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Statistical Inference in the Cumulative Exposure Lognormal model with Hybrid Censoring

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ABSTRACT This research aims to analyze data coming from step stress life testing experiments where the stress level is incremented at a preset time to obtain failure data faster. To analyze step stress data, a model that extrapolates the information attained from the accelerated tests to normal conditions needs to be fit to the life test data. We used the Cumulative Exposure Model (CEM) to model simple step stress lognormal life test data where hybrid censoring is present and applied the maximum likelihood estimation method to find the point and interval estimates of the parameters. Bootstrap intervals (bootstrap-t intervals and percentile intervals) were also constructed. We then performed a simulation study to assess the proposed methods of estimation under different hybrid censoring schemes. The Bias and MSE of the maximum likelihood estimators (MLEs) along with the coverage probability and average lengths of the corresponding confidence intervals were investigated. Finally, an illustrative example has been used to demonstrate the application of the methods discussed in this paper.

Keywords: Maximum Likelihood Estimation, Bootstrap intervals, Cumulative Exposure Model (CEM), Simple step stress, Hybrid censoring.

Mathematics Subject Classification: 62N01; 62N05; 62F40

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1. Introduction

Owing to the competing nature of the market today, the manufacturers are pressured to produce and develop higher technology products that are both productive and reliable (Meeker W Q, 1998). It is difficult to test the reliability and performance of products that have a long mean lifetime to failure under normal circumstances. Instead, accelerated tests which run at stress higher than in normal conditions are used to obtain information on reliability in a limited time. One type of accelerated testing is step stress accelerated life testing (SSALT) known as overstressing testing where the product is exposed to higher successive stress levels at specified times (Nelson, 2004). In simple step stress accelerated life testing (Simple SSALT), the product is subjected to two stress levels. Balakrishnan and Kundu (2007) considered a simple step stress model with exponentially distributed lifetimes in the presence of type I censoring. Balakrishnan et al. (2007) considered a simple step stress model for exponentially distributed lifetimes with type II censoring. Balakrishnan and Xie (2007) considered a cumulative exposure model for simple step stress exponentially distributed life test data with hybrid censoring. Balakrishnan et al.(2009) considered a cumulative exposure model where the life test data are step stress lognormally distributed with the presence of type-I censoring. Lin and Chou (2012) considered a cumulative exposure model for k step stress lognormally distributed lifetimes when the data is type-I censored. Samanta et al. (2019) considered a step stress exponentially distributed life test model with two stress levels. In this paper we consider simple step stress life testing experiment where the lifetimes are lognormally distributed with the presence of hybrid censoring.

2. The Cumulative Exposure Lognormal Model

According to Nelson (1980), the CEM mainly assumes that the residual lifetime of the testing units is dependent on the accumulative exposure despite how it came to be. The CEM is the most widely used model for analyzing SSALT data.

The CDF of CEM for simple step stress data is given by:

$$\mathrm{F}_{\mathrm{CEM}}(\mathrm{t}) = \begin{cases} \mathrm{F}_1(t),\ t < \tau \\ \mathrm{F}_2\left(t - \tau + \tau^*\right), t > \tau \end{cases}$$

Where τ is the stress changing time, $\tau^* = F_2^{-1}(F_1(t))$ is the equivalent testing time of τ under a higher stress level, $F_1(t)$ is the distribution of the units' lifetime before changing the stress and $F_2(t)$ is the distribution of the lifetime after changing the stress.

For a cumulative exposure lognormal model with two stress levels, the lifetimes of the units follow a lognormal distribution with parameters (μ_1, σ) for the initial stress level and (μ_2, σ) after the stress has been increased. The pdf of the lognormal distribution is given by:

$$f(t;\mu,\sigma) = \frac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log t-\mu}{\sigma}\right)^2}, t > 0, -\infty < \mu \langle \infty, \sigma \rangle 0$$

The pdf of the lognormal simple step stress model is given by:

$$g(t) = \begin{cases} g_1(t) = \dfrac{1}{\sigma t\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log t-\mu_1}{\sigma}\right)^2}, 0 \le t < \tau \\ g_2(t) = \dfrac{1}{\sigma\left[t-\alpha(\mu)\right]\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log\left[t-\alpha(\mu)\right]-\mu_2}{\sigma}\right)^2}, \tau \le t < \infty \end{cases} \tag{2.1}$$

$$\alpha(\mu) = \tau\left(1 - e^{\mu_2-\mu_1}\right) \tag{2.2}$$

The CDF is given by:

$$G(t) = \begin{cases} G_1(t) = \Phi\left(\dfrac{\log t-\mu_1}{\sigma}\right), 0 \le t < \tau \\ G_2(t) = \Phi\left(\dfrac{\log\left[t-\alpha(\mu)\right]-\mu_2}{\sigma}\right), \tau \le t < \infty \end{cases} \tag{2.3}$$

The CDF of the lognormal simple step stress model with stress changing time of 30 has been shown in Figure 1.

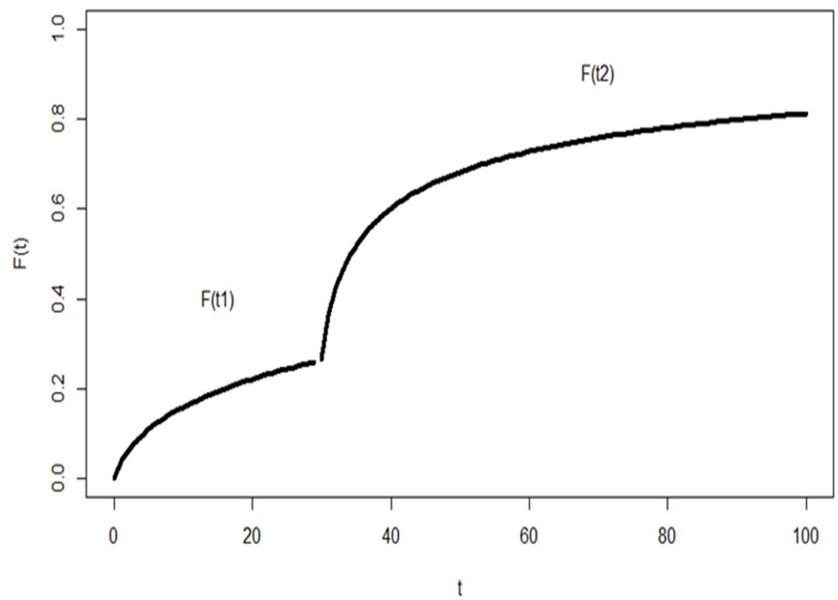

Figure 1. Cumulative Exposure lognormal model for simple step stress data

In this paper, we consider the simple step stress lognormal model where hybrid censoring is present in the data. In hybrid censoring, the experiment terminates if the predetermined experiment time has been reached before r failures has occurred or if r failures has occurred before reaching the predetermined experiment time (Balakrishnan, N. & Kundu, D., 2013). We present the maximum likelihood method for estimating the model parameters in section 3.

3. Likelihood Estimation of Model Parameters

Let N_1 be the number of the units that failed in the life testing experiment at the initial stress level, N_2 be the number of failures at the higher stress level and m be the total number of failures in the experiment. The likelihood function for the simple step stress lognormal data with hybrid censoring can be constructed by considering two cases:

Case 1: $T^* = min\left(t_1, t_{(r)}\right) = t_1$, where $N_1 + N_2 < r$

The observed time to failure of the n units in this case will be in the form:

$$t_{1:n} < t_{2:n} < \ldots < t_{N_1:n} < \tau < t_{N_1+1:n} < \ldots < t_{m:n} < t_1 \tag{3.1}$$

The likelihood function is constructed as:

$$L(\theta|t) = \left\{\prod_{k=1}^{N_1} g_1(t_{k:n})\right\}\left\{\prod_{k=N_1+1}^{m} g_2(t_{k:n})\right\}\left(1 - G_2(t_1)\right)^{n-m} \tag{3.2}$$

Substituting the pdf and CDF of the step stress model:

$$L(\theta|t)=\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{m}\frac{1}{\prod_{k=1}^{N_1}t_k}\frac{1}{\prod_{k=N_1}^{m}\left[t_{k:n}-\alpha(\mu)\right]}e^{\frac{-1}{2}\sum_{k=1}^{N_1}\left(\frac{logt_k-\mu_1}{\sigma}\right)^2} \times e^{-\frac{1}{2}\sum_{k=N_1+1}^{m}\left(\frac{log[t_k-\alpha(\mu)]-\mu_2}{\sigma}\right)^2}\left\{1-\Phi\left(\frac{\log\left[t_1-\alpha(\mu)\right]-\mu_2}{\sigma}\right)^{n-m}\right\} \tag{3.3}$$

Case 2: $T^*=min\left(t_1,t_{(r)}\right)=t_{(r)}$, where $N_1+N_2=r$

The observed time to failure of the n units in case 2 will be in the form:

$$t_{1:n}<t_{2:n}<\ldots<t_{N1:n}<\tau<t_{N_1+1:n}<\ldots<t_{(r)} \tag{3.4}$$

The likelihood function is constructed as:

$$L(\theta|t)=\left\{\prod_{k=1}^{N_1}g_1(t_{k:n})\right\}\left\{\prod_{k=N_1+1}^{r}g_2(t_{k:n})\right\}\left(1-G_2\left(t_r\right)\right)^{n-r} \tag{3.5}$$

Substituting the pdf and CDF of the step stress model:

$$L(\theta|t)=\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{r}\frac{1}{\prod_{k=1}^{N_1}t_k}\frac{1}{\prod_{k=N_1}^{r}\left[t_{k:n}-\alpha(\mu)\right]}e^{\frac{-1}{2}\sum_{k=1}^{N_1}\left(\frac{logt_k-\mu_1}{\sigma}\right)^2} \times e^{-\frac{1}{2}\sum_{k=N_1+1}^{r}\left(\frac{log[t_k-\alpha(\mu)]-\mu_2}{\sigma}\right)^2}\left\{1-\Phi\left(\frac{\log\left[t_r-\alpha(\mu)\right]-\mu_2}{\sigma}\right)^{n-r}\right\} \tag{3.6}$$

4. Bootstrap Intervals

Bootstrapping is a computer intensive non-parametric method used to make statistical inference about the parameter of interest. Bootstrap sampling can also be performed parametrically when some knowledge about the distribution of the population is available. To estimate the parameter of interest by bootstrap sampling, B samples of size n are drawn from the parametric estimate of the population and the statistic of interest is evaluated for all the bootstrap samples (Efron, B. & Tibshirani, R,1998).

The parametric bootstrap procedures can be used as a replacement for the mathematical approximations that are difficult to compute and obtain by the means of Monte Carlo simulation. These procedures are used when the given data has a specified distribution, and if the chosen distribution of the given data is the right distribution, the parametric bootstrap procedures tend to give good confidence intervals even if the samples are of small sizes. There are several bootstrap confidence intervals that have been proposed so far. In our study we will construct two bootstrap intervals, the bootstrap-t interval, and the percentile interval.

The bootstrap-t interval: In the bootstrap-t method, we compute the statistic $t^* = \dfrac{\hat{\theta}^* - \hat{\theta}}{s_{\hat{\theta}^*}}$ for each generated bootstrap sample where $\hat{\theta}^*$ is the value of the statistic $\hat{\theta}$ evaluated for the bootstrap sample and $s_{\hat{\theta}^*}$ is the standard error of the bootstrap sample. The bootstrap-t interval is given by:

$$\left(\hat{\theta} - q_{1-\alpha} s_{\hat{\theta}}, \hat{\theta} - q_{\alpha} s_{\hat{\theta}}\right)$$

Where $q_{1-\alpha}$ and q_{α} are the $(1-\alpha)$ & α percentiles of t^* respectively.

The percentile interval: The percentile interval is given by:

$$\left(q_{\alpha}, q_{1-\alpha}\right)$$

5. Simulation Study

A Monte Carlo simulation has been performed where different sample sizes n, different predetermined number of failures r, and different stress changing time have been considered. The simulation results are based on 2000 simulated samples and 1000 bootstrap samples. The different choices of hybrid censoring schemes with different sample sizes and stress changing times are presented in table 1. The Bias, MSE, average length and coverage probabilities which are used to assess the performance of the point and interval estimates of the model parameters are displayed in tables 2 and 3 respectively.

Table 1. Different hybrid censoring schemes

Scheme	N	r	(τ, t_1)
1	30	$r_1 = 15$, $r_2 = 23$, $r_3 = 27$	(30,60). (50,80)
2	50	$r_1 = 25$, $r_2 = 38$, $r_3 = 45$	(30,60), (50,80)
3	80	$r_1 = 40$, $r_2 = 60$, $r_3 = 72$	(30,60), (50,80)

Table 2. Bias and MSE of the MLE of the parameters for different censoring schemes

N	r	(τ, t_1)	Bias			MSE		
			$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\sigma}$	$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\sigma}$
30	15	(30,60)	-0.145	-0.013	-0.247	0.815	0.521	0.621
	23		-0.042	0.094	-0.140	0.748	0.414	0.474
	27		-0.021	0.100	-0.116	0.725	0.408	0.446
50	25		-0.082	0.013	-0.149	0.511	0.307	0.380
	38		-0.016	0.075	-0.082	0.441	0.253	0.269
	45		-0.002	0.079	-0.068	0.431	0.249	0.257
80	40		-0.045	0.013	-0.098	0.341	0.182	0.231
	60		-0.009	0.047	-0.060	0.278	0.151	0.159
	72		-0.0001	0.050	-0.050	0.270	0.149	0.153
30	15	(50,80)	-0.035	-0.011	-0.177	0.638	0.594	0.548
	23		-0.020	0.100	-0.128	0.667	0.447	0.459
	27		-0.003	0.107	-0.109	0.653	0.440	0.438
50	25		-0.026	0.009	-0.108	0.410	0.357	0.328
	38		-0.0003	0.078	-0.074	0.397	0.279	0.263
	45		0.010	0.083	-0.062	0.389	0.275	0.254
80	40		-0.005	0.001	-0.064	0.273	0.216	0.203
	60		0.010	0.042	-0.046	0.243	0.169	0.156
	72		0.017	0.046	-0.037	0.238	0.167	0.152

Table 3. Average length and coverage probability for the intervals

N	R	$\tau,t_1)$		Approx. CI			Bootstrap-t CI			Percentile CI		
				μ_1	μ_2	σ	μ_1	μ_2	Σ	μ_1	μ_2	σ
30	15	(30,6)	E.L	3.580	2.693	2.876	4.109	3.193	3.964	3.295	2.946	2.750
			C.P	0.892	0.934	0.862	0.955	0.957	0.957	0.936	0.932	0.853
	23		E.L	3.307	2.474	2.530	3.824	2.590	3.202	3.390	2.559	2.541
			C.P	0.917	0.936	0.900	0.943	0.947	0.951	.919	.929	.882
	27		E.L	3.312	2.486	2.522	3.573	2.571	2.877	3.394	2.549	2.510
			C.P	0.924	0.938	0.909	0.937	0.950	0.945	0.93	0.93	0.908
50	25		E.L	2.858	2.104	2.322	2.969	2.266	2.658	2.661	2.210	2.208
			C.P	0.918	0.939	0.896	0.926	0.947	0.935	0.921	0.938	0.872
	38		E.L	2.574	1.925	1.978	2.826	1.946	2.118	2.586	1.947	1.970
			C.P	0.933	0.945	0.917	0.946	0.941	0.933	0.927	0.935	0.898
	45		E.L	2.574	1.930	1.970	2.703	1.925	2.047	2.570	1.940	1.938
			C.P	0.939	0.946	0.931	0.947	0.942	0.933	0.937	0.936	0.915
80	40		E.L	2.293	1.668	1.874	2.442	1.731	2.029	2.206	1.713	1.820
			C.P	0.930	0.951	0.922	0.948	0.945	0.945	0.944	0.944	0.9
	60		E.L	2.031	1.521	1.563	2.098	1.639	1.662	2.051	1.538	1.576
			C.P	0.936	0.946	0.933	0.948	0.957	0.939	0.946	0.940	0.921
	72		E.L	2.028	1.523	1.555	2.105	1.618	1.608	2.034	1.534	1.549
			C.P	0.942	0.946	0.940	0.953	0.957	0.942	0.956	0.941	0.928
30	15	$(50,80)$	E.L	3.302	2.937	2.783	3.401	3.583	3.910	2.918	3.238	2.668
			C.P	0.938	0.929	0.870	0.969	0.959	0.970	0.982	0.934	0.889
	23		E.L	3.066	2.561	2.485	3.368	2.803	3.240	3.109	2.663	2.480
			C.P	0.922	0.926	0.898	0.944	0.955	0.963	0.927	0.924	0.893
	27		E.L	3.072	2.570	2.480	3.227	2.727	3.064	3.120	2.637	2.465
			C.P	0.925	0.929	0.908	0.941	0.952	0.96	0.938	0.928	0.915
50	25		E.L	2.599	2.279	2.216	2.511	2.606	2.472	2.361	2.491	2.114
			C.P	0.932	0.940	0.901	0.936	0.961	0.941	0.953	0.938	0.895
	38		E.L	2.390	1.992	1.946	2.406	2.046	2.037	2.390	2.026	1.932
			C.P	0.930	0.936	0.915	0.926	0.94	0.925	0.926	0.934	0.903
	45		E.L	2.392	1.996	1.940	2.376	2.021	1.989	2.382	2.014	1.912
			C.P	0.931	0.941	0.921	0.935	0.939	0.925	0.938	0.939	0.914
80	40		E.L	2.082	1.783	1.785	2.170	1.887	1.953	1.962	1.905	1.723
			C.P	0.947	0.948	0.928	0.958	0.952	0.953	0.951	0.945	0.915
	60		E.L	1.894	1.577	1.545	1.975	1.697	1.616	1.896	1.594	1.545
			C.P	0.944	0.938	0.934	0.956	0.961	0.949	0.946	0.938	0.922
	72		E.L	1.892	1.579	1.538	1.936	1.677	1.611	1.885	1.586	1.526
			C.P	0.944	0.938	0.939	0.954	0.959	0.956	0.955	0.936	0.931

6. Simulation Results

From table 2, it is noticed that the bias decreases as the sample size n and the predetermined number of failures r increase, however the bias for $\hat{\mu}_2$ increases slightly with increasing the predetermined number of failures r. The MSE decreases with increasing the sample size n and r.

Considering the average interval length shown in table 3, it is apparent that the expected lengths of the three confidence intervals decrease as the sample size n and predetermined number of failures r increase however, the bootstrap-t interval has the highest average length compared to the approximate and percentile CIs. According to the coverage probability presented in table 3, it can be seen that the bootstrap-t interval gives the best results for all the considered sample sizes compared to the approximate and percentile confidence intervals . For good coverage probabilities, the approximate and percentile confidence intervals can be used when the sample size is at least 80.

7. An illustrative example

A step stress lognormal sample of size 30 with hybrid censoring was generated where the predetermined number of failures was chosen to be 15 with stress changing time of 30 and experiment time of $T_1^* = \min(t_{15}, 60)$. The MLEs of the model parameters and their corresponding standard errors were obtained along with the hessian and variance-covariance matrix. Different confidence intervals and their corresponding lengths were found.

The following data are generated from a lognormal simple step stress sample of size 30 with τ=30 and fixed time $t_1 = 60$ with true parameter values $\mu_1 = log(200)$, $\mu_2 = log(5)$ and $\sigma = 3$.

Table 4. Simulated data from the cumulative exposure lognormal model

Lifetimes								
Under normal stress level								
0.5259	11.2563	12.0893	21.9279					
Under higher stress level								
30.0079	30.0537	30.9522	31.13089	31.3562	32.0215	32.15492	32.7659	32.8116
33.8831	35.6947	36.1690	36.4736	43.93030	48.6155	49.9542	52.2285	61.9079
68.0899	157.7506	158.6676	223.0157	432.5115	1338.9093	3101.1047	3276.2830	

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The MLEs for the model parameters and their corresponding standard errors are given in Tble 5:

Table 5. Maximum likelihood estimators of the cumulative exposure lognormal model parameters

$\hat{\mu}_1$	$\hat{\mu}_2$	$\hat{\sigma}$
6.553779	1.869605	2.870388
$SE(\hat{\mu}_1)$	$SE(\hat{\mu}_2)$	$SE(\hat{\sigma})$
1.995996	0.4186727	0.8634164

The hessian matrix was found as below:

$$\begin{pmatrix} 1.8111902 & -0.9171184 & -2.1090687 \\ -0.9171184 & 2.9909645 & 0.6566688 \\ -2.1090687 & 0.6566688 & 3.6810780 \end{pmatrix}$$

The variance-covariance matrix was derived by finding the inverse of the hessian matrix and is given below:

$$\begin{pmatrix} 1.9959957 & .3756656 & 1.0765879 \\ .3756656 & .4186727 & .1405500 \\ 1.0765879 & .1405500 & .8634164 \end{pmatrix}$$

The confidence intervals for the model parameters and their corresponding lengths are computed and given in the following table:

Table 6. Confidence intervals for the parameters of the cumulative lognormal model

	Approx. CI	Length	Bootstrap-t CI	Length	Percentile CI	Length
μ_1	(3.7847, 9.3228)	5.5381	(2.7867, 10.2372)	7.4505	(4.3141, 8.8444)	4.5304
μ_2	(0.6014, 3.1378)	2.5364	(0.4814, 3.4261)	2.9447	(0.568, 3.0299)	2.4614
σ	(1.0492, 4.6916)	3.6424	(1.1578, 5.2226)	4.0649	(1.092, 4.1651)	3.0731

The length of the bootstrap-t confidence interval is the longest while percentile interval has the shortest length.

8. Conclusion

In this research, we considered step stress lognormal life test data with hybrid censoring where the cumulative exposure model has been fit to the data. The study's main interest is finding good estimators for the model parameters. We obtained the maximum likelihood estimators numerically by using the nlm function in R since the likelihood equations for the lognormal distribution cannot be found explicitly. To study and examine the performance of the point and interval estimates of the model parameters, we performed a simulation study. In terms of coverage probabilities, the bootstrap-t interval gives the best results for all the considered sample sizes and thus can be used to estimate the model parameters for small and large sample sizes. The approximate and percentile confidence intervals can be used with large sample sizes.

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