half Logistic-Generalized Burr III Poisson (EHL-GBP) Distribution

The cdf and pdf of the EHL-GBP distribution are given by

$$F_{EHL-G}(x) = 1 - \frac{\exp\left(\theta\left[1-\left(\frac{1-[1-(1+x^{-a})^{-\gamma}]^{\beta}}{1+[1-(1+x^{-a})^{-\gamma}]^{\beta}}\right)^{\delta}\right]\right)-1}{\exp(\theta)-1}$$

and

$$f_{EHL-GBP}(x) = 2a\gamma\delta\beta\theta x^{-a-1}(1+x^{-a})^{-\gamma-1}[1-(1+x^{-a})^{-\gamma}]^{\beta+1}$$
$$\times \frac{(1-[1-(1+x^{-a})^{-\gamma}]^{\beta})^{\delta-1}}{(1+[1-(1+x^{-a})^{-\gamma}]^{\beta})^{\delta+1}} \frac{\exp\left(\theta\left[1-\left(\frac{1-[1-(1+x^{-a})^{-\gamma}]^{\beta}}{1+[1-(1+x^{-a})^{-\gamma}]^{\beta}}\right)^{\delta}\right]\right)}{\exp(\theta)-1}$$

for θ, δ, β and $\lambda > 0$.

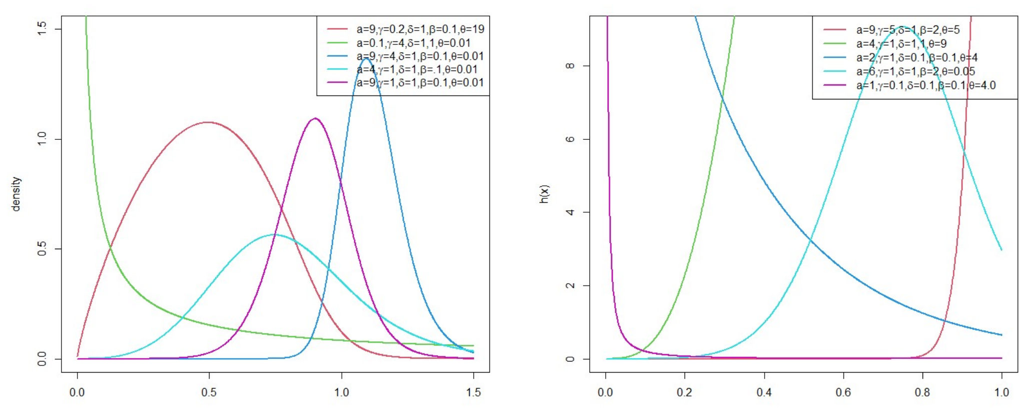

Figure 7: Plots of the pdf and hrf for the EHL-GBP distribution

The EHL-GBP distribution applies to data sets that are heavy tailed, left or right skewed as shown in Figure 7. The hrf function exhibits increasing, decreasing, unimodal, J and reverse-J shapes.

3.6 Exponentiated Half Logistic-Generalized Burr III Geometric (EHL-GBG) Distribution

The cdf and pdf of the EHL-GBG distribution are given by

$$F_{EHL-GBG}(x) = 1 - \frac{(1-\theta)\left[1-\left(\frac{1-[1-(1+x^{-a})^{-\gamma}]^{\beta}}{1+[1-(1+x^{-a})^{-\gamma}]^{\beta}}\right)^{\delta}\right]}{1-\theta\left[1-\left(\frac{1-[1-(1+x^{-a})^{-\gamma}]^{\beta}}{1+[1-(1+x^{-a})^{-\gamma}]^{\beta}}\right)^{\delta}\right]}$$

and

$$f_{EHL-GBG}(x) = 2a\gamma\delta\beta\theta x^{-a-1}(1+x^{-a})^{-\gamma-1}[1-(1+x^{-a})^{-\gamma}]^{\beta+1}$$
$$\times \frac{(1-[1-(1+x^{-a})^{-\gamma}]^{\beta})^{\delta-1}}{(1+[1-(1+x^{-a})^{-\gamma}]^{\beta})^{\delta+1}} \frac{\left(1-\left(\theta\left[1-\left(\frac{1-[1-(1+x^{-a})^{-\gamma}]^{\beta}}{1+[1-(1+x^{-a})^{-\gamma}]^{\beta}}\right)^{\delta}\right]\right)\right)^{-2}}{(1-\theta)^{-1}},$$

for $\delta, \beta, \lambda > 0$ and $0 < \theta < 1$.

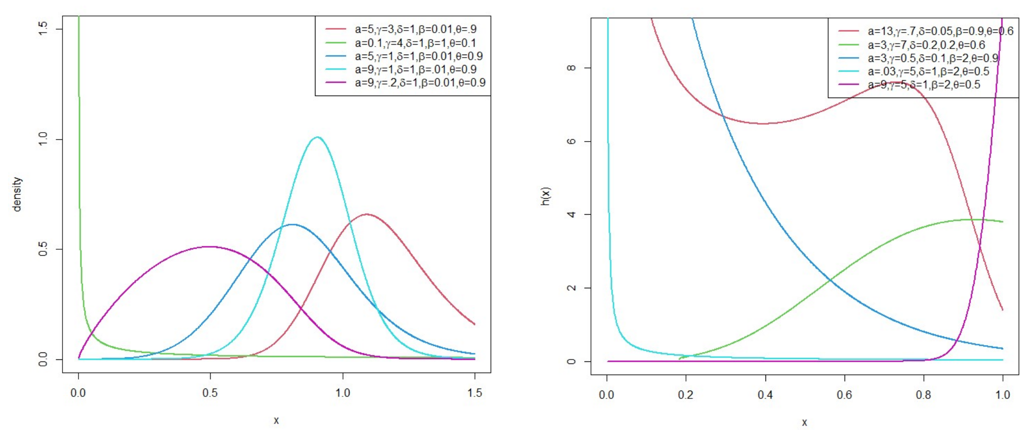

Figure 8: Plots of the pdf and hrf for the EHL-GBG distribution

The EHL-GBG distribution pdf exhibits extreme tails, almost symmetric and reverse-J shapes. The hrf function exhibits decreasing, increasing, bathtub followed by upside bathtub, J and reverse-J shapes as shown in Figure 8.

4 Simulation Study

In this section, we examine the performance of the EHL-GWP distribution by conducting various simulations for different sizes (n=50, 100, 200, 400 and 800) via the R package. We simulate 1000 samples for the true parameter values given in the Tables 1.1 and 1.2 in the appendix. The bias and RMSE are given by: $Bias(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta$, and $RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i-\theta)^2}{N}}$, respectively. From the Monte Carlo simulations results, we notice that the mean estimates of the parameters approach to the true parameter values as the sample size n increases. As well as the RMSE and bias values decay towards zero, when sample size n is increasing.

5 Applications

In this section, we demonstrate the applicability of the EHL-GWP distribution to three lifetime data sets. We compared the EHL-GWP distribution to several non-nested models. We use R software to estimate model parameters and standard errors. We assessed model performance using -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Cramer von Mises (W^*), and Andersen-Darling (A^*) (see Chen and Balakrishnan [8] for details), Kolmogorov-Smirnov (K-S) statistic (and its p-value). Tables 2, and 3 shows model parameters estimates (standard errors in parentheses) and several goodness-of-fit statistics. We also provide fitted densities and probability plots (as described by Chambers et al. [7]) to demonstrate how well our model fits the selected data sets.

The non-nested models considered in this paper are the exponentiated power Lindley Poisson (EPLP) by Pararai et al. [23], exponentiated Burr XII Poisson by da Silva et al.[14], beta odd Lindley-Uniform (BOL-U) distribution by Chipepa et al. [10] and exponential Lindley odd log-logistic Weibull (ELOLLW) by Korkmaz[19]. The pdfs of the non-nested models are as follows:

$$f_{EPLP}(x;\alpha,\beta,\theta,\omega)=\frac{\alpha\beta^2\omega\theta(1+y^\alpha)y^{\alpha-1}e^{-\beta y^\alpha}}{(1+\beta)(e^\theta-1)}[1-\left(1+\frac{\beta y^\alpha}{\beta+1}\right)e^{-\beta y^\alpha}]^{\omega-1}$$
$$\times\exp\left(\theta[1-\left(1+\frac{\beta y^\alpha}{\beta+1}\right)e^{-\beta y^\alpha}]^{\omega}\right),$$

for $\alpha,\beta,\theta,\omega>0$,

$$g_{EBXIIP}(x;c,s,k,\alpha,\lambda)=\frac{\alpha\lambda ck\ {}^{-c}x^{c-1}}{1-e^{-\lambda}}\left(1+(\frac{x}{s})^c\right)^{-k-1}\left(1-\left(1+(\frac{x}{s})^c\right)^{-k}\right)^{\alpha-1}$$
$$\times\exp[-\lambda\left(1-\left(1+(\frac{x}{s})^c\right)^{-k}\right)^{\alpha}]$$

for $c,s,k,\alpha,\lambda>0$,

$$f_{ELOLLW}(x;\alpha,\beta,\gamma,\theta,\lambda)=\frac{\alpha\theta^2\gamma\lambda^\gamma x^{\gamma-1}e^{-(\lambda x)^\gamma}(e^{-(\lambda x)})^{\alpha\theta-1}(1-e^{-(\lambda x)^\gamma})^{\alpha-1}}{(\theta+\beta)((1-e^{-(\lambda x)^\gamma})^\alpha+e^{-\alpha(\lambda x)^\gamma})^{\theta-1}}$$
$$\times(1-\beta\log[\frac{e^{-(\lambda x)^\gamma}}{(1-e^{-(\lambda x)^\gamma})^\alpha+e^{-\alpha(\lambda x)^\gamma}}]),$$

for $\alpha,\beta,\gamma,\theta,\lambda>0$,

$$f_{BOL-U}(x;a,b,\lambda,\theta)=\frac{1}{B(a,b)}\left[1-\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{a-1}$$
$$\times\left[\frac{\lambda+(1-x/\theta)}{(1+\lambda)(1-x/\theta)}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\}\right]^{b-1}\frac{\lambda^2}{(1+\lambda)}\frac{\theta^2}{(\theta-x)^3}\exp\left\{-\lambda\frac{x}{(\theta-x)}\right\},$$

for $a,b,\lambda>0$, $0<x<\theta$. For the EBXIIP and ELOLLW distributions we consider $k=1$ and $\alpha=1$, respectively.

5.1 Active Repair Times Data

This data set represents active repair times (hours) for an airborne communication transceiver. The data was also analyzed by Jorgensen [18]. The data are 0.50, 0.60, 0.60, 0.70, 0.70, 0.70, 0.80, 0.80, 1.00, 1.00, 1.00, 1.00, 1.10, 1.30, 1.50, 1.50, 1.50, 1.50, 2.00, 2.00, 2.20, 2.50, 2.70, 3.00, 3.00, 3.30, 4.00, 4.00, 4.50, 4.70, 5.00, 5.40, 5.40, 7.00, 7.50, 8.80, 9.00, 10.20, 22.00, 24.50.

Table 2: MLEs and and Standard Errors in parentheses for active repair times data set

Model	α	β	δ	θ
EHL-GWP	0.0575 (0.0143)	7.9793 (0.171)	5.3082×10^{3} (5.2144×10^{-5})	8.4103 (4.7364)
	δ	β	λ	θ
EHL-GLLoGP	0.4215 (0.1825)	0.09361 (0.088)	9.4098 (6.7795)	2.3325×10^{-8} (0.0176)
	a	b	α	β
EPLP	7.3644×10^{-8} (0.0322)	0.5608 (0.114)	1.2931 (0.4485)	2.7570 (1.4633)
	c	s	α	λ
EBXIIP	1.9730 (0.2068)	0.4673 (1.003)	7.2001 (1.4061)	2.1832×10^{-9} (0.4914)
	a	b	λ	θ
BOL-U	1.1045 (0.2302)	1.0599 (0.219)	6.7740×10^{5} (3.2121×10^{-7})	2.7006×10^{6} (8.0572×10^{-8})
	β	λ	θ	γ
ELOLLW	4.4238×10^{-5} (1.5701)	0.0483 (0.025)	4.9289 (4.2745×10^{-4})	0.9604 (0.0958)

Table 3: Goodness-of-fit Statistics for active repair times data set

Model	$-2\log L$	AIC	$AICC$	BIC	W^{*}	A^{*}	K-S	p-value
EHL-GWP	181.5	189.5	190.6	196.2	0.0706	0.4729	0.1113	0.7044
EHL-GLLoGP	185.0	193.0	194.1	199.7	0.1499	0.9056	0.1488	0.3387
EPLP	185.8	193.8	194.9	200.5	0.1002	0.7247	0.1278	0.5306
EBXIIP	194.5	202.5	203.6	209.2	0.0804	0.4965	0.2591	0.0093
BOL-U	190.9	198.9	200.1	205.7	0.1490	1.0715	0.1577	0.2728
ELOLLW	191.0	199.0	200.2	205.8	0.1455	1.0483	0.1290	0.5181

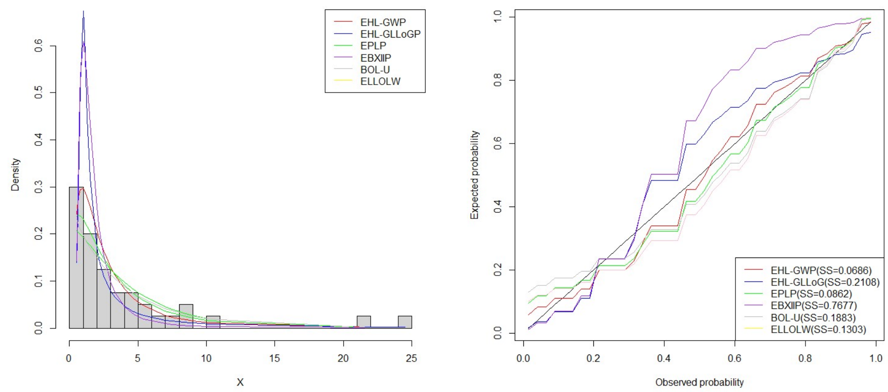

Figure 11: Fitted densities and probability plots for active repair times data

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 0.000205 & 0.001652 & 0.000001 & -0.059321 \\ 0.001652 & 0.029419 & 0.000006 & -0.717552 \\ 0.000001 & 0.000006 & 0.000000 & -0.000236 \\ -0.059321 & -0.717552 & -0.000236 & 22.433446 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [0.0575 \pm 0.0281]$, $\beta \in [7.9793 \pm 0.3362]$, $\delta \in [5.3082 \times 10^3 \pm 0.0001]$ and $\theta \in [8.4103 \pm 9.2833]$.

From the results of the goodness-of-fit statistics as shown in Table 3, we conclude that the EHL-GWP model performs better than the selected non-nested models. Figures 11 shows the fitted densities and probability plots for the EHL-GWP model.

5.2 Cancer Patients Data

This data set represents remission times of a random sample of 128 bladder cancer patients. The data set was analysed by Lee and Wang [21]. The observations are; 0.08, 4.98, 25.74, 3.7, 10.06, 2.69, 7.62, 1.26, 7.87, 4.4, 2.02, 21.73, 2.09, 6.97, 0.5, 5.17, 14.77, 4.18, 10.75, 2.83, 11.64, 5.85, 3.31, 2.07, 3.48, 9.02, 2.46, 7.28, 32.15, 5.34, 16.62, 4.33, 17.36, 8.26, 4.51, 3.36, 4.87, 13.29, 3.64, 9.74, 2.64, 7.59, 43.01, 5.49, 1.4, 11.98, 6.54, 6.93, 6.94, 0.4, 5.09, 14.76, 3.88, 10.66, 1.19, 7.66, 3.02, 19.13, 8.53, 8.65, 8.66, 2.26, 7.26, 26.31, 5.32, 15.96, 2.75, 11.25, 4.34, 1.76, 12.03, 12.63, 13.11, 3.57, 9.47, 0.81, 7.39, 36.66, 4.26, 17.14, 5.71, 3.25, 20.28, 22.69, 23.63, 5.06, 14.24, 2.62, 10.34, 1.05, 5.41, 79.05, 7.93, 4.5, 2.02, 0.2, 7.09, 25.82, 3.82, 14.83, 2.69, 7.63, 1.35, 11.79, 6.25, 3.36, 2.23, 9.22, 0.51, 5.32, 34.26, 4.23, 17.12, 2.87, 18.1, 8.37, 6.76, 3.52, 13.8, 2.54, 7.32, 0.9, 5.41, 46.12, 5.62, 1.46, 12.02, 12.07.

Table 4: MLEs and and Standard Errors in parentheses fitted for cancer patients data set

Model	α	β	δ	θ
EHL-GWP	0.3110 (0.1763)	0.7523 (0.621)	5.0079 (4.078)	10.2685 (5.4982)
	δ	β	λ	θ
EHL-GLLoGP	2.2581 (0.4291)	0.8484 (0.196)	1.2822 (0.135)	3.4061×10^{-8} (0.0122)
	a	b	α	β
EPLP	2.2382×10^{-7} (0.0305)	0.5667 (0.1017)	0.8183 (0.3112)	2.7647 (1.2868)
	c	s	α	λ
EBXIIP	0.9877 (0.0858)	1.5057 (0.249)	1.9748 (0.21)	7.8518×10^{-9} (0.0083)
	a	b	λ	θ
BOL-U	1.1988 (0.1199)	1.3360 (0.1576	6.9105×10^{5} (3.0371×10^{-7})	7.6215×10^{6} (2.7537×10^{-8})
	β	λ	θ	γ
ELOLLW	3.9683×10^{-5} (1.1720)	0.0164 (0.003)	6.9697 (6.2309×10^{-6})	1.0478 (0.0676)

Table 5: Goodness-of-fit Statistics fitted for cancer patients data set

Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	K-S	p-value
EHL-GWP	181.5	189.5	190.6	196.2	0.0706	0.4729	0.1113	0.7044
EHL-GLLoGP	185.0	193.0	194.1	199.7	0.1499	0.9056	0.1488	0.3387
EPLP	185.8	193.8	194.9	200.5	0.1002	0.7247	0.1278	0.5306
EBXIIP	194.5	202.5	203.6	209.2	0.0804	0.4965	0.2591	0.0093
BOL-U	190.9	198.9	200.1	205.7	0.1490	1.0715	0.1577	0.2728
ELOLLW	191.0	199.0	200.2	205.8	0.1455	1.0483	0.1290	0.5181

Table 3: Parameter estimates and goodness-of-fit statistics for various models fitted for cancer patients data set

The estimated variance-covariance matrix is given by

$$\begin{bmatrix} 0.031078 & -0.106448 & -0.706539 & -0.685157 \\ -0.106448 & 0.386030 & 2.519119 & 1.826943 \\ -0.706539 & 2.519119 & 16.634906 & 13.611971 \\ -0.685159 & 1.826943 & 13.611971 & 30.229626 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [0.3110 \pm 0.3455]$, $\beta \in [0.7523 \pm 1.2178]$, $\delta \in [5.0078 \pm 7.9940]$ and $\theta \in [10.2685 \pm 10.7764]$.

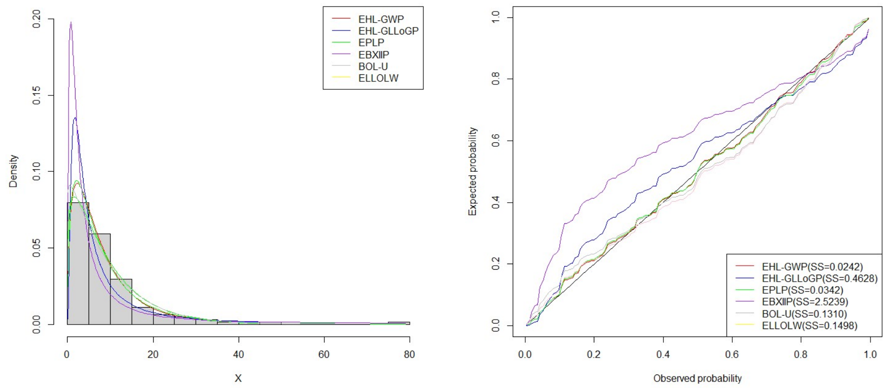

Figure 14: Fitted densities and probability plots for cancer patients data

From the values of the goodness-of-fit statistics A^{*}, W^{*}, K-S and the p-value of the K-S statistic as shown in Table 5, we conclude that the EHL-GWP model performs better than the non-nested models considered in this paper. Figures 14 shows the fitted densities and probability plots for the EHL-GWP model. We observe that the EHL-GWP model can fit extreme tailed data compared to the non-nested models.

6 Conclusions

A new class of distributions called the EHL-GGPS distribution is developed. The special case of EHL-GWP distribution can be applied in reliability and medicine. We derived mathematical properties of the new distribution. EHL-GGPS class of distribution can be expressed as a linear combination of the Exp-G distribution. Finally, we applied the EHL-GWP distribution to real data sets in order to illustrate the flexibility of the proposed class of distributions. We can conclude that the EHL-GWP model performs better than selected non-nested models on considered data sets.

Appendix

The following URL contains derivations of statistical properties and elements of the score vector. https://drive.google.com/file/d/15TNGou2SmQFoTBLUvD9dRETCjZDnjBKG/view?usp=sharing

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