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The Marshall-Olkin-Type II-Topp-Leone-G Family of Distributions: Model, Properties and Applications

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ABSTRACT
We develop a new family of distributions called the Marshall-Olkin-Type II-Topp-Leone-G (MO-TII-TL-G) distribution, which is an infinite linear combination of the exponential-G family of distributions. The statistical properties of the new distributions are studied and its model parameters are estimated using the maximum likelihood method. A simulation study is carried out to determine the performance of the maximum likelihood estimates and lastly, real data examples are provided to demonstrate the usefulness of the proposed model in comparison to several other models.

Keywords: Marshall-Olkin-G, Type II-Topp Leone, Maximum Likelihood Estimation.

Mathematics Subject Classifications: 62E99; 60E05

1 Introduction

Several techniques have been proposed to generate new families of distributions. For instance, the Marshall-Olkin-G (MO-G) distribution developed by Marshall and Olkin [18]. It is one of the techniques available in the statistical literature that models lifetime data and generalizes known distributions. The (MO-G) distribution is flexible in comparison to other distributions like the

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exponential, Weibull and gamma and it applies to data that has both monotonic and non-monotonic hazard rates. It plays an important role in reliability analysis in many areas such as engineering, economics, biology and hydrology. This MO-G distribution has cumulative distribution function (cdf) and probability density function (pdf) given by

$$F_{MO-G}(x;\delta,\xi) = 1 - \frac{\delta\bar{G}(x;\xi)}{1-\bar{\delta}\bar{G}(x;\xi)} \tag{1}$$

and

$$f_{MO-G}(x;\delta,\xi) = \frac{\delta g(x;\xi)}{[1-\bar{\delta}\bar{G}(x;\xi)]^2}, \tag{2}$$

respectively, where $\delta > 0$ is the tilt parameter and $\bar{G}(x;\xi) = 1 - G(x;\xi)$ is the survival function of the baseline distribution.

Generalizations of the Marshall-Olkin distribution include Kumaraswamy Marshall-Olkin-G by Alizadeh et al. [5], Beta Marshall-Olkin-G by Alizadeh et al. [4], Marshall-Olkin Log-logistic Extended Weibull by Lepetu et al. [16], Marshall-Olkin-Extended Burr Type III distribution by Kumar et al. [15], Marshall-Olkin Log-logistic Erlang-Truncated Exponential by Oluyede et al. [22] and Marshall-Olkin-Gompertz-G by Chipepa and Oluyede [12] to name a few.

Elgarhy et al. [13] proposed the Type II Topp-Leone (TII-TL-G) generated family of distributions with cdf and pdf, respectively, specified by

$$F_{TII-TL-G}(x;b,\xi) = 1 - [1 - G^2(x;\xi)]^b \tag{3}$$

and

$$f_{TII-TL-G}(x;b,\xi) = 2bg(x;\xi)G(x;\xi)[1 - G^2(x;\xi)]^{b-1},$$

where $b > 0$ and $G(x;\xi)$ is the cdf of the baseline distribution. Some notable generalizations of the TII-TL-G distribution include the Type II Topp-Leone inverse Rayleigh distribution by Mohammed and Yahia [19], Type II Topp-Leone generalized Rayleigh by Yahia and Mohammed [28], Type II Topp-Leone inverted Kumaraswamy by ZeinEldin et al. [29], Type II Topp-Leone inverse Exponential by Al-Marzouki [1], Type II Topp-Leone power Lomax by Al-Marzouki et al. [2] and Type II Topp-Leone Dagum by Sakthivel and Dhivakar [24].

In this note, we are motivated by the interesting properties of the MO-G and TII-TL-G distributions to develop a new family of distributions which is a combination of these two distributions. We develop this new family of distributions which is flexible because it can applied to data sets of varying skewness and kurtosis. Also, it can model different types of hazard rate functions including monotonic as well as non-monotonic shapes. We hope the new distribution will receive much attention from statisticians.

In this paper, we develop and study the new family of distributions, the MO-TII-TL-G family of distributions. In Section 2, we develop the new family of distributions and provide its density expansion. Some special cases of the MO-TII-TL-G family of distributions are presented in Section 3. In section 4, we present some of the statistical properties of the proposed distrbution. Section 5 contains the maximum likelihood estimates of the model parameters. Simulation study results are given in Section 6. Applications of the proposed model to real data examples are presented in Section 7, followed by concluding remarks.

2 Marshall-Olkin-Type II-Topp-Leone-G Family of Distributions

We derive the new MO-TII-TL-G family of distributions by using the generalizations in equations (1) and (3). The cdf, pdf and hazard rate function (hrf) of the MO-TII-TL-G family of distributions are given by

$$F_{MO-TII-TL-G}(x;\delta,b,\xi)=1-\frac{\delta[1-G^2(x;\xi)]^b}{1-\bar{\delta}[1-G^2(x;\xi)]^b},$$

$$f_{MO-TII-TL-G}(x;\delta,b,\xi)=\frac{2\delta b g(x;\xi)G(x;\xi)[1-G^2(x;\xi)]^{b-1}}{(1-\bar{\delta}[1-G^2(x;\xi)]^b)^2} \quad (4)$$

and

$$h_{MO-TII-TL-G}(x;\delta,b,\xi)=\frac{2\delta b g(x;\xi)G(x;\xi)[1-G^2(x;\xi)]^{b-1}}{(1-\bar{\delta}[1-G^2(x;\xi)]^b)(\delta[1-G^2(x;\xi)]^b)},$$

respectively, for b, $\delta>0$, $\bar{\delta}=1-\delta$ and ξ is a vector of parameters from the baseline distribution function G(.).

2.1 Expansion of Density Function

The series expansion of the MO-TII-TL-G family of distributions is derived by making use of the general results of the Marshall and Olkin's family of distributions by Barreto-Souza et al. [8]. The pdf of the MO-TII-TL-G given by

$$f_{MO-TII-TL-G}(x;\delta,b,\xi)=\frac{\delta f_{TII-TL-G}(x;b,\xi)}{(1-\bar{\delta}\bar{F}_{TII-TL-G}(x;b,\xi))^2},$$

can be written as

$$f_{MO-TII-TL-}\ (x;\delta,b,\xi)=\frac{f_{TII-TL-G}(x;b,\xi)}{\delta(1-\frac{\delta-1}{\delta}F_{TII-TL-G}(x;b,\xi))^2},$$

where $f_{TII-TL-G}$ and $F_{TII-TL-G}$ are the pdf and cdf of the TII-TL-G family of distributions, respectively. We also make use of the series expansion

$$(1-z)^{-k}=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)}{\Gamma(k)j!}z^j, \quad (5)$$

which is valid for $|z|<1$, $k>0$. If $\delta\in(0,1)$ we obtain

$$f_{MO-TII-TL-G}(x;\delta,b,\xi)=f_{TII-TL-G}(x;b,\xi)\sum_{j=0}^{\infty}\sum_{k=0}^{j}w_{j,k}F_{TII-TL-G}(x;b,\xi)^{j-k},$$

where $w_{j,k}=w_{j,k}(\delta)=\delta(j+1)(1-\delta)^j(-1)^{j-k}\binom{j}{k}$. For $\delta>1$, we have

$$f_{MO-TII-TL-G}(x;\delta,b,\xi) = f_{TII-TL-G}(x;b,\xi)\sum_{j=0}^{\infty} v_j F_{TII-TL-G}^{j}(x;b,\xi),$$

where $v_j = v_j(\delta) = \frac{(j+1)(1-\frac{1}{\delta})}{\delta}$.

For $\delta \in (0,1)$, equation (4) becomes

$$\begin{aligned} f_{MO-TII-TL-G}(x;\delta,b,\xi) &= 2bg(x;\xi)G(x;\xi)[1-G^2(x;\xi)]^{b-1} \\ &\times \sum_{j=0}^{\infty}\sum_{k=0}^{j} w_{j,k}(1-[1-G^2(x;\xi)]^{b})^{j-k}. \end{aligned}$$

By applying the generalized binomial series expansions

$$(1-[1-G^2(x;\xi)]^{b})^{j-k} = \sum_{l=0}^{\infty} (-1)^l \binom{j-k}{l} [1-G^2(x;\xi)]^{bl}$$

and

$$[1-G^2(x;\xi)]^{b(l+1)-1} = \sum_{m=0}^{\infty} (-1)^m \binom{b(l+1)-1}{m} G^{2m}(x;\xi),$$

we can write

$$\begin{aligned} f_{MO-TII-TL-G}(x;\delta,b,\xi) &= \sum_{j,l,m=0}^{\infty}\sum_{k=0}^{j} \frac{2b(-1)^{l+m}w_{j,k}}{2m+2}\binom{j-k}{l}\binom{b(l+1)-1}{m} \\ &\times (2m+2)g(x;\xi)G^{2m+1}(x;\xi) \\ &= \sum_{m=0}^{\infty} w_m^* g_m(x;\xi). \end{aligned} \tag{6}$$

It then follows that for $\delta \in (0,1)$, the MO-TII-TL-G family of distributions can be expressed as an infinite linear combination of the Exponentiated-G (Exp-G) distribution with power parameter $(2m+2)$ and linear component

$$w_m^* = \sum_{j,l}^{\infty}\sum_{k=0}^{j} \frac{2b(-1)^{l+m}w_{j,k}}{2m+2}\binom{j-k}{l}\binom{b(l+1)-1}{m}. \tag{7}$$

Furthermore, for $\delta > 1$, equation (4) can be written as

$$\begin{aligned} f_{MO-TII-TL-G}(x;\delta,b,\xi) &= 2bg(x;\xi)G(x;\xi)[1-G^2(x;\xi)]^{b-1} \\ &\times \sum_{k=0}^{j} v_j(1-[1-G^2(x;\xi)]^{b-1})^{j}. \end{aligned}$$

Applying the series expansions

$$(1-[1-G^2(x;\xi)]^{b-1})^{j} = \sum_{l=0}^{\infty} (-1)^l \binom{j}{l} [1-G^2(x;\xi)]^{bl}$$

and

$$[1-G^2(x;\xi)]^{b(l+1)-1} = \sum_{m=0}^{\infty} (-1)^m \binom{b(l+1)-1}{m} G^{2m}(x;\xi),$$

we get

$$\begin{aligned} f_{MO-TII-TL-}(x;\delta,b,\xi) &= \sum_{j,l,m=0}^{\infty} \frac{2b(-1)^{l+m}v_j}{2m+2}\binom{j}{l}\binom{b(l+1)-1}{m} \\ &\times (2m+2)g(x;\xi)G^{2m+1}(x;\xi) \end{aligned}$$

$$= \sum_{m=0}^{\infty} v_m^* g_m(x;\xi). \quad (8)$$

Therefore, for $\delta > 1$ the MO-TII-TL-G family of distributions can be expressed as an infinite linear combination of the Exponentiated-G (Exp-G) distribution with power parameter $(2m+2)$ and linear component

$$v_m^* = \sum_{j,l}^{\infty} \frac{2b(-1)^{l+m} v_j}{2m+2} \binom{j}{l} \binom{b(l+1)-1}{m}. \quad (9)$$

3 Some Special Cases

In this Section, we present some special cases of the MO-TII-TL-G family of distributions. We considered cases when the baseline distributions are Weibull, uniform and log-logistic distributions.

3.1 Marshall-Olkin-Type II-Topp-Leone-Weibull (MO-TII-TL-W) Distribution

Consider the Weibull distribution as the baseline distribution with pdf and cdf given by $g(x;\lambda) = \lambda x^{\lambda-1} e^{-x^{\lambda}}$ and $G(x;\lambda) = 1 - e^{-x^{\lambda}}$, respectively, for $\lambda > 0$. The cdf, pdf and hrf of the MO-TII-TL-W distribution are given by

$$F_{MO-TII-TL-W}(x;\delta,b,\lambda) = 1 - \frac{\delta[1-(1-e^{-x^{\lambda}})^2]^b}{1-\bar{\delta}[1-(1-e^{-x^{\lambda}})^2]^b},$$

$$f_{MO-TII-TL-}(x;\delta,b,\lambda) = \frac{2\delta b\lambda x^{\lambda-1} e^{-x^{\lambda}}(1-e^{-x^{\lambda}})[1-(1-e^{-x^{\lambda}})^2]^{b-1}}{(1-\bar{\delta}[1-(1-e^{-x^{\lambda}})^2]^b)^2}$$

and

$$h_{MO-TII-TL-}(x;\delta,b,\lambda) = \frac{2\delta b\lambda x^{\lambda-1} e^{-x^{\lambda}}(1-e^{-x^{\lambda}})[1-(1-e^{-x^{\lambda}})^2]^{b-1}}{(1-\bar{\delta}[1-(1-e^{-x^{\lambda}})^2]^b)(\delta[1-(1-e^{-x^{\lambda}})^2]^b)},$$

respectively, for δ, b, $\lambda > 0$.

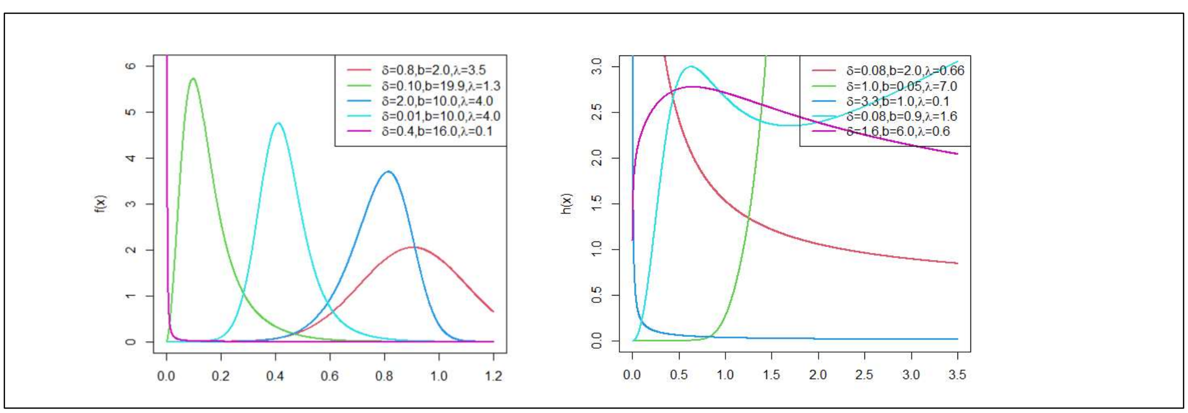

Figure 1 Pdf and hrf plots for the MO-TII-TL-W distribution
Figure 1 shows the plots of the pdfs and hrfs for the MO-TII-TL-W distribution. The pdf exhibit different shapes including right or left-skewed, reverse-J and unimodal. The hrf exhibit reverse-J, increasing, decreasing, upside-down bathtub and upside-down bathtub followed by bathtub shapes.

3.2 Marshall-Olkin-Type II-Topp-Leone-Log-Logistic (MO-TII-TL-LLoG) Distribution

Suppose that the Log-logistic distribution is the baseline distribution with pdf and cdf given $g(x;c,k) = cx^{c-1}(1+x^c)^{-2}$ and $G(x;c) = 1-(1+x^c)^{-1}$, respectively, for c $>$ 0. Therefore, the MO-TII-TL-LLoG distribution have cdf, pdf and hrf given by

$$F_{MO-TII-TL-LLo}\ (x;\delta,b,c) = 1 - \frac{\delta[1-(1-(1+x^c)^{-1})^2]^b}{1-\bar{\delta}[1-(1-(1+x^c)^{-1})^2]^b},$$

$$f_{MO-TII-TL-L}\quad (x;\delta,b,c) = \frac{2\delta bcx^{c-1}(1+x^c)^{-2}(1-(1+x^c)^{-1})}{(1-\bar{\delta}[1-(1-(1+x^c)^{-1})^2]^b)^2} \times [1-(1-(1+x^c)^{-1})^2]^{b-1}$$

and

$$h_{MO-TII-TL-LLo}\ (x;\delta,b,c) = \frac{2\delta bcx^{c-1}(1+x^c)^{-2}(1-(1+x^c)^{-1})}{(1-\bar{\delta}[1-(1-(1+x^c)^{-1})^2]^b)} \times \frac{[1-(1-(1+x^c)^{-1})^2]^{b-1}}{(\delta[1-(1-(1+x^c)^{-1})^2]^b)},$$

respectively, for δ, b, c $>$ 0. Plots for the MO-TII-TL-LLoG pdf shows that the distribution can take various shapes that include: reverse-J, J, almost symmetric and left or right-skewed. The hazard rate function exhibits both monotonic and non-monotonic hazards rate shapes.

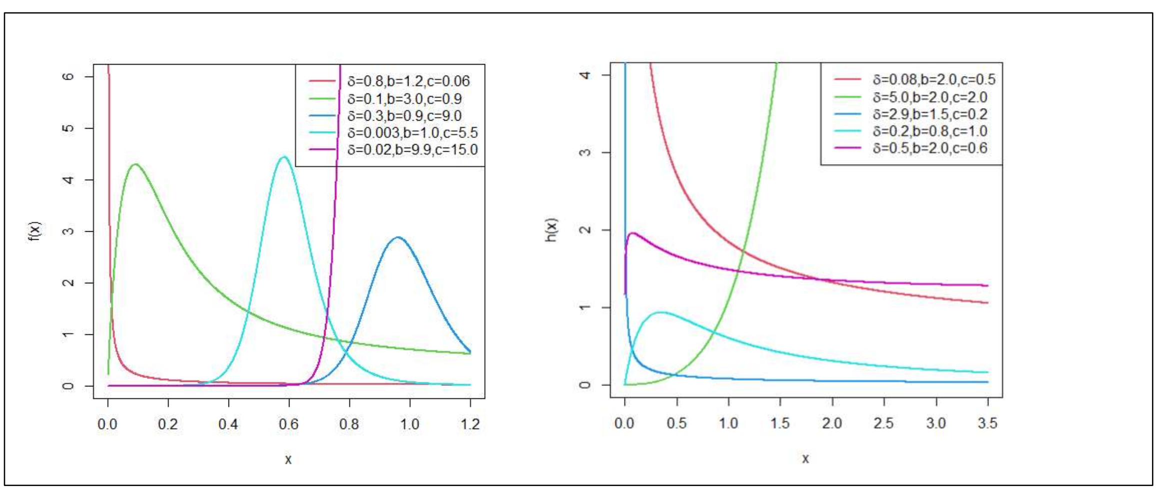

Figure 2 Pdf and hrf plots for the MO-TII-TL-LLoG distribution

3.3 Marshall-Olkin-Type II-Topp-Leone-Uniform (MO-TII-TL-U) Distribution

Consider the Uniform distribution as the baseline distribution with pdf and cdf given by $g(x;\theta) = 1/\theta$ and $G(x;\theta) = x/\theta$, respectively, for $0 < x < \theta$. Therefore, the MO-TII-TL-U distribution have cdf, pdf and hrf given by

$$F_{MO-TII-TL-}\ (x;\delta,b,\theta) = 1 - \frac{\delta[1-(x/\theta)^2]^b}{1-\bar{\delta}[1-(x/\theta)^2]^b},$$

$$f_{MO-TII-TL-}\ (x;\delta,b,\theta) = \frac{2\delta\ (1/\theta)(x/\theta)[1-(x/\theta)^2]^{b-1}}{(1-\bar{\delta}[1-(x/\theta)^2]^b)^2}$$

and

$$h_{MO-TII-TL-}\ (x;\delta,b,\theta) = \frac{2\delta\ (1/\theta)(x/\theta)[1-(x/\theta)^2]^{b-1}}{(1-\bar{\delta}[1-(x/\theta)^2]^b)(\delta[1-(x/\theta)^2]^b)}\ ,$$

respectively, for δ, b, $\lambda > 0$. Figures 3 shows that the pdf of the MO-TII-TL-U can take various shapes that includes right or left-skewed, J and unimodal. The hrf shows J, increasing, upside-down bathtub and upside-down bathtub followed by bathtub shapes.

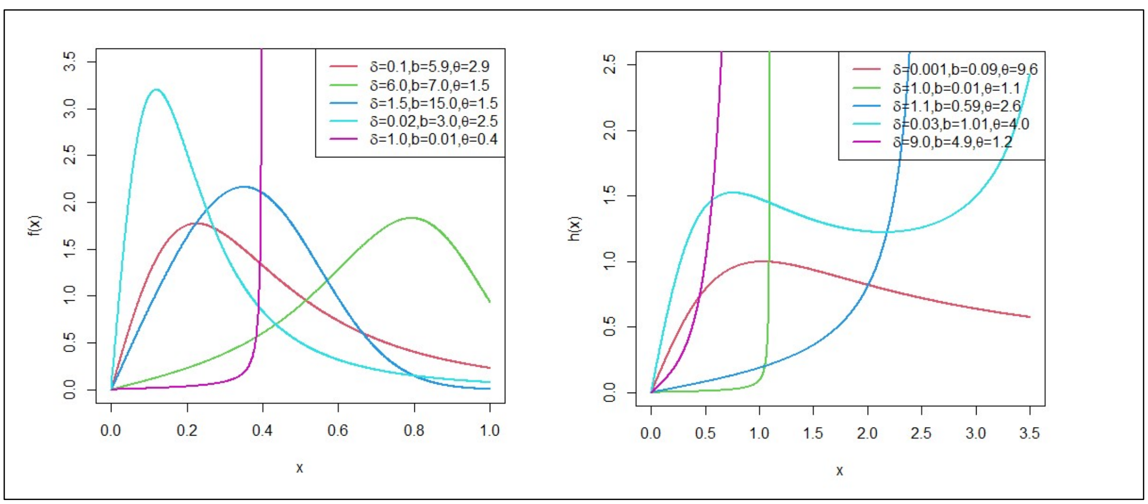

Figure 3 Pdf and hrf plots for the MO-TII-TL-U distribution

4 Statistical Properties

The distribution of the i^{th} order statistics, Rényi entropy, moments and the quantile function of the MO-TII-TL-G family of distributions are presented in this section.

4.1 Distribution of Order Statistics

Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed (iid) random variables from the MO-TII-TL-G family of distributions. The pdf of the i^{th} order statistic X $_{i:n}$, is given by

$$f_{i:n}(x;\delta,b,\xi) = \delta n!\, f_{TII-TL-G}(x;b,\xi) \sum_{k=0}^{n-i} \frac{(-1)^k}{(i-1)!(n-1)!} \times \frac{F_{TII-TL-G}^{k+i-1}(x;b,\xi)}{[1-\bar{\delta}F_{TII-TL-G}(x;b,\xi)]^{k+i-1}}.$$

If $\delta \in (0,1)$, we have

$$f_{i:n}(x;\delta,b,\xi) = f_{TII-TL-G}(x;b,\xi) \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^{j} B_{j,k,l} F_{TII-TL-G}^{j+k-l+i-1}(x;b,\xi), \qquad (10)$$

where $B_{j,k,l} = B_{j,k,l}(\delta) = \delta n!\,(-1)^{j+k-1}(1-\delta)^j \binom{j}{k}\binom{k+i+j}{j}$.

Substituting the pdf and cdf of the TII-TL-G distribution into equation (10) we get

$$f_{i:n}(x;\delta,b,\xi) = 2bg(x;\xi)G(x;\xi)[1-G^2(x;\xi)]^{b-1} \times \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^{j} B_{j,k,l}(1-[1-G^2(x;\xi)]^b)^{j+k-l+i-1}.$$

Using the series of expansions,

$$(1-[1-G^2(x;\xi)]^b)^{j+k-l+i-1} = \sum_{m=0}^{\infty} (-1)^m \binom{j+k-l+i-1}{m} [1-G^2(x;\xi)]^{bm}$$

and

$$[1-G^2(x;\xi)]^{b(m+1)-1} = \sum_{p=0}^{\infty} (-1)^p \binom{b(m+1)-1}{p} G^{2p}(x;\xi),$$

we get

$$\begin{aligned} f_{i:n}(x;\delta,b,\xi) &= \sum_{j,m,p=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^{j} \frac{2b(-1)^{m+p}B_{j,k,l}}{2p+2} \binom{j+k-l+i-1}{m} \\ &\quad \times \binom{b(m+1)-1}{p} (2p+2)g(x;\xi)G^{2p+1}(x;\xi) \\ &= \sum_{p=0}^{\infty} B_p^* g_p(x;\xi), \end{aligned}$$

where

$$B_p^* = \sum_{j,m=0}^{\infty} \sum_{k=0}^{n-i} \sum_{l=0}^{j} \frac{2b(-1)^{m+p}B_{j,k,l}}{2p+2} \binom{j+k-l+i-1}{m} \binom{b(m+1)-1}{p}$$

and $g_p(x;\xi) = (2p+2)g(x;\xi)G^{2p+1}(x;\xi)$ is an Exp-G distribution with power parameter $(2p+2)$. Therefore, the distribution of the order statistics of the MO-TII-TL-G family of distributions can be obtained from those of the Exp-G distribution with parameter $(2p+2)$.

For $\delta > 1$, we write $1-\bar{\delta}F_{TII-TL-G}(x;b,\xi) = \delta[1-\frac{(\delta-1)F_{TII-TL-G}(x;b,\xi)}{\delta}]$, so that

$$f_{i:n}(x;\delta,b,\xi) = f_{TII-TL-G}(x;b,\xi) \sum_{j=0}^{\infty} \sum_{k=0}^{n-i} U_{j,k} F_{TII-TL-G}^{j+k+i-1}(x;b,\xi),$$

where $U_{j,k} = U_{j,k}(\delta) = \frac{(-1)^k(\delta-1)^j n!}{\delta^{k+j+i}(i-1)!(n-i)!}\binom{k+i+j}{j}$.

$$f_{i:n}(x;\delta,b,\xi) = 2bg(x;\xi)G(x;\xi)[1-G^2(x;\xi)]^{b-1} \times \sum_{j=0}^{\infty}\sum_{k=0}^{n-i} U_{j,k}(1-[1-G^2(x;\xi)]^b)^{j+k+i-1}.$$

Applying the series of expansions,

$$(1-[1-G^2(x;\xi)]^b)^{j+k+i-1} = \sum_{m=0}^{\infty}(-1)^m\binom{j+k+i-1}{m}[1-G^2(x;\xi)]^{bm}$$

and

$$[1-G^2(x;\xi)]^{b(m+1)-1} = \sum_{p=0}^{\infty}(-1)^p\binom{b(m+1)-1}{p}G^{2p}(x;\xi),$$

we obtain

$$\begin{aligned} f_{i:n}(x;\delta,b,\xi) &= \sum_{j,m,p=0}^{\infty}\sum_{k=0}^{n-i}\frac{2b(-1)^{m+p}U_{j,k}}{2p+2}\binom{j+k+i-1}{m} \\ &\times \binom{b(m+1)-1}{p}(2p+2)g(x;\xi)G^{2p+1}(x;\xi) \\ &= \sum_{p=0}^{\infty}U_p^* g_p(x;\xi), \end{aligned}$$

where

$$U_p^* = \sum_{j,m=0}^{\infty}\sum_{k=0}^{n-i}\frac{2b(-1)^{m+p}U_{j,k}}{2p+2}\binom{j+k+i-1}{m}\binom{b(m+1)-1}{p}$$

and $g_p(x;\xi) = (2p+2)g(x;\xi)G^{2p+1}(x;\xi)$ is an Exp-G distribution with power parameter $(2p+2)$.

Also, for $\delta > 1$, the distribution of the order statistics of MO-TII-TL-G family of distributions can be obtained from those of the Exp-G since the distribution of the i^{th} order statistic is an infinite linear combination of Exp-G densities with parameter $(2p+2)$.

4.2 Entropy

Entropy measures variation of uncertainty of a random variable. Rényi entropy [23] is a generalization of Shannon entropy [26]. Rényi entropy is defined to be

$$I_R(v) = \frac{1}{1-v}\log\left(\int_0^{\infty} f_{MO-TII-TL-}^{v}(x;\delta,b,\xi)dx\right),$$

where $v > 0$, and $v \neq 1$. Using expansion (5), for $\delta \in (0,1)$

$$f_{MO-TII-TL-}^{v}(x;\delta,b,\xi) = \frac{\delta^v f_{TII-TL-G}^{v}(x;b,\xi)}{\Gamma(2v)}\sum_{j=0}^{\infty}(1-\delta)^j\Gamma(2v+j) \times \frac{[1-F_{TII-TL-G}(x;b,\xi)]^j}{j!}$$

and for $\delta > 1$

$$f^{v}(x;\delta,b,\xi) = \frac{\delta^{v} f^{v}_{TII-TL-G}(x;b,\xi)}{\delta^{v}\Gamma(2v)} \sum_{j=0}^{\infty} (\delta-1)^{j}\Gamma(2v+j) \times \frac{[F_{TII-TL-G}(x;b,\xi)]^{j}}{j!}$$

Therefore, the Rényi entropy for $\delta \in (0,1)$ is

$$I_R(v) = \frac{1}{1-v}\log\left(\sum_{j=0}^{\infty} e_j \int_0^{\infty} f^{v}_{TII-TL-G}(x;b,\xi)[1-F_{TII-TL-G}(x;b,\xi)]^{j}dx\right),$$

where $e_j = e_j(\delta) = \frac{\delta^{v}(1-\delta)^{j}\Gamma(2v+j)}{\Gamma(2v)j!}$.

Thus

$$I_R(v) = \frac{1}{1-v}\log\left(\sum_{j=0}^{\infty} e_j \int_0^{\infty} (2b)^{v} g^{v}(x;\xi)G^{v}(x;\xi)[1-G^{2}(x;\xi)]^{v(b-1)} \times [1-G^{2}(x;\xi)]^{bj}dx\right).$$

Using the generalized binomial expansion

$$[1-G^{2}(x;\xi)]^{b(j+v)-v} = \sum_{k=0}^{\infty} (-1)^{k}\binom{b(j+v)-v}{k} G^{2k}(x;\xi),$$

we can write

$$\begin{aligned} I_R(v) &= \frac{1}{1-v}\log\Bigg(\sum_{j,k=0}^{\infty} e_j \frac{(2b)^{v}(-1)^{k}}{(\frac{2k+v}{v}+1)^{v}}\binom{b(j+v)-v}{k} \\ &\quad \times \int_0^{\infty} [(\frac{2k+v}{v}+1)g(x;\xi)G^{\frac{2k+v}{v}}(x;\xi)]^{v}dx\Bigg) \\ &= \frac{1}{1-v}\log\left(\sum_{k=0}^{\infty} e_k^{*} e^{(1-v)I_{REG}}\right), \end{aligned}$$

where

$$e_k^{*} = \sum_{j=0}^{\infty} e_j \frac{(2b)^{v}(-1)^{k}}{(\frac{2k+v}{v}+1)^{v}}\binom{b(j+v)-v}{k}$$

and

$$I_{REG} = \frac{1}{1-v}\log\left(\int_0^{\infty} [(\frac{2k+v}{v}+1)g(x;\xi)G^{\frac{2k+v}{v}}(x;\xi)]^{v}dx\right)$$

is the Rényi entropy of the Exp-G distribution with power parameter $\frac{2k+v}{v}+1$.

For $\delta > 1$,

$$I_R(v) = \frac{1}{1-v}\log\left(\sum_{j=0}^{\infty} h_j \int_0^{\infty} f^{v}_{TII-TL-G}(x;b,\xi)F^{j}_{TII-TL-G}(x;b,\xi)dx\right),$$

where $h_j = h_j(\delta) = \frac{(\delta-1)^{j}\Gamma(2v+j)}{\delta^{v+j}\Gamma(2v)j!}$.

Now

$$I_R(v) = \frac{1}{1-v}\log(\sum_{j=0}^{\infty} h_j \int_0^{\infty} (2b)^v g^v(x;\xi)G^v(x;\xi)[1-G^2(x;\xi)]^{v(b-1)} \times (1-[1-G^2(x;\xi)]^b)^j dx).$$

Applying the generalized binomial expansions

$$(1-[1-G^2(x;\xi)]^b)^j = \sum_{k=0}^{\infty} (-1)^k \binom{j}{k} [1-G^2(x;\xi)]^{bk}$$

and

$$[1-G^2(x;\xi)]^{b(k+v)-v} = \sum_{l=0}^{\infty} (-1)^l \binom{b(k+v)-v}{l} G^{2l}(x;\xi),$$

we get

$$I_R(v) = \frac{1}{1-v}\log(\sum_{j,k,l=0}^{\infty} h_j \frac{(2b)^v(-1)^{k+l}}{(\frac{2k+v}{v}+1)^v}\binom{j}{k}\binom{b(k+v)-v}{l} \times \int_0^{\infty} [(\frac{2l+v}{v}+1)g(x;\xi)G^{\frac{2l+v}{v}}(x;\xi)]^v dx)$$
$$= \frac{1}{1-v}\log(\sum_{l=0}^{\infty} h_l^* e^{(1-v)I_{REG}}),$$

where

$$h_l^* = \sum_{j,k=0}^{\infty} h_j \frac{(2b)^v(-1)^{k+l}}{(\frac{2k+v}{v}+1)^v}\binom{j}{k}\binom{b(k+v)-v}{l}$$

and

$$I_{REG} = \frac{1}{1-v}\log(\int_0^{\infty} [(\frac{2l+v}{v}+1)g(x;\xi)G^{\frac{2l+v}{v}}(x;\xi)]^v dx)$$

is the Rényi entropy of the Exp-G distribution with power parameter $\frac{2l+v}{v}+1$.

4.3 Moments and Generating Functions

Let $X \sim$ MO-TII-TL-G(δ, b, ξ) and $Y \sim$ EXP-G$(2m+1)$, then the r^{th} moment of X can be obtained as follows. For $\delta \in (0,1)$,

$$E[X^r] = \sum_{m=0}^{\infty} w_m^* E[Y^r],$$

where w_m^* is given by equation (7) and $E[Y^r]$ is the r^{th} moment of Y which follows an Exp-G distribution with power parameter $(2m+1)$.

For $\delta > 1$,

$$E[X^r] = \sum_{m=0}^{\infty} v_m^* E[Y^r],$$

where v_m^* is given by equation (9) and $E[Y^r]$ is the r^{th} moment of Y which follows an Exp-G distribution with power parameter $(2m+1)$.

The moment generating function (mgf) of X is given by: For $\delta \in (0,1)$

$$M_X(t) = \sum_{m=0}^{\infty} w_m^* E[e^{tY}],$$

where $E[e^{tY}]$ is the mgf of the Exp-G distribution with power parameter $(2m+1)$ and w_m^* is given by equation (7).

For $\delta > 1$,

$$M_X(t) = \sum_{m=0}^{\infty} v_m^* E[e^{tY}],$$

where $E[e^{tY}]$ is the mgf of the Exp-G distribution with power parameter $(2m+1)$ and v_m^* is given by equation (9).

A table of moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS), and coefficient of kurtosis (CK) for selected parameter values of the MO-TII-TL-W distribution are given in Table 1.

Table 1 Table of Moments for Selected Parameters of the MO-TII-TL-W Distribution

	$(\delta$, b, $\lambda)$				
	(1.5,0.9,0.3)	(2,0.9,0.2)	(1.5,0.7,0.5)	(1.1,2,0.2)	(1.2,2.9,0.4)
E(X)	0.0849	0.0526	0.0955	0.1085	0.2142
$E(X^2)$	0.0491	0.0294	0.0599	0.0565	0.1158
$E(X^3)$	0.0345	0.0204	0.0435	0.0380	0.0779
$E(X^4)$	0.0265	0.0156	0.0341	0.0286	0.0584
$E(X^5)$	0.0215	0.0126	0.0280	0.0229	0.0466
SD	0.2047	0.1632	0.2253	0.2116	0.2645
CV	2.4110	3.1042	2.3583	1.9504	1.2352
CS	2.7000	3.6886	2.4518	2.3399	1.2518
CK	9.5510	16.5919	7.9505	7.8199	3.5090

3D plots of skewness and kurtosis for the MO-TII-TL-W distribution are given in Figures 4 and 5. We observe that

- When we fix the parameters for b and λ the skewness and kurtosis of MO-TII-TL-W increases as δ increases.
- When we fix the parameters for δ and λ the skewness and kurtosis MO-TII-TL-W increases as b increases.

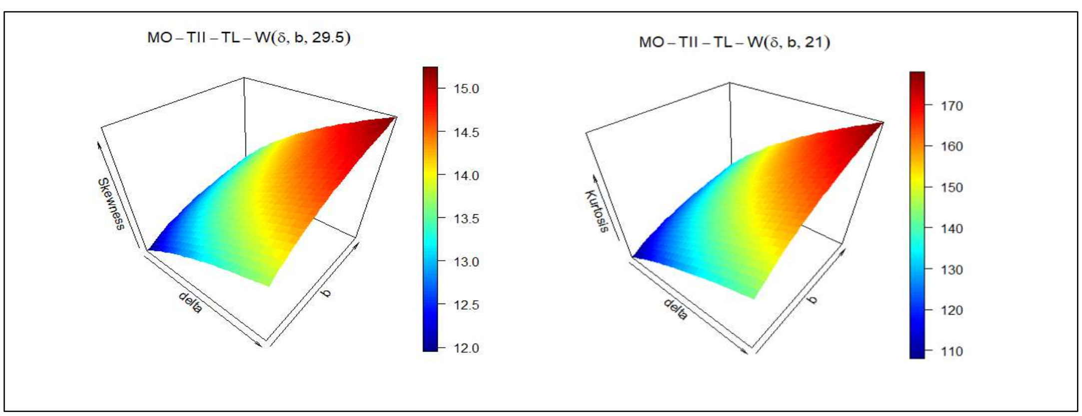

Figure 4 Plots of skewness and kurtosis for the MO-TII-TL-W distribtion

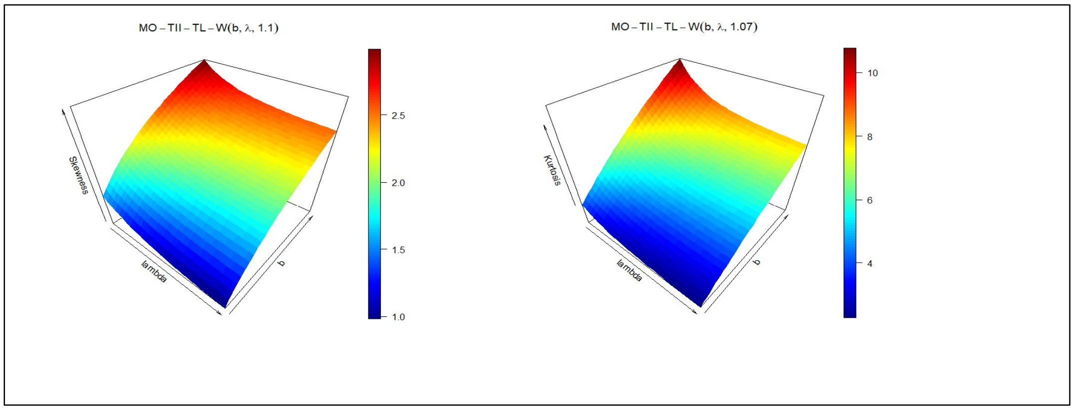

Figure 5 Plots of skewness and kurtosis for the MO-TII-TL-W distribtion

4.4 Quantile Function

The quantile function of the MO-TII-TL-G family of distributions is obtained by solving the non-linear equation:

$$1-\frac{\delta[1-G^2(x;\xi)]^b}{1-\bar{\delta}[1-G^2(x;\xi)]^b}=u,$$

for $0 \leq u \leq 1$. Simplifying the equation, we otain

$$(1-u)-(1-u)\bar{\delta}[1-G^2(x;\xi)]^b=\delta[1-G^2(x;\xi)]^b,$$

such that

$$\frac{1-u}{\bar{\delta}(1-u)}=[1-G^2(x;\xi)]^b.$$

Further simplifying the equation yields

$$(1-[\frac{1-u}{\bar{\delta}(1-u)}]^{\frac{1}{b}})^{\frac{1}{2}}=G^2(x;\xi).$$

Therefore, the quantiles of the MO-TII-TL-G family of distributions may be obtained by solving the non-linear equation

$$x(u)=G^{-1}(1-[\frac{1-u}{\bar{\delta}(1-u)}]^{\frac{1}{b}})^{\frac{1}{2}}.$$

The equation can be solved using software like R, SAS or MATLAB. Quantiles for selected parameter values for the MO-TII-TL-W distribution are shown in Table 2.

Table 2 Table of Quantiles for Selected Parameters of the MO-TII-TL-W Distribution

	$(\delta$, b, $\lambda)$				
U	(1.1,1.5,0.9)	(1.2,0.1,1.1)	(0.5,0.3,0.8)	(0.4,1.7,0.8)	(0.5,0.2,1.2)
0.1	0.2795	1.7615	0.4427	0.1116	0.7135
0.2	0.4515	2.9591	0.8084	0.1910	1.0827
0.3	0.6166	4.1925	1.2195	0.2730	1.4424
0.4	0.7886	5.5368	1.7135	0.3653	1.8308
0.5	0.9777	7.0553	2.3378	0.4750	2.2780
0.6	1.1964	8.8376	3.1709	0.6132	2.8240
0.7	1.4658	11.0446	4.3652	0.8005	3.5371
0.8	1.8316	14.0305	6.2799	1.0836	4.5652
0.9	2.4383	18.9112	10.1294	1.6166	6.3653

5 Maximum Likelihood Estimation

Let $X_i \sim MO-TII-TL-G(\delta, b, \xi)$ and $\Delta = (\delta, b, \xi)^T$ be the parameter vector. The log-likelihood $\ell = \ell(\Delta)$ from a random sample of size n is given by

$$\ell(\Delta) = n\log(2b\delta) + \sum_{i=1}^{n} \log[g(x_i;\xi)] + \sum_{i=1}^{n} \log[G(x_i;\xi)] \\ +(b-1)\sum_{i=1}^{n} \log[1-G^2(x_i;\xi)] - 2\sum_{i=1}^{n} \log[1-\bar{\delta}[1-G^2(x_i;\xi)]^b].$$

The score vector $U = (\frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\xi_k})$ has elements given by

$$\frac{\partial\ell}{\partial\delta} = \frac{n}{\delta} - 2\sum_{i=1}^{n} \frac{[1-G^2(x_i;\xi)]^b}{[1-\bar{\delta}[1-G^2(x_i;\xi)]^b]},$$

$$\frac{\partial\ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n} \log[1-G^2(x_i;\xi)] - 2\sum_{i=1}^{n} \frac{[1-G^2(x_i;\xi)]^b \log[1-G^2(x;\xi)]}{[1-\bar{\delta}[1-G^2(x;\xi)]^b]}$$

and

$$\frac{\partial\ell}{\partial\xi_k} = \sum_{i=1}^{n} \frac{\frac{\partial}{\partial\xi_k} g(x_i;\xi)}{g(x_i;\xi)} + \sum_{i=1}^{n} \frac{\frac{\partial}{\partial\xi_k} G(x_i;\xi)}{G(x_i;\xi)} + (b-1)\sum_{i=1}^{n} \frac{\frac{\partial}{\partial\xi_k}[1-G^2(x_i;\xi)]}{[1-G^2(x_i;\xi)]} \\ -2\sum_{i=1}^{n} \frac{\frac{\partial}{\partial\xi_k}[1-\bar{\delta}[1-G^2(x_i;\xi)]^b]}{[1-\bar{\delta}[1-G^2(x_i;\xi)]^b]}.$$

The maximum likelihood estimates of the parameters, denoted by $\widehat{\Delta}$ is obtained by solving the nonlinear equation $(\frac{\partial\ell}{\partial\delta}, \frac{\partial\ell}{\partial b}, \frac{\partial\ell}{\partial\xi_k})^T$=0 using a numerical method such as the Newton-Raphson procedure. The multivariate normal distribution $N_{q+2}(\underline{0}, J(\widehat{\Delta})^{-1})$, where the mean vector $\underline{0}$ = (0, 0, $\underline{0})^{\mathrm{T}}$ and $J(\widehat{\Delta})^{-1}$ is the observed Fisher information matrix evaluated at $\widehat{\Delta}$, can be used to construct confidence intervals and confidence regions for the model parameters.

6 Simulation Study

In this section, a simulation study was carried out to assess the performance of the maximum likelihood estimates. Different simulations were conducted for the sample sizes n= 100, 200, 400, 800, 1000 and 1200, for N=1000 for each sample. We estimate the mean, root mean square error (RMSE), and average bias (ABIAS). The ABIAS and RMSE for the estimated parameter, are given by

$$ABIAS(\hat{\theta}) = \frac{1}{N}\sum_{i=1}^{N} (\hat{\theta}_i - \theta) \quad and \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N} (\hat{\theta}_i - \theta)^2}{N}},$$

respectively. The results of the simulation study are shown in Tables 3 and 4 and from the results we see that as the sample size increases, the mean approximates the true parameter values, the RMSE and bias decay towards zero for all parameters. Consequently, we conclude that the MO-TII-TL-W model gives out consistent model parameter estimates.

<u>Table 3</u> Monte Carlo Simulation Results for MO-TII-TL-W Distribution: Mean, RMSE and Average Bias

		$\delta = 0.5, b = 0.5, \lambda = 0.5$			$\delta = 0.2, b = 0.6, \lambda = 0.6$		
	n	Mean	RMSE	ABIAS	Mean	RMSE	ABIAS
	100	0.7653	0.6525	0.2653	0.3205	0.2903	0.1205
	200	0.6435	0.3659	0.1435	0.2677	0.1616	0.0677
δ	400	0.5632	0.2150	0.0632	0.2336	0.0995	0.0336
	800	0.5272	0.1346	0.0272	0.2163	0.0572	0.0163
	1000	0.5264	0.1268	0.0264	0.2125	0.0532	0.0125
	1200	0.5170	0.1094	0.0170	0.2097	0.0494	0.0097
	100	0.6564	0.3851	0.1564	0.7973	0.4672	0.1973
	200	0.5834	0.2546	0.0834	0.7196	0.3185	0.1196
β	400	0.5380	0.1610	0.0380	0.6589	0.2088	0.0589
	800	0.5186	0.1089	0.0186	0.6342	0.1292	0.0342
	1000	0.5182	0.1026	0.0182	0.6251	0.1205	0.0251
	1200	0.5099	0.0893	0.0099	0.6180	0.1115	0.0180
	100	0.4664	0.0704	-0.0336	0.5773	0.0642	-0.0227
	200	0.4782	0.0532	-0.0218	0.5796	0.0486	-0.0204
λ	400	0.4859	0.0361	-0.0141	0.5844	0.0344	-0.0156
	800	0.4910	0.0268	-0.0090	0.5888	0.0233	-0.0112
	1000	0.4909	0.0246	-0.0092	0.5899	0.0221	-0.0101

	1200	0.4937	0.0220	-0.0063	0.5915	0.0205	-0.0085

Table 4 Monte Carlo Simulation Results for MO-TII-TL-W Distribution: Mean, RMSE and Average Bias

		$\delta = 0.2, b = 1.1, \lambda = 1.1$			$\delta = 0.9, b = 0.2, \lambda = 0.2$		
	n	Mean	RMSE	ABIAS	Mean	RMSE	Bias
	100	0.2988	0.3088	0.0988	1.6470	1.3322	0.7470
	200	0.2477	0.1600	0.0477	1.1732	0.4518	0.2732
δ	400	0.2223	0.0953	0.0223	1.0286	0.2356	0.1286
	800	0.2117	0.0603	0.0117	0.9581	0.1338	0.0581
	1000	0.2093	0.0537	0.0093	0.9372	0.0447	0.0372
	1200	0.2044	0.0480	0.0044	0.9185	0.0992	0.0185
	100	1.3267	0.7280	0.2267	0.3230	0.2169	0.1230
	200	1.2255	0.5035	0.1255	0.2430	0.0853	0.0430
β	400	1.1573	0.3345	0.0573	0.2196	0.0484	0.0196
	800	1.1354	0.2251	0.0354	0.2076	0.0233	0.0076
	1000	1.1290	0.2025	0.0290	0.2020	0.0056	0.0020
	1200	1.1095	0.1855	0.0095	0.2104	0.0222	0.0104
	100	1.0952	0.1276	-0.0048	0.1811	0.0293	-0.0189
	200	1.0996	0.0933	-0.0004	0.1912	0.0161	-0.0088
λ	400	1.0991	0.0660	-0.0009	0.1951	0.0106	-0.0049
	800	1.0978	0.0456	-0.0022	0.1976	0.0048	-0.0024
	1000	1.0984	0.0410	-0.0016	0.1990	0.0030	-0.0010
	1200	1.0999	0.0378	-0.0001	0.1966	0.0056	-0.0034

7 Applications

In this section, we present two real data examples to illustrate the applicability of the MO-TII-TL-W distribution. Model parameters were estimated using the maximum likelihood estimation technique, with the aid of the R software for fitting the data and model diagnostics. The performance of the models were assessed using the following goodness-of-fit statistics: -2loglikelihood (-2 log L), Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Cramér-von Mises (W^*) and Andersen-Darling (A^*) (as described by Chen and Balakrishnan [11]), Kolmogorov-Smirnov (K-S) statistic and its p-value. The smaller the goodness-of-fit statistics, the better the model.

We present the model parameters estimates (standard errors in parenthesis) and the goodness-of-fit-statistics in Tables 5, 6, 7 and 8. In addition, the plots of the fitted densities and probability plots (Chambers et al. [10]) are provided to demonstrate how well our model fits the two data sets.

The non-nested models compared to the MO-TI-TL-W distribution are the Weibull-exponential (WE) distribution by Oguntunde et al. [21], Odd Lindley Fréchet (OLiFr) distribution by Mansour et al. [17], Type II generalized Topp-Leone-Rayleigh (TIGTL-R) distribution by Hassan et al. [14], Topp-Leone generalized exponential (TL-GE) distribution by Sangsanit and Bodhisuwan [25], Marshall-Olkin extended Fréchet (MOEFr) and Marshall-Olkin extended generalized exponential (MOEGE) distribution given by Barreto-Souza et al. [8]. The pdfs of the non-nested models are given by:

$$f_{WE}(x;\alpha,\beta,\lambda)=\alpha\beta(\lambda e^{-\lambda}\ )[\frac{(1-e^{-\lambda x})^{\beta-1}}{(e^{-\lambda x})^{\beta+1}}]e^{-\alpha[\frac{(1-e^{-\lambda x})}{(e^{-\lambda x})}]^{\beta}},$$

for $\alpha,\beta,\lambda>0$,

$$f_{OLiFr}(x;\theta,\alpha,\beta)=\frac{\theta\alpha^{\beta}\theta^{2}x^{-\beta-1}e^{-(\frac{\alpha}{x})^{\beta}}}{(1+\theta)(1-e^{-(\frac{\alpha}{x})^{\beta}})^{3}}\exp(\frac{-\theta e^{-(\frac{\alpha}{x})^{\beta}}}{1-e^{-(\frac{\alpha}{x})^{\beta}}}),$$

for $\theta,\alpha,\beta>0$,

$$f_{TIIGTL-R}(x;\alpha,\beta,\delta)=4\alpha\beta\delta x e^{-\delta^{2}}[1-e^{-\delta^{2}}]^{2\beta-1}(1-(1-e^{-\delta^{2}})^{2\beta})^{\alpha-1},$$

for $\alpha,\beta,\delta>0$,

$$f_{TL-G}\ (x;\alpha,\beta,\lambda)=2\alpha\beta\lambda e^{-\lambda x}(1-(1-e^{-\lambda x})^{\beta}(1-e^{-\lambda x})^{\beta\alpha-1}\times(2-(1-\ e^{-\lambda x})^{\beta}))^{\alpha-1},$$

for $\alpha,\beta,\lambda>0$,

$$f_{MOEFr}(x;\alpha,\delta,\lambda)=\frac{\alpha\lambda\delta^{\lambda}x^{-(\lambda+1)e^{-(\delta/x)^{\lambda}}}}{[1-\bar{\alpha}(1-e^{-(\delta/x)^{\lambda}})]^{2}},$$

for $\alpha,\delta,\lambda>0$, and

$$f_{MOEGE}(x;\alpha,\gamma,\lambda)=\frac{\alpha\gamma e^{-\lambda x}(1-e^{-\lambda x})^{\gamma-1}}{(1-\bar{\alpha}[1-e^{-\lambda x}])^{2}}$$

for $\alpha,\gamma,\lambda>0$.

7.1 Silicon Nitride Data

The first data set is on fracture toughness of silicon nitride measured in MPa $m^{1/2}$. The data set was analyzed by Nadarajah and Kotz [20] and also by Ali et al. [3]. The data are 5.50, 5.00, 4.90, 6.40, 5.10, 5.20, 5.20, 5.00, 4.70, 4.00, 4.50, 4.20, 4.10, 4.56, 5.01, 4.70, 3.13, 3.12, 2.68, 2.77, 2.70, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.80, 3.73, 3.71, 3.28, 3.90, 4.00, 3.80, 4.10, 3.90, 4.05, 4.00, 3.95, 4.00, 4.50, 4.50, 4.20, 4.55, 4.65, 4.10, 4.25, 4.30, 4.50, 4.70, 5.15, 4.30, 4.50, 4.90, 5.00, 5.35, 5.15, 5.25, 5.80, 5.85, 5.90, 5.75, 6.25, 6.05, 5.90, 3.60, 4.10, 4.50, 5.30, 4.85, 5.30, 5.45, 5.10, 5.30, 5.20, 5.30, 5.25, 4.75, 4.50, 4.20, 4.00, 4.15, 4.25, 4.30, 3.75, 3.95, 3.51, 4.13, 5.40, 5.00, 2.10, 4.60, 3.20, 2.50, 4.10, 3.50, 3.20, 3.30, 4.60, 4.30, 4.30, 4.50, 5.50, 4.60, 4.90, 4.30, 3.00, 3.40, 3.70, 4.40, 4.90, 4.90, 5.00.

Table 5 Parameter estimates for various models fitted for silicon nitride data set

Model	δ	β	λ
MO-TII-TL-W	71.2032 (0.0014)	0.3663 (0.1045)	1.6974 (0.1695)
	α	β	λ
WE	25.6129 (0.0014)	4.1621 (0.2700)	0.0798 (0.0032)
	θ	α	β
OLiFr	0.1591 (0.1063)	2.1951 (0.4366)	3.3991 (0.2320)
	α	β	δ
TIIGTL-R	3585.2000 (2.1799×10^{-7})	1.2646 (0.0966)	1.8076×10^{-3} (4.4088×10^{-4})
	α	β	λ
TL-GE	0.3892 (0.2426)	56.9230 (42.6845)	0.7549 (0.0779)
	α	δ	λ
MOEFr	2407.7000 (7.7867×10^{-6})	1.4344 (0.1296)	7.0579 (0.5495)
	α	γ	λ
MOEGE	0.0107 (7.0715×10^{-3})	20.7600 (3.5685×10^{-5})	1.7321 (0.1434)

Table 6 Goodness-of-fit statistics for various models fitted for silicon nitride data set

Model	-2 log L	AIC	AICC	BIC	W*	A*	KS	P-value
MO-TII-TL-W	336.9	342.9	343.1	351.2	0.0460	0.3294	0.0473	0.9527
WE	337.2	343.2	343.4	351.5	0.0820	0.4981	0.0689	0.6252
OLIFr	339.2	345.2	345.4	353.5	0.1426	0.8740	0.0818	0.4037
TIIGTL-R	337.5	343.5	343.7	351.8	0.0942	0.5795	0.0725	0.5599
TL-GE	359.6	365.6	365.8	373.9	0.4620	2.7741	0.1414	0.01713
MOEFr	356.6	362.6	362.9	371.0	38.2495	235.4782	0.9989	$< 2.2 \times 10^{-16}$
MOEGE	340.3	346.3	346.5	354.6	0.4722	2.9106	0.9893	$< 2.2 \times 10^{-16}$

The estimated variance-covariance for the MO-TII-TL-W model on silicon nitride data is given by

$$\begin{bmatrix} 2.0274 \times 10^{-6} & -1.4830 \times 10^{-4} & 2.4119 \times 10^{-4} \\ -1.4830 \times 10^{-4} & 0.0109 & -0.0176 \\ 2.4119 \times 10^{-4} & -0.0176 & 0.0287 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [71.2032 \pm 0.0028]$, b $\in [0.3663 \pm 0.2048]$ and $\lambda \in [1.6974 \pm 0.3323]$.

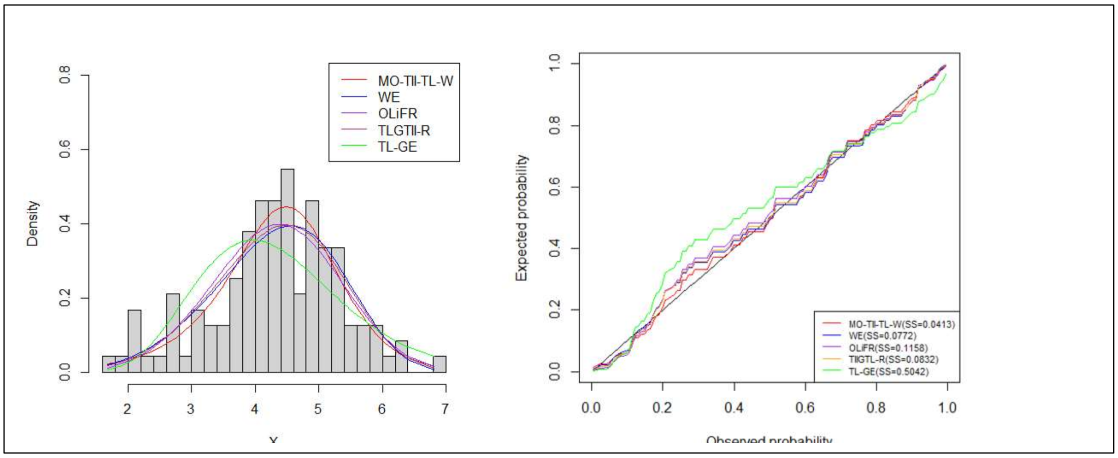

Figure 6 Fitted densities and probability plots for silicon nitride data

Based on the results of the goodness-of-fit statistics and the p-value shown in Tables 5 and 6, we conclude that the MO-TII-TL-W distribution performs better than the non-nested models considered in this paper. Figure 6 which shows the fitted densities and probability plots also shows that the MO-TII-TL-W model fit the silicon nitride data set better than the non-nested models.

7.2 Kevlar 49/Epoxy Strands Failure at 90% Data

The second data set is on 101 observations representing the stress-rupture life of kevlar 49/epoxy strands which are subjected to constant sustained pressure at the 90% stress level until all had failed. The failure times are in hours and are shown below (see Andrews and Herzberg [6] or Barlow et al. [7], for details): 0.01, 0.01, 0.02, 0.02, 0.02, 0.03, 0.03, 0.04, 0.05, 0.06, 0.07, 0.07, 0.08, 0.09, 0.09, 0.10, 0.10, 0.11, 0.11, 0.12, 0.13, 0.18, 0.19, 0.20, 0.23, 0.24, 0.24, 0.29, 0.34, 0.35, 0.36, 0.38, 0.40, 0.42, 0.43, 0.52, 0.54, 0.56, 0.60, 0.60, 0.63, 0.65, 0.67, 0.68, 0.72, 0.72, 0.72, 0.73, 0.79, 0.79, 0.80, 0.80, 0.83, 0.85, 0.90, 0.92, 0.95, 0.99, 1.00, 1.01, 1.02, 1.03, 1.05, 1.10, 1.10, 1.11, 1.15, 1.18, 1.20, 1.29, 1.31, 1.33, 1.34, 1.40, 1.43, 1.45, 1.50, 1.51, 1.52, 1.53, 1.54, 1.54, 1.55, 1.58, 1.60, 1.63, 1.64, 1.80, 1.80, 1.81, 2.02, 2.05, 2.14, 2.17, 2.33, 3.03, 3.03, 3.34, 4.20, 4.69, 7.89.

The estimated variance-covariance for the MO-TII-TL-W model on kelvar data is given by

$$\begin{bmatrix} 17.6587 & 5.8581 & -0.3385 \\ 5.8581 & 2.0343 & -0.1162 \\ -0.3385 & -0.1162 & 0.0079 \end{bmatrix}$$

and the 95% confidence intervals for the model parameters are given by $\delta \in [5.0534 \pm 8.2364]$, b $\in [4.4047 \pm 2.7956]$ and $\lambda \in [\, 0.4231 \pm 0.1744]$.

Table 7 Parameter estimates for various models fitted for kelvar data set

Model	δ	β	λ
MO-TII-TL-W	5.0534 (4.2022)	4.4047 (1.4263)	0.4231 (0.0890)
	α	β	λ
WE	147.49 (1.6905 x 10^{-6})	0.9232 (0.0711)	4.5027 x 10^{-3} (1.7793 x 10^{-3})
	θ	α	β
OLiFr	0.2650 (0.1627)	0.0377 (0.0353)	0.6349 (0.0480)
	α	β	δ
TIIGTL-R	41.4010 (1.5381 x 10^{-6}	0.2278 (0.0178)	2.7764 x10^{-4} (1.6758 x10^{-4})
	α	β	λ
TL-GE	0.4742 (0.5673)	1.7241 (1.9169)	0.5110 (0.1540)
	α	δ	λ
MOEFr	312.8500 (4.4813 x 10^{-6})	6.5773 x10^{-3} (2.6809 x 10^{-3})	1.2631 (0.1033)
	α	γ	λ
MOEGE	0.5942 (0.3306)	0.7307 (0.1849)	1.0456 (0.2071)

Table 8 Goodness-of-fit statistics for various models fitted for kelvar data set

Model	-2 log L	AIC	AICC	BIC	W*	A*	KS	P-value
MO-TII-TL-W	204.1	210.1	210.4	217.9	0.1580	0.9161	0.0799	0.5400
WE	206.0	212.0	212.2	219.8	0.1973	1.1051	0.0903	0.3820
OLIFr	207.2	213.2	213.5	221.1	0.2858	1.5263	0.1065	0.2021
TIIGTL-R	206.0	212.0	212.2	219.8	0.1922	1.0825	0.0898	0.3895

TL-GE	205.4	211.4	211.7	219.3	0.1518	0.8982	0.0801	0.5366
MOEFr	225.2	231.2	231.5	239.1	33.5614	199.4422	0.9960	<2.2 x 10^{-16}
MOEGE	204.8	210.8	211	218.6	0.1568	1.4421	0.4186	8.882 x 10^{-16}

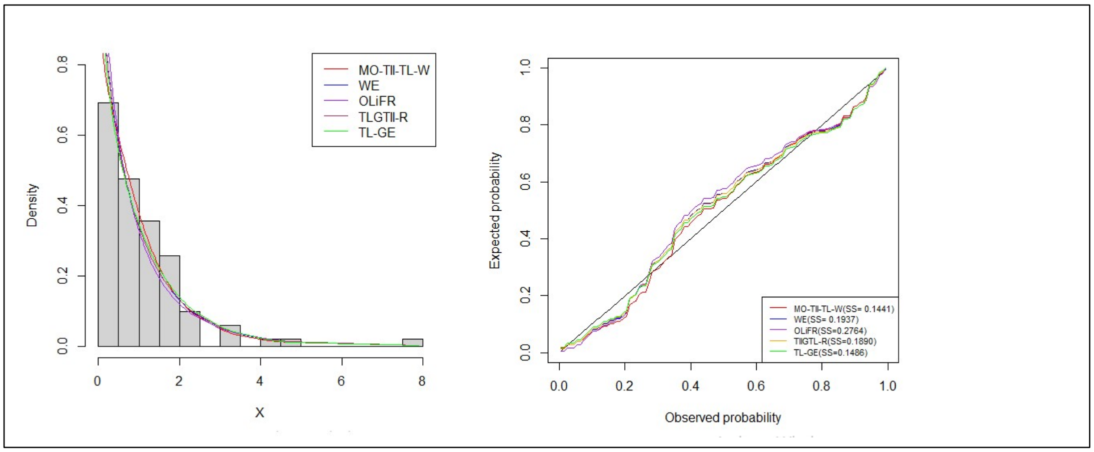

Figure 7 Fitted densities and probability plots for kelvar data

Furthermore, the results of the goodness-of-fit statistics and the p-value shown in Tables 7 and 8 also confirms that the MO-TII-TL-W distribution performs better than the non-nested models considered in this paper. Figure 7 which shows the fitted densities and probability plots also indicates that the MO-TII-TL-W model fit the kelvar data set better than the non-nested models.

8 Concluding Remarks

A new generalized distribution referred to as the Marshall-Olkin-Type II-Topp-Leone-G family of distributions is developed and presented. The MO-TII-TL-G family of distributions has hazard function with flexible behavior and can be expressed as an infinite linear combination of the Exp-G distribution. Closed form expressions for the moments, distribution of order statistics and entropy were obtained. Maximum likelihood estimation method was used to estimate the model parameters. The performance of the special case of the MO-TII-TL-G distribution was examined by conducting various simulations for different sample sizes and lastly, the special case of the MO-TII-TL-G distribution was fitted to two real data sets to illustrate the applicability and usefulness of the proposed family of distributions.

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