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The Marshall-Olkin-Odd Power Generalized Weibull-G Family of Distributions with Applications of COVID-19 Data

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ABSTRACT
Attempts have been made to define new families of distributions that provide more flexibility for modelling data that is skewed in nature. In this work, we propose a new family of distributions called Marshall-Olkin-odd power generalized Weibull (MO-OPGW-G) distribution based on the generator pioneered by Marshall and Olkin [20]. This new family of distributions allows for a flexible fit to real data from several fields, such as engineering, hydrology and survival analysis. The mathematical and statistical properties of these distributions are studied and its model parameters are obtained through the maximum likelihood method. We finally demonstrate the effectiveness of these models via simulation experiments and applications to COVID-19 daily deaths data sets.

Keywords: Marshall-Olkin-G, Maximum Likelihood Estimation, Power Generalized Weibull Distribution, Simulation.

1 Introduction

Statistical distributions are very useful when it comes to modeling real life data. However, some of the well known distributions have limitations and problems when it comes to modeling of heavy-tailed or highly skewed data. Thus, to deal with these problems, statisticians proposed techniques to develop new families of probability distributions in order to improve flexibility of classical distributions. Some examples of families of distributions in the literature are Gamma-G by Zografos and Balakrishnan [34], the extended generalized log-logistic family by Gleaton and Lynch [10], Kumaraswamy odd log-logistic family by Alizadeh et al. [2], Marshall-Olkin alpha power-G by Nassar et al. [25], the Lindley family of distributions by Cakmakyapan and Ozel [5],

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the Odd Lindley-G family by Gomes-Silva et al. [9], power Lindley-G by Hassan and Nassr [12], the Topp Leone odd Lindley-G family of distributions by Reyad et al. [28], the Marshall-Olkin generalized-G by Yousof et al. [32], the Marshall-Olkin Generalized-G Poisson family of distributions by Korkmaz et al. [17], the Marshall-Olkin-Kumarswamy-G by Handique and Chakraborty [11] , the Marshall-Olkin Topp Leone-G by Khaleel et al. [15] and the new power generalized Weibull-G by Moakofi et al. [22] to mention a few.

The Marshall-Olkin transformation was applied to several well-known distributions including Weibull (Ghittany et al. [8], Zhang and Xie [33]). More recently, general results on the Marshall-Olkin family of distributions were given by Barreto-Souza et al. [3]. Moakofi et al. [21] developed the Marshall-Olkin Lindley-Log-logistic distribution. Krishna et al. [16] established Marshall-Olkin Fréchet distribution and its applications. Santos-Neto et al. [29] introduces a new class of models called the Marshall-Olkin extended Weibull family of distributions which defines at least twenty-one special models. Lepetu et al. [19] proposed the Marshall-Olkin Log-Logistic Extended Weibull distribution. Usman and Haq [30] studied the Marshall-Olkin extended inverted Kumaraswamy distribution and Javed et al. [13] developed the Marshall-Olkin Kappa distribution.

Marshall and Olkin [20], introduced a new distribution with cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{MO-G}(x;\delta,\xi) = 1 - \frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)}, \tag{1}$$

and

$$f_{MO-G}(x;\delta,\xi) = \frac{\delta g(x;\xi)}{\left[1-\bar{\delta}\bar{G}(x;\xi)\right]^2}, \tag{2}$$

respectively, where δ is the tilt parameter and $G(x;\xi)$ is the baseline cdf. The distribution with the exponential, Weibull and gamma distributions as baseline distributions is more flexible than the corresponding baseline distributions.

In a recent note, Moakofi et al. [23] developed the odd power generalized Weibull-G (OPGW-G) distribution with cdf and pdf given by

$$F_{OPGW-}\ (x;\alpha,\beta,\xi) = 1 - \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right) \tag{3}$$

and

$$f_{OPGW-G}(x;\alpha,\beta,\xi) = \alpha\beta\left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1}$$
$$\times \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\frac{g(x;\xi)}{(1-G(x;\xi))^2},$$

respectively, for $\alpha,\beta > 0$ and parameter vector $\underline{\psi}$.

The basic motivations for developing the MO-OPGW-G family of distributions are;

• to construct and generate distributions with symmetric, left-skewed, right-skewed, reversed-J shapes;
• to define special models that posseses various types of hazard rate functions including monotonic as well as non-monotonic shapes;
• to provide better fits than other generated distributions under the same transformation;
• to construct heavy-tailed distributions for modeling different real data sets;
• to make the kurtosis more flexible compared to that of the baseline distribution.

In this paper, we develop the new family of distributions, namely, the MO-OPGW-G family of distributions. In Section 2, we present the new generalized family of distributions, its density expansion, sub-families, moments and moment generating function. Some special case are presented in Section 3. Structural properties including the distribution of order statistics, Rényi entropy and quantile function are presented in Section 4. In Section 5, we present the maximum likelihood estimates. Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators for each parameter in Section 6. Applications of the proposed model to real data are given in Section 7, followed by concluding remarks.

2 The Model and Some Properties

We develop the MO-OPGW-G family of distributions using the generalization proposed by Marshall and Olkin [20], and taking the baseline distribution to be the OPGW-G distribution. The cdf, pdf and hazard rate function (hrf) of the MO-OPGW-G family of distributions are given by

$$F_{MO-OPGW-G}(x;\delta,\alpha,\beta,\xi)=\frac{1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)}{1-\bar{\delta}\left[\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]}, \quad (4)$$

$$f_{MO-OPGW-G}(x;\delta,\alpha,\beta,\xi)=\delta\alpha\beta\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1}$$
$$\times\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\frac{g(x;\xi)}{(1-G(x;\xi))^{2}}$$
$$\times\left[1-\bar{\delta}\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]^{-2}, \quad (5)$$

and

$$h(x)=\delta\alpha\beta\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta-1}\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha-1}$$
$$\times\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\frac{g(x;\xi)}{(1-G(x;\xi))^{2}}$$
$$\times\left[1-\bar{\delta}\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]^{-2}$$
$$\times\left(1-\frac{1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)}{1-\bar{\delta}\left[\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]}\right)^{-1}, \quad (6)$$

respectively, for $\alpha,\beta,\delta>0$, $\bar{\delta}=1-\delta$ and ξ is a vector of parameters from the baseline distribution function G(.).

2.1 Sub-Families

In this subsection, some sub-families of the MO-OPGW-G family of distributions are presented.

• When $\delta=1$, we obtain the odd power generalized Weibull-G (OPGW-G) family of distributions (Moakofi et al. [23]) with the cdf

$$F(x;\alpha,\beta,\xi)=1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right),$$

for $\alpha,\beta>0$, and parameter vector ξ.

• When $\beta=1$, we obtain the new family called Marshall-Olkin Odd Weibull-G (MO-OW-G) family of distributions with the cdf

$$F(x;\delta,\alpha,\xi)=\frac{1-e\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right)}{1-\delta\overline{\left[\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]}},$$

for $\alpha,\delta>0$, and parameter vector ξ.

• When $\alpha=2$, we obtain the new family of distributions with the cdf

$$F(x;\delta,\beta,\xi)=\frac{1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{2}\right]^{\beta}\right)}{1-\bar{\delta}\left[\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{2}\right]^{\beta}\right)\right]},$$

for $\beta,\delta>0$, and parameter vector ξ.

• If $\alpha=1$, we obtain the new family called Marshall-Olkin odd Nadarajah Haghighi-G (MO-ONH-G) family of distributions with the cdf

$$F(x;\delta,\beta,\xi)=\frac{1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right]^{\beta}\right)}{1-\bar{\delta}\left[\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right]^{\beta}\right)\right]},$$

for $\beta,\delta>0$, and parameter vector ξ.

• If $\alpha=\beta=1$, we obtain the new family called Marshall-Olkin odd exponential-G (MO-OE-G) family of distributions with the cdf

$$F(x;\delta,\xi)=\frac{1-\exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right)}{1-\bar{\delta}\left[\exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right)\right]},$$

for $\delta>0$ and parameter vector ξ.

• If $\beta=1,\alpha=2$ we obtain the new family called Marshall-Olkin odd Rayleigh-G (MO-OR-G) family of distributions with the cdf

$$F(x;\delta,\xi)=\frac{1-\exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{2}\right)}{1-\bar{\delta}\left[\exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{2}\right)\right]},$$

for $\delta>0$ and parameter vector ξ.

• When $\delta=\beta=1$, we obtain the Weibull-G (W-G) family of distributions (Bourguignon et al. [4]) with the cdf

$$F(x;\alpha,\xi)=1-\exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right),$$

for $\alpha>0$ and parameter vector ξ.

• When $\delta=1,\alpha=2$, we obtain the new family of distributions with the cdf

$$F(x;\beta,\xi) = 1 - \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^2\right]^\beta\right),$$

for $\beta > 0$, and parameter vector ξ.

• If $\delta = \alpha = 1$, we obtain the odd Nadarajah Haghighi-G (ONH-G) family of distributions (Nascimento et al. [24]) with the cdf

$$F(x;\beta,\xi) = 1 - \exp\left(1 - \left[1 + \left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right]^\beta\right),$$

for $\beta > 0$, and parameter vector ξ.

• If $\delta = \alpha = \beta = 1$, we obtain the odd exponential-G (OE-G) family of distributions (Bourguignon et al. [4]) with the cdf

$$F(x;\xi) = 1 - \exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)\right),$$

for parameter vector ξ.

• If $\delta = \beta = 1, \alpha = 2$ we obtain the odd Rayleigh-G (OR-G) family of distributions (Bourguignon et al. [4]) with the cdf

$$F(x;\xi) = 1 - \exp\left(-\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^2\right),$$

for parameter vector ξ.

2.2 Expansion of Density Function

In this section, we derive the statistical properties of the MO-OPGW-G family of distributions using general results for the Marshall and Olkin's family of distributions by Barreto-Souza et al.[3]. Considering

$$f_{MO-OPGW-G}(x;\delta,\alpha,\beta,\xi) = \frac{\alpha f_{OPGW-G}(x;\alpha,\beta,\xi)}{(1-\bar{\delta}\bar{F}_{OPGW-}\ (x;\alpha,\beta,\xi))^2}, \tag{7}$$

we can write equation (6) as

$$f_{MO-OPGW-G}(x;\delta,\alpha,\beta,\xi) = \frac{f_{OPGW-G}(x;\alpha,\beta,\xi)}{\delta[1-\left(\frac{\delta-1}{\delta}\right)F_{OPGW-G}(x;\alpha,\beta,\xi)]^2}, \tag{8}$$

where $f_{OPGW-G}(x;\alpha,\beta,\xi)$ and $F_{OPGW-G}(x;\alpha,\beta,\xi)$ are as given in equations (4) and (3), respectively. We apply the series expansion

$$(1-z)^{-k} = \sum_{j=0}^{\infty} \frac{\Gamma(k+j)}{\Gamma(k)j!} z^j, \tag{9}$$

which is valid for $|z| < 1$ and $k > 0$. If $\delta \in (0,1)$, to obtain

$$f_{MO-OPGW-G}(x;\delta,\alpha,\beta,\xi) = f(x;\alpha,\beta,\xi)\sum_{j=0}^{\infty}\sum_{k=0}^{j} w_{j,k}F(x;\alpha,\beta,\xi)^{j-k}, \tag{10}$$

where $w_{j,k} = w_{j,k}(\delta) = \delta(j+1)(1-\delta)^j(-1)^{j-k}\binom{j}{k}$. For $\delta > 1$, we have

$$f_{MO-OPGW-G}(x;\delta,\alpha,\beta,\xi) = f(x;\alpha,\beta,\xi)\sum_{j=0}^{\infty} v_j F^j(x;\alpha,\beta,\xi), \tag{11}$$

where $v_j = v_j(\delta) = \frac{(j+1)(1-1/\delta)}{\delta}$. For $\delta \in (0,1)$, equation (6) becomes

$$f_{MO-OPGW-}\ (x;\delta,\alpha,\beta,\xi) = \sum_{v=0}^{\infty} e_{v+1}^{*} g_{v+1}(x;\xi), \tag{12}$$

where

$$e^*_{v+1} = \sum_{j=0}^{\infty} \sum_{k=0}^{j} \sum_{m,l,i,p,q=0}^{\infty} \binom{j-k}{m}\binom{l}{i} \binom{\beta(i+1)-1}{p}\binom{\alpha(p+1)-1}{q} \times \binom{-\alpha(p+1)-1+q}{v} \frac{w_{j,k}\alpha\beta}{v+1} \frac{(-1)^{m+i+q+v}(m+1)^l}{l!}, \quad (13)$$

and $g_{v+1}(x;\xi) = (v+1)G^v_{v+1}(x;\xi)g_{v+1}(x;\xi)$ is the exponentiated-G (Exp-G) distribution with the power parameter $(v+1) > 0$. It follows that for $\delta \in (0,1)$, the pdf of the MO-OPGW-G family of distributions can be expressed as a linear combination of the Exp-G densities.

Furthermore, for $\delta > 1$, equation (6) can be written as

$$f_{MO-OPGW-G}(x;\delta,\alpha,\beta,\xi) = \sum_{w=0}^{\infty} b^*_{w+1} g_{w+1}(x;\xi). \quad (14)$$

Therefore, for $\delta > 1$, the MO-OPGW-G family of distributions can be expressed as a linear combination of the Exp-G distribution with power parameter $(w+1) > 0$ and linear component

$$b^*_{w+1} = \sum_{j=0}^{\infty} \sum_{m,l,i,p,q=0}^{\infty} \binom{j}{m}\binom{l}{i} \binom{\beta(i+1)-1}{p}\binom{\alpha(p+1)-1}{q} \times \binom{-\alpha(p+1)-1+q}{w} \frac{v_j\alpha\beta}{w+1} \frac{(-1)^{m+i+q+v}(m+1)^l}{l!}. \quad (15)$$

Details of the derivations are provided in the appendix.

2.3 Moments and Generating Functions

Let $X \sim MO-OPGW-G(\delta,\alpha,\beta,\xi)$, $Y_{v+1} \sim E-G(v+1,\xi)$ and $Y_{w+1} \sim E-G(w+1,\xi)$, then the r^{th} moment can be obtained from equations (12) and (14). For $\delta \in (0,1)$,

$$E(X^r) = \sum_{v=0}^{\infty} e^*_{v+1} E(Y^r_{v+1}),$$

where e^*_{v+1} is as defined in equation (13) and $E(Y^r_{v+1})$ denotes the r^{th} moment of an Exp-G distribution with power parameter $(v+1) > 0$. For $\delta > 1$

$$E(X^r) = \sum_{w=0}^{\infty} b^*_{w+1} E(Y^r_{w+1}),$$

where b^*_{w+1} is as defined in equation (15) and $E(Y^r_{w+1})$ denotes the r^{th} moment of an Exp-G distribution with power parameter $(w+1) > 0$. The incomplete moments can be obtained as follows:

For $\delta \in (0,1)$

$$I_X(t) = \int_0^t x^s f_{MO-OPGW-G}(x;\delta,\alpha,\beta,\xi)dx = \sum_{v=0}^{\infty} e^*_{v+1} I_{v+1}(t),$$

where $I_{v+1}(t) = \int_0^t x^s g_{v+1}(x;\xi)dx$ and e^*_{v+1} is as defined in equation (13). Also, For $\delta > 1$

$$I_X(t) = \int_0^t x^s f_{MO-OPGW-G}(x;\delta,\alpha,\beta,\xi)dx = \sum_{w=0}^{\infty} b^*_{w+1} I_{w+1}(t),$$

where $I_{w+1}(t) = \int_0^t x^r g_{w+1}(x;\xi)dx$ and b^*_{w+1} is as defined in equation (15). The moment generating function (mgf) of X is given by:

For $\delta \in (0,1)$

$$M_X(t) = \sum_{v=0}^{\infty} e^*_{v+1} E(e^{tY_{v+1}}),$$

where $E(e^{tY_{v+1}})$ is the mgf of the Exp-G distribution with power parameter $(v+1) > 0$ and e^*_{v+1} is as defined in equation (13). For $\delta > 1$

$$M_X(t) = \sum_{w=0}^{\infty} b^*_{w+1} E(e^{tY_{w+1}}),$$

where $E(e^{tY_{w+1}})$ is the mgf of the Exp-G distribution with power parameter $(w+1) > 0$ and b^*_{w+1} is as defined in equation (15).

3 Some Special Cases

In this section, we present some special cases of the MO-OPGW-G family of distributions. We considered cases when the baseline distributions are Burr XII and Kumaraswamy distributions.

3.1 Marshall-Olkin-Odd Power Generalized Weibull-Burr XII (MO-OPGW-BXII) Distribution

By taking the Burr XII distribution as the baseline distribution with pdf and cdf given by $g(x;c,k) = ckx^{c-1}(1+x^c)^{-k-1}$ and $G(x;c,k) = 1-(1+x^c)^{-k}$, respectively, for $c,k>0$, we obtain the MO-OPGW-BXII distribution with cdf and pdf given by

$$F_{MO-OPGW-BXII}(x) = \frac{1-\exp\left(1-\left[1+\left(\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k}}\right)^{\alpha}\right]^{\beta}\right)}{1-\bar{\delta}[\exp\left(1-\left[1+\left(\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k}}\right)^{\alpha}\right]^{\beta}\right)]} \quad (16)$$

and

$$f_{MO-OPGW-BXII}(x) = \delta\alpha\beta\left[1+\left(\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k}}\right)^{\alpha}\right]^{\beta-1}\left(\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k}}\right)^{\alpha-1}$$
$$\times\exp\left(1-\left[1+\left(\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k}}\right)^{\alpha}\right]^{\beta}\right)\frac{ckx^{c-1}(1+x^c)^{-k-1}}{\left((1+x^c)^{-k}\right)^2}$$
$$\times\left[1-\bar{\delta}\exp\left(1-\left[1+\left(\frac{1-(1+x^c)^{-k}}{(1+x^c)^{-k}}\right)^{\alpha}\right]^{\beta}\right)\right]^{-2}, \quad (17)$$

respectively, for $\delta,\alpha,\beta,c,k>0$. The MO-OPGW-BXII distribution reduces to MO-OPGW-log-logistic (MO-OPGW-LLoG) and MO-OPGW-Lomax (MO-OPGW-Lx) distribution by setting k=1 and c=1, respectively.

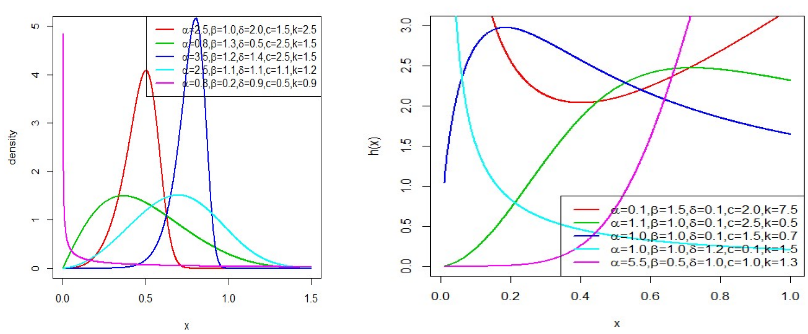

Figure 3: Pdf and hrf graphs for the MO-OPGW-BXII distribution

The pdf of the MO-OPGW-BXII distribution takes various shapes and addresses variations in skewness and kurtosis as shown in Figure 3. The hrf exhibits monotonic, bathtub and upside bathtub shapes.

We plot 3D diagrams of skewness and kurtosis for the submodel of the MO-OPGW-BXII distribution, which is the MO-OPGW-LLoG distribution and the results are given in Figures 1 and 2. We observe that:

• When we fix the parameters α and δ, the skewness and kurtosis of MO-OPGW-LLoG increases as c and β increases.

• When we fix the parameters $\alpha$and δ, the skewness and kurtosis of MO-OPGW-LLoG increases as β and c increases.

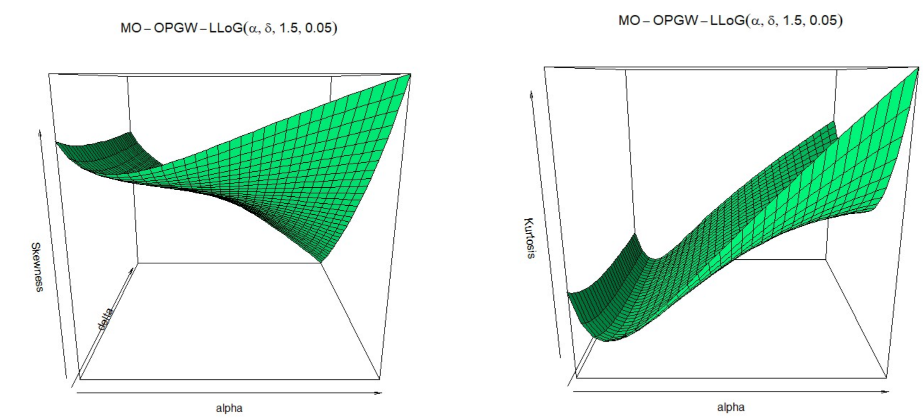

Figure 1: Plots of skewness and kurtosis for the MO-OPGW-LLoG distribution

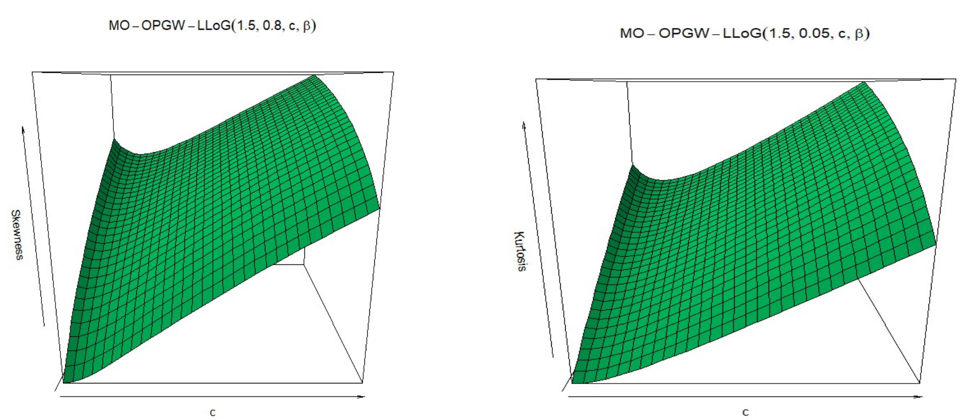

Figure 2: Plots of skewness and kurtosis for the MO-OPGW-LLoG distribution

3.2 Marshall-Olkin-Odd Power Generalized Weibull-Kumaraswamy (MO-OPGW-Kw) Distribution

Suppose the baseline distribution is the Kumaraswamy distribution with pdf and cdf given by $g(x;a,b) = abx^{a-1}(1-x^a)^{b-1}$ and $G(x;a,b) = 1-(1-x^a)^b$, for $a,b>0$, respectively, we obtain the MO-OPGW-Kw distribution with cdf and pdf given by

$$F_{MO-OPGW-K}(x) = \frac{1-\exp\left(1-\left[1+\left(\frac{1-(1-x^a)^b}{(1-x^a)^b}\right)^{\alpha}\right]^{\beta}\right)}{1-\bar{\delta}[\exp\left(1-\left[1+\left(\frac{1-(1-x^a)^b}{(1-x^a)^b}\right)^{\alpha}\right]^{\beta}\right)]} \quad (18)$$

and

$$f_{MO-OPGW-Kw}(x) = \delta\alpha\beta\left[1+\left(\frac{1-(1-x^a)^b}{(1-x^a)^b}\right)^{\alpha}\right]^{\beta-1}\left(\frac{1-(1-x^a)^b}{(1-x^a)^b}\right)^{\alpha-1}$$
$$\times\exp\left(1-\left[1+\left(\frac{1-(1-x^a)^b}{(1-x^a)^b}\right)^{\alpha}\right]^{\beta}\right)\frac{abx^{a-1}(1-x^a)^{b-1}}{\left((1-x^a)^b\right)^2}$$
$$\times\left[1-\bar{\delta}\exp\left(1-\left[1+\left(\frac{1-(1-x^a)^b}{(1-x^a)^b}\right)^{\alpha}\right]^{\beta}\right)\right]^{-2}, \quad (19)$$

respectively, for $\delta,\alpha,\beta,a,b>0$.

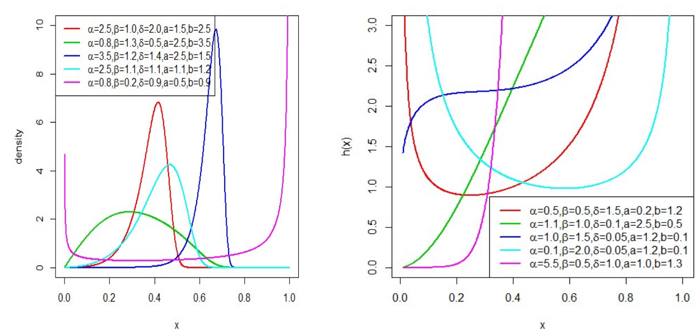

Figure 4: Pdf and hrf graphs for the MO-OPGW-Kw distribution

Plots of the pdf and hrf of the MO-OPGW-Kw distribution are shown in Figure 4. The pdf addresses various forms of skewness and kurtosis. Furthermore, the hrf exhibits both monotonic and non-monotonic shapes.

4 Order Statistics and Entropy

We derive the distribution of the i^{th} order statistic and Rényi entropy of the MO-OPGW-G distribution in this section.

4.1 Distribution of Order Statistics

Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed (i.i.d) random variables distributed according to (6). The pdf of the i^{th} order statistic $X_{i:n}$, is given by

$$f_{i:n}(x;\delta,\alpha,\beta,\xi)=\delta n!\,f_{OPGW-G}(x;\alpha,\beta,\xi)\sum_{l=0}^{n-i}\frac{(-1)^l}{(i-1)!(n-i)!}\frac{F_{OPGW-G}^{l+i-1}(x;\alpha,\beta,\xi)}{[1-\bar{\delta}F_{OPGW-G}(x;\alpha,\beta,\xi)]^{l+i-1}}.\quad(20)$$

If $\delta\in(0,1)$, we have

$$f_{i:n}(x;\delta,\alpha,\beta,\xi)=f_{OPGW-G}(x;\alpha,\beta,\xi)\sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}U_{j,l,k}F_{OPGW-G}^{j+l-k+i-1}(x;\alpha,\beta,\xi),\quad(21)$$

where

$$U_{j,l,k}=U_{j,l,k}(\delta)=\frac{\delta n!(-1)^l(1-\delta)^j(-1)^{j-k}}{(i-1)!(n-i)!}\binom{j}{k}\binom{l+i+j}{j}.\quad(22)$$

For $\delta>1$, we write $\left(1-\bar{\delta}\bar{F}_{OPGW-G}(x;\alpha,\beta,\xi)\right)=\delta\{1-(\delta-1)F_{OPGW-G}(x;\alpha,\beta,\xi)/\delta\}$, such that

$$f_{i:n}(x;\delta,\alpha,\beta,\xi)=f_{OPGW-G}(x;\alpha,\beta,\xi)\sum_{j=0}^{\infty}\sum_{l=0}^{n-i}c_{j,l}F_{OPGW-}^{j+l+i-1}(x;\alpha,\beta,\xi),\quad(23)$$

where

$$c_{j,l}=c_{j,l}(\delta)=\frac{(-1)^l(\delta-1)^j n!}{\delta^{l+j+i}(i-1)!(n-i)!}\binom{l+i+j}{j}.\quad(24)$$

For $\delta\in(0,1)$, using equation (21) and substituting $f_{OPGW-G}(x)$ by equation (22) and $F_{OPGW-G}(x)$ by equation (3), we get

$$f_{i:n}(x;\delta,\alpha,\beta,\xi)=\sum_{v=0}^{\infty}e_{v+1}^{**}g_{v+1}(x;\xi),$$

where $g_{v+1}(x;\xi)=(v+1)(G(x;\xi))^v g(x;\xi)$ is the exponentiated-G (Exp-G) density function with power parameter $(v+1)>0$ and linear component

$$e_{v+1}^{**}=\sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{k=0}^{j}\sum_{m,l,i,p,q=0}^{\infty}\binom{j+l-k+i-1}{m}\binom{l}{i}\binom{\beta(i+1)-1}{p}\binom{\alpha(p+1)-1}{q}$$
$$\times\binom{-\alpha(p+1)-1+q}{v}U_{j,l,k}\alpha\beta\frac{1}{v+1}\frac{(-1)^{m+i+q+v}(m+1)^l}{l!}.$$

Furthermore, for $\delta>1$, we get

$$f_{i:n}(x;\delta,\alpha,\beta,\xi)=\sum_{v=0}^{\infty}w_{v+1}^{**}g_{v+1}(x;\xi),$$

where $g_{v+1}(x;\xi)=(v+1)(G(x;\xi))^v g(x;\xi)$ is the exponentiated-G (Exp-G) density function with power parameter $(v+1)>0$ and linear component

$$w_{v+1}^{**}=\sum_{j=0}^{\infty}\sum_{l=0}^{n-i}\sum_{m,l,i,p,q=0}^{\infty}\binom{j+l+i-1}{m}\binom{l}{i}\binom{\beta(i+1)-1}{p}\binom{\alpha(p+1)-1}{q}$$
$$\times\binom{-\alpha(p+1)-1+q}{v}C_{j,l}\alpha\beta\frac{1}{v+1}\frac{(-1)^{m+i+q+v}(m+1)^l}{l!}.$$

Details of the derivations are provided in the appendix.

4.2 Entropy

An Entropy is a measure of variation of uncertainty for a random variable X with pdf g(x). Here we present the measures of entropy, namely Rényi entropy [27]. Rényi entropy is defined by

$$I_R(\nu)=(1-\nu)^{-1}\log\left[\int_0^{\infty}g^{\nu}(x)dx\right],$$

where $\nu>0$ and $\nu\neq1$. Using expansion (9), for $\delta\in(0,1)$

$$f_{MO-OPGW-G}^{\nu}(x;\delta,\alpha,\beta,\xi)=\frac{\delta^{\nu}f_{OPGW-}^{\nu}(x;\xi)}{\Gamma(2\nu)}\sum_{j=0}^{\infty}(1-\alpha)^j\Gamma(2\nu+j)\frac{[1-F_{OPGW-}(x;\xi)]^j}{j!}$$

and for $\delta>1$

$$f_{MO-OPGW-G}^{\nu}(x;\delta,\alpha,\beta,\xi)=\frac{f_{OPGW-G}^{\nu}(x)}{\delta^{\nu}\Gamma(2\nu)}\sum_{j=0}^{\infty}(\delta-1)^j\Gamma(2\nu+j)\frac{F_{OPGW-}^{j}(x;\xi)}{j!}.$$

Thus, Rényi entropy for $\delta\in(0,1)$ and $\delta>1$ are given by

$$I_R(\nu)=(1-\nu)^{-1}\log\left(\sum_{j=0}^{\infty}e_j\int_0^{\infty}f_{OPGW-}^{\nu}(x;\xi)(1-F_{OPGW-G}(x;\xi))^j dx\right)\quad(25)$$

and

$$I_R(\nu) = (1-\nu)^{-1}\log\left(\sum_{j=0}^{\infty} h_j \int_0^{\infty} f_{OPGW-}^{\nu}\ (x;\xi) F_{OPGW-G}^{j}(x;\xi)dx\right), \tag{26}$$

where

$$e_j = e_j(\delta) = \frac{\delta^{\nu}(1-\delta)^j \Gamma(2\nu+j)}{\Gamma(2\nu)j!}$$

and

$$h_j = h_j(\delta) = \frac{(\delta-1)^j \Gamma(2\nu+j)}{\delta^{\nu+j}\Gamma(2\nu)j!}.$$

Now, for $\delta \in (0,1)$ and using equation (25), we have

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1}\log[\sum_{j,m,l,i,k,q=0}^{\infty} e_j \frac{((j+1))^m(-1)^{l+k+q}}{m!}\binom{m}{l}(\alpha\beta)^{\nu} \\ &\times \binom{\beta(\nu+l)-\nu}{i}\binom{\alpha(\nu+i)-\nu}{k}\binom{k-(\alpha(\nu+i)+\nu)}{q} \\ &\times \frac{1}{(\frac{q}{\nu}+1)^{\nu}}\int_0^{\infty} ((\frac{q}{\nu}+1)g(x;\xi)[G(x;\xi)]^{\frac{q}{\nu}})^{\nu}dx] \\ &= (1-\nu)^{-1}\log[\sum_{q=0}^{\infty} e_q^{*}\exp(1-\nu)I_{REG}], \end{aligned} \tag{27}$$

where

$$\begin{aligned} e_q^{*} &= \sum_{j,m,l,i,k=0}^{\infty} e_j \frac{((j+1))^m(-1)^{l+k+q}}{m!}\binom{m}{l}(\alpha\beta)^{\nu}\frac{1}{(\frac{q}{\nu}+1)^{\nu}} \\ &\times \binom{\beta(\nu+l)-\nu}{i}\binom{\alpha(\nu+i)-\nu}{k}\binom{k-(\alpha(\nu+i)+\nu)}{q} \end{aligned} \tag{28}$$

and $I_{REG} = \int_0^{\infty} ((\frac{q}{\nu}+1)g(x;\xi)[G(x;\xi)]^{\frac{q}{\nu}})^{\nu}dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{q}{\nu}+1$. Furthermore, for $\delta > 1$, we can write

$$\begin{aligned} I_R(\nu) &= (1-\nu)^{-1}\log[\sum_{j,w,m,l,i,k,q=0}^{\infty} e_j j_W \frac{((w+1))^m(-1)^{l+w+k+q}}{m!}\binom{m}{l}(\alpha\beta)^{\nu} \\ &\times \binom{\beta(\nu+l)-\nu}{i}\binom{\alpha(\nu+i)-\nu}{k}\binom{k-(\alpha(\nu+i)+\nu)}{q}\frac{1}{(\frac{q}{\nu}+1)^{\nu}} \\ &\times \int_0^{\infty} ((\frac{q}{\nu}+1)g(x;\xi)[G(x;\xi)]^{\frac{q}{\nu}})^{\nu}dx] \\ &= (1-\nu)^{-1}\log[\sum_{q=0}^{\infty} h_q^{*}\exp(1-\nu)I_{REG}], \end{aligned} \tag{29}$$

where

$$\begin{aligned} h_q^{*} &= \sum_{j,w,m,l,i,k=0}^{\infty} e_j j_W \frac{((w+1))^m(-1)^{l+w+k+q}}{m!}\binom{m}{l}(\alpha\beta)^{\nu}\frac{1}{(\frac{q}{\nu}+1)^{\nu}} \\ &\times \binom{\beta(\nu+l)-\nu}{i}\binom{\alpha(\nu+i)-\nu}{k}\binom{k-(\alpha(\nu+i)+\nu)}{q} \end{aligned} \tag{30}$$

and $I_{REG} = \int_0^{\infty} ((\frac{q}{\nu}+1)g(x;\xi)[G(x;\xi)]^{\frac{q}{\nu}})^{\nu}dx$ is the Rényi entropy of the Exp-G distribution with power parameter $\frac{q}{\nu}+1$.

Details of the derivations are provided in the appendix.

4.3 Quantile Function

The quantile function for the MO-OPGW-G family of distributions is obtained by solving the non-linear equation:

$$F_{MO-OPGW-G}(x;\delta,\alpha,\beta,\xi)=\frac{1-\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)}{1-\bar{\delta}[\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)]}=u$$

for $0\le u\le 1$, that is,

$$\exp\left(1-\left[1+\left(\frac{G(x;\xi)}{1-G(x;\xi)}\right)^{\alpha}\right]^{\beta}\right)=\frac{1-u}{1-u\bar{\delta}},$$

so that

$$\frac{G(x;\xi)}{1-G(x;\xi)}=\left[\left(1-\ln\left(\frac{1-u}{1-u\bar{\delta}}\right)\right)^{\frac{1}{\beta}}-1\right]^{\frac{1}{\alpha}}.$$

Therefore, the quantiles of the MO-OPGW-G family of distributions is given by

$$Q_X(u)=G^{-1}\left[\left(\left(\left[\left(1-\ln\left(\frac{1-u}{1-u\bar{\delta}}\right)\right)^{\frac{1}{\beta}}-1\right]^{\frac{1}{\alpha}}\right)^{-1}+1\right)^{-1}\right]. \tag{31}$$

5 Maximum Likelihood Estimation

If $X_i\sim MO-OPGW-G(\alpha,\beta,\delta,\xi)$ with the parameter vector $\Delta=(\alpha,\beta,\delta,\xi)^T$. The total log-likelihood $\ell=\ell(\Delta)$ from a random sample of size n is given by

$$\ell=n\log(\delta\alpha\beta)+(\beta-1)\sum_{i=1}^{n}\log\left[1+\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)^{\alpha}\right]$$
$$+\sum_{i=1}^{n}\left(1-\left[1+\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)^{\alpha}\right]^{\beta}\right)]+(\alpha-1)\sum_{i=1}^{n}\log\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)$$
$$-2\sum_{i=1}^{n}\log\left[1-\bar{\delta}\exp\left(1-\left[1+\left(\frac{G(x_i;\xi)}{1-G(x_i;\xi)}\right)^{\alpha}\right]^{\beta}\right)\right]$$
$$+\sum_{i=1}^{n}\ln(g(x_i;\xi))-2\sum_{i=1}^{n}\ln(1-G(x_i;\xi)).$$

The score vector $U=\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\delta},\frac{\partial\ell}{\partial\xi_k}\right)$ elements are given in the appendix.

The maximum likelihood estimates of the parameters, denoted by $\widehat{\boldsymbol{\Delta}}$ is obtained by solving the nonlinear equation $\left(\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\delta},\frac{\partial\ell}{\partial\xi_k}\right)^T=\mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The multivariate normal distribution $N_{q+3}(\underline{0},J(\Delta)^{-1})$, where the mean vector $\underline{0}=(0,0,0,\underline{0})^T$ and $J(\Delta)^{-1}$ is the observed Fisher information matrix evaluated at Δ, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions.

6 Simulation Study

A simulation study to examine the consistency of the maximum likelihood estimates (MLE) is conducted in this section. We used the following sets of initial values $\alpha=1.0,\delta=0.5,c=0.05,\beta=1.1$ and $\alpha=1.0,\delta=1.0,c=0.05,\beta=1.1$ for sample sizes n= 25, 50, 100, 200, 400, 800 and 1000. We estimate the mean, average bias and root mean square error (RMSE). The bias and RMSE for the estimated parameter, say, $\widehat{\Delta}$, say, are given by:

$$Bias(\widehat{\Delta}) = \frac{1}{N}\sum_{i=1}^{N} (\widehat{\Delta}_i - \Delta), \quad and \quad RMSE(\widehat{\Delta}) = \sqrt{\frac{\sum_{i=1}^{N} (\widehat{\Delta}_i - \Delta)^2}{N}},$$

respectively. From the results in Table 1, the mean values approximate the true parameter values, RMSE and bias decay towards zero for all the parameter values. We therefore, conclude that our model give consistent maximum likelihood estimates (MLEs).

Table 1: Monte Carlo Simulation Results for MO-OPGW-LLoG Distribution: Mean, RMSE and Average Bias

		$\alpha = 1.0, \delta = 0.5, c = 0.05, \beta = 1.1$			$\alpha = 1.0, \delta = 1.0, c = 0.05, \beta = 1.1$		
	n	Mean	RMSE	Bias	Mean	RMSE	Bias
	25	0.9725	0.1064	-0.0275	0.9730	0.1060	-0.0270
	50	0.9911	0.0725	-0.0089	0.9900	0.0710	-0.0100
	100	0.9977	0.0533	-0.0023	0.9982	0.0537	-0.0018
α	200	0.9958	0.0392	-0.0042	0.9977	0.0404	-0.0023
	400	0.9979	0.0266	-0.0021	0.9999	0.0277	-0.0001
	800	0.9974	0.0189	-0.0026	0.9987	0.0196	-0.0013
	1000	0.9983	0.0170	-0.0017	1.0002	0.0180	0.0002
	25	0.9710	0.7674	0.4710	1.7278	1.2141	0.7278
	50	0.7011	0.4071	0.2011	1.3342	0.6890	0.3342
	100	0.6077	0.2476	0.1077	1.1976	0.4619	0.1976
δ	200	0.5631	0.1740	0.0631	1.1147	0.3272	0.1147
	400	0.5272	0.1026	0.0272	1.0490	0.1897	0.0490
	800	0.5190	0.0731	0.0190	1.0342	0.1382	0.0342
	1000	0.5112	0.0634	0.0112	1.0222	0.1218	0.0222
	25	0.3657	4.3221	0.3157	0.5385	4.9922	0.4885
	50	0.0933	0.1053	0.0433	0.0976	0.1158	0.0476
	100	0.0696	0.0472	0.0196	0.0714	0.0525	0.0214
C	200	0.0620	0.0314	0.0120	0.0625	0.0342	0.0125
	400	0.0550	0.0169	0.0050	0.0547	0.0177	0.0047
	800	0.0534	0.0120	0.0034	0.0533	0.0130	0.0033
	1000	0.0522	0.0100	0.0022	0.0519	0.0107	0.0019
	25	1.0665	0.1086	-0.0335	1.0560	0.1210	-0.0440
	50	1.0886	0.0702	-0.0114	1.0838	0.0739	-0.0162
	100	1.0965	0.0502	-0.0035	1.0948	0.0533	-0.0052
β	200	1.0955	0.0366	-0.0045	1.0960	0.0393	-0.0040
	400	1.0978	0.0247	-0.0022	1.0992	0.0264	-0.0008
	800	1.0974	0.0175	-0.0026	1.0983	0.0185	-0.0017
	1000	1.0983	0.0157	-0.0017	1.0998	0.0169	-0.0002

7 Inference

Usefulness of the proposed model is shown in this section. We consider the MO-OPGW-LLoG distribution as an example to demonstrate the flexibility of the new family of distributions. We apply the model to two real data examples and compare it to various distributions. We use R software via the **nlm** package to estimate the model parameters, via the maximum likelihood estimation technique. Various goodness-of-fit statistics are used to assess model performance, that is Cramer-von-Mises (W^*) and Andersen-Darling (A^*), -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (AICC), Bayesian Information Criterion (BIC), Kolmogorov-Smirnov (K-S) statistic (and its p-value), and sum of squares (SS). The model with the smallest values of the goodness-of-fit statistics and a bigger p-value for the K-S statistic is regarded as the best model.

We compare the MO-OPGW-LLoG distribution to several models, namely, Marshall-Olkin-inverse Weibull (MO-IW) by Pakungwati et al. [26], Marshall-Olkin-log-logistic-LLoG (MO-LLoG) by Wenhao [31], Topp-Leone-Marshall-Olkin-Log-logistic (TL-MO-LLoG) and Topp-Leone-Marshall-Olkin-Weibull (TL-MO-W) by Chipepa et al. [7], odd exponentiated half-logistic Burr XII (OEHL-BXII) by Aldahlan and Afify [1] and odd generalized half logistic Weibull-Weibull (OGHLW-W) distribution by Chipepa et al. [6].

The pdfs of the non-nested models are:

$$f_{MO-IW}(x;\alpha,\theta,\lambda)=\frac{\alpha\lambda\theta^{-\lambda}x^{-\lambda-1}e^{-(\theta x)^{-\lambda}}}{[\alpha-(\alpha-1)e^{-(\theta x)^{-\lambda}}]^2},$$

for $\alpha,\theta,\lambda>0$,

$$f_{MO-LLoG}(x;\alpha,\beta,\gamma)=\frac{\alpha^{\beta}\beta\gamma x^{\beta-1}}{(x^{\beta}+\alpha^{\beta}\gamma)^2},$$

for $\alpha,\beta,\gamma>0$,

$$f_{TL-MO-LLo}\ (x;b,\delta,\lambda)=\frac{2b\delta^2\lambda x^{\lambda-1}(1+x^{\lambda})^{-3}}{[1-\bar{\delta}(1+x^{\lambda})^{-1}]^3}[1-\frac{\delta^2[1+x^{\lambda}]^{-2}}{[1-\bar{\delta}(1+x^{\lambda})^{-1}]^2}]^{b-1},$$

respectively, for $b,\delta,\lambda>0$,

$$f_{TL-MO}\quad (x;b,\delta,\lambda,\gamma)=\frac{2b\delta^2\lambda\gamma\ \ ^{\gamma-1}e^{-2\lambda x^{\gamma}}}{(1-\bar{\delta}e^{-\lambda x^{\gamma}})^3}[1-\frac{\delta^2e^{-2\lambda x^{\gamma}}}{(1-\bar{\delta}e^{-\lambda x^{\gamma}})^2}]^{b-1},$$

for $b,\delta,\lambda,\omega>0$,

$$f_{OEHLBXII}(x;\alpha,\lambda,a,b)=\frac{2\alpha\lambda abx^{a-1}\exp(\lambda[1-(1+x^a)^b])(1-\exp(\lambda[1-(1+x^a)^b]))^{\alpha-1}}{(1+x^a)^{-b-1}(1+\exp(\lambda[1-(1+x^a)^b]))^{\alpha+1}}.$$

for $\alpha,\lambda,a,b>0$ and

$$f_{OGHLW-W}(x;\alpha,\beta,\lambda,\gamma)=\frac{2\alpha\beta\lambda\gamma x^{\gamma-1}e^{-\lambda x^{\gamma}}(1-e^{-\lambda x^{\gamma}})^{\beta-1}\exp\{-\alpha[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}]^{\beta}\}}{e^{-(\beta+1)\lambda x^{\gamma}}(1+\exp\{-\alpha[\frac{1-e^{-\lambda x^{\gamma}}}{e^{-\lambda x^{\gamma}}}]^{\beta}\})^2},$$

for $\alpha,\beta,\lambda,\gamma>0$.

Data analyses results are shown in Tables 2, 3, 4 and 5. Histogram of data, fitted densities and probability plots are shown in Figures 5 and 6.

7.1 Data set 1

The first data set represents the number of daily deaths due to COVID-19 in Europe from the 1st of March 2020 to 30th of March 2020 (see https://covid19.who.int/ for details). The observations are: 6, 18, 29, 28, 47, 55, 40, 150, 129, 184, 236, 237, 336, 219, 612, 434, 648, 706, 838, 1129, 1421, 118, 116, 1393, 1540, 1941, 2175, 2278, 2824, 2803, 2667.

Table 2: MLEs and and Standard Errors in parentheses for Data Set 1

Model	α	δ	c	β
MO-OPGW-LLoG	4.4895 (0.0102)	20.6798 (11.2017)	1.3610 (0.0432)	0.0386 (0.0038)
OPGW-LLoG	3.1377 (1.0650×10^{-5})	1 -	2.4574 (1.3598×10^{-5})	0.0172 (0.0019)
MO-IW	α	λ	θ	-
	34.0364 (69.9333)	0.9718 (0.1634)	0.1159 (0.1983)	- -
	α	β	γ	-
MO-LLoG	20.4725 (4.3538)	1.0060 (0.1469)	16.7061 (5.3034)	- -
TL-MO-LLoG	B	δ	λ	γ
	0.2345 (0.0448)	3.2354×10^{7} (2.3242×10^{-10})	2.2040 (0.0635)	- -
TL-MO-W	0.4220 (0.0709)	2.4312 (0.0019)	0.001 (3.4223×10^{-5})	1.1997 (0.0304)
OEHL-BXII	α	λ	a	b
	0.3397 (0.0821)	1.2340×10^{-4} (1.8432×10^{-4})	16.1790 (2.9341×10^{-4})	0.0766 (0.0115)
OGHLW-W	α	β	λ	γ
	2.9675×10^{-5} (2.2273×10^{-5})	0.0977 (0.0024)	74.8450 (3.1052×10^{-6})	0.0592 (0.0056)

Table 3: Goodness-of-fit Statistics for Data Set 1

Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	**K-S**	**p-value**
MO-OPGW-LLoG	474.3	482.3	483.8	488.0	0.0707	0.4898	0.1032	0.8629
OPGW-LLoG	507.0	513.0	513.9	517.3	0.0744	0.5408	0.3611	0.0004
MO-IW	476.5	482.5	483.4	486.9	0.0777	0.5563	0.1093	0.8146
MO-LLoG	476.9	482.9	483.8	487.2	0.0791	0.5648	0.1052	0.8474
TL-MO-LLoG	473.2	479.2	480.1	483.5	0.0867	0.5635	0.1402	0.5301
TL-MO-W	471.9	479.9	481.4	485.6	0.0856	0.5508	0.1467	0.4729
OEHL-BXII	480.6	488.6	490.1	494.3	0.0811	0.5240	0.1120	0.7908
OGHLW-W	473.7	481.7	483.3	487.5	0.0851	0.5605	0.1351	0.5768

The estimated variance-covariance matrix is

$$\begin{bmatrix} 1.0377 \times 10^{-4} & -0.1141 & 0.0004 & -3.5296 \times 10^{-5} \\ -0.1141 & 125.4775 & -0.4837 & 0.0388 \\ 4.3991 \times 10^{-4} & -0.4837 & 0.0018 & -0.0001 \\ -3.5296 \times 10^{-5} & 0.0388 & -0.0001 & 1.4495 \times 10^{-5} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\alpha \in [4.4895 \pm 0.0199]$, $\delta \in [20.6798 \pm 21.9553]$, $c \in [1.3610 \pm 0.0846]$ and $\beta \in [0.0386 \pm 0.0075]$.

Based on the results shown in Table 2, we conclude that the MO-OPGW-LLoG distribution performs better than the non-nested models considered on COVID-19 daily deaths in Europe since it has the lowest values for the goodness-of-fit statistics $-2\log L$, AIC, $AICC$, BIC, A^*, W^* and K-S (and the largest p-value for the K-S statistic). Figure 5 shows the flexibility enjoyed when using the MO-OPGW-LLoG distribution in fitting the COVID-19 deaths data set compared to the selected non nested models.

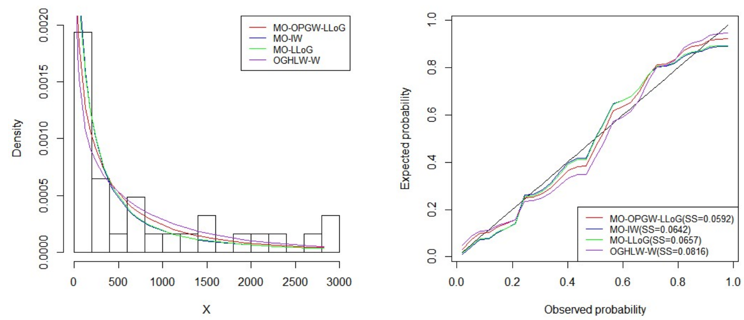

Figure 5: Fitted pdfs and probability plots for COVID-19 daily deaths in Europe

7.2 Data set 2

The second data set represents COVID-19 deaths in China for the period 23 January 2020 to 28 March 2020 (see https://www.worldometers.info/coronavirus/country/china for details). The observations are: 8, 16, 15, 24, 26, 26, 38, 43, 46, 45, 57, 64, 65, 73, 73, 86, 89, 97, 108, 97, 146, 121, 143, 142, 105, 98, 136, 114, 118, 109, 97, 150, 71, 52, 29, 44, 47, 35, 42, 31, 38, 31, 30, 28, 27, 22, 17, 22, 11, 7, 13, 10, 14, 13, 11, 8, 3, 7, 6, 9, 7, 4, 6, 5, 3, 5.

The estimated variance-covariance matrix is

$$\begin{bmatrix} 3.7134\times 10^{-5} & -0.0322 & 6.6624\times 10^{-5} & -1.0133\times 10^{-5} \\ -0.0322 & 28.0095 & -0.0578 & 0.0088 \\ 6.6624\times 10^{-5} & -0.0578 & 0.0001 & -1.8183\times 10^{-5} \\ -1.0133\times 10^{-5} & 0.0088 & -1.8183\times 10^{-5} & 4.0021\times 10^{-6} \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by

$\alpha \in [4.1732 \pm 0.0119]$, $\delta \in [14.7160 \pm 10.3731]$, $c \in [2.4710 \pm 0.0214]$ and $\beta \in [0.0361 \pm 0.0039]$.

Table 4: MLEs and and Standard Errors in parentheses for Data Set 2

Model	α	δ	c	β
MO-OPGW-LLoG	4.1732 (0.0061)	14.7160 (5.2924)	2.4710 (0.0109)	0.0361 (0.0020)
OPGW-LLoG	4.5538 (4.9521×10^{-6})	1 -	2.8264 (7.9788×10^{-6})	0.0171 (0.0013)
MO-IW	α	λ	θ	-

	15.8531 (17.7468)	1.4275 (0.1759)	0.2270 (0.1314)	- -
MO-LLoG	α	β	γ	-
	7.5970 (1.1972)	1.5154 (0.1517)	8.8164 (0.6808)	- -
TL-MO-LLoG	B	δ	λ	γ
	0.2644 (0.0349)	1.8210×10^{6} (7.2990×10^{-9})	2.9896 (0.0699)	- -
TL-MO-W	0.8102 (1.0422)	1.1618 (1.0716)	0.0038 (0.0204)	1.2167 (1.0248)
OEHL-BXII	α	λ	a	b
	0.2786 (0.0216)	2.4954×10^{-5} (9.6561×10^{-6})	3.5466 (4.0306×10^{-3})	0.6525 (0.0236)
OGHLW-W	α	β	λ	γ
.	2.7256×10^{-5} (1.1679×10^{-5})	0.1954 (0.0022)	39.4860 (1.0859×10^{-5})	0.0864 (0.0058)

Table 5: Goodness-of-fit Statistics for Data Set 2

Model	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	**K-S**	**p-value**
MO-OPGW-LLoG	657.0	657.7	665.8	0.1051	0.8076	0.0843	0.7361	657.0
OPGW-LLoG	706.6	712.6	713.0	719.1	0.1475	1.0037	0.3070	7.9170×10^{-6}
MO-IW	653.6	659.6	659.9	666.1	0.1567	1.0832	0.0937	0.6083
MO-LLoG	655.1	661.1	661.5	667.7	0.1486	1.0763	0.0876	0.6922
TL-MO-LLoG	449.7	655.7	656.1	662.3	0.1473	1.0054	0.1014	0.5054
TL-MO-W	646.9	654.9	655.6	663.7	0.1082	0.7941	0.0894	0.6668
OEHL-BXII	661.6	669.6	670.2	678.3	0.1536	0.9967	0.1218	0.2815
OGHLW-W	650.1	658.1	658.8	666.9	0.1419	0.9918	0.0871	0.6983

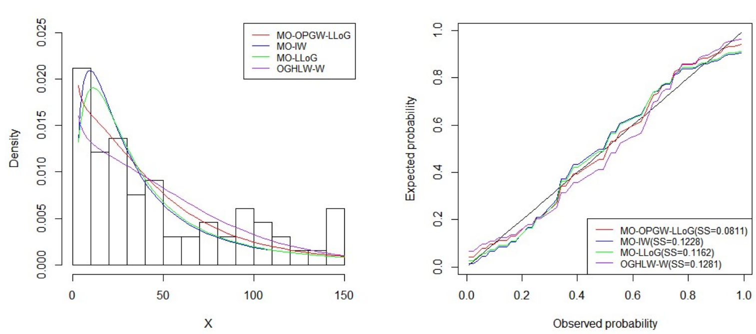

Figure 6: Fitted pdfs and probability plots for COVID-19 daily deaths in China

Based on the results shown in Table 2, we further conclude that the MO-OPGW-LLoG distribution performs better than the non-nested models since it has the lowest values for the goodness-of-fit statistics $-2\log L$, AIC, $AICC$, BIC, A^*, W^* and K-S (and the largest p-value for the K-S statistic).

8 Concluding Remarks

A new family of distributions was developed. The proposed distribution is an infinite linear combination of Exp-G distribution. Several statistical properties of the new family were also derived. A stimulation study to examine the consistency of the maximum likelihood estimates was conducted. A special case of the proposed family (MO-OPGW-LLoG distribution) was applied to two real data examples and compared to a variety of non nested models. The proposed family is versatile and flexible in data modeling as indicated in data analysis results presented in Section 7.

Appendix

The link provided contains derivations for the statistical properties:
https://drive.google.com/file/d/1WZtpVhpJ3J0-oWTCgiDQD_wUyMw6t8y2/view?usp=sharing

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