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# An Alternative Method for Proposing New Families of Distributions: Estimation of Parameters Through Classical and Bayesian Analysis

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## ABSTRACT

In this article, a novel method named as an alternative family of distributions for proposing new continuous probability distributions is suggested. The novelty of the proposed scheme is highlighted by considering Log-Logistic distribution as a special case. Important properties of the AL family of distributions are derived. The inferential section of the AL family, primarily, includes the maximum likelihood estimation. The diversity of parameter estimation is encountered for complete and censoring phenomena along with Bayesian Paradigm. The ironic property of the maximum likelihood estimates of the unknown parameters of the proposed family of distributions is assessed by rigorous simulation study for complete and censored phenomena. Further, three real data sets have been used to illustrate the applicability of the new scheme.

***Keywords:*** Log-logistic distribution, Log-logistic distribution, Type-I and Type-II censoring schemes, reliability analysis

## 1. Introduction

In statistical distribution theory, it is a common practice to add new parameter(s) to a distribution function or compounding a distribution with other distribution to make it more flexible and useful to analyze a given lifetime phenomena. The central idea is to propose lifetime distributions which can broadly accommodate practical applications where the underlying hazard functions are of either decreasing, constant or increasing shapes. Many lifetime distributions have been proposed in the literature and different attempts have made to develop new distributions with a specific failure rate property by compounding two well-known distributions.

One of such techniques for developing new distributions, formulated as DUS transformation, is projected by [3]. We denote the probability distribution function (PDF) and cumulative distribution function (CDF) as $g(x)$ and $G(x)$, respectively, of some baseline distribution, the PDF of the DUS transformation is denoted by $f(x)$ and is given as

$$f(x)=\frac{1}{e-1}\left[g(x)e^{G(X)}\right]. \tag{1}$$

And the CDF can be deduced as

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$$F(x)=\frac{1}{e-1}\left[e^{G(X)}-1\right] \tag{2}$$

[3] then used exponential distribution as a special case. Numerous statistical properties are defined and then showed the applicability of the proposed method in various field of study. [4] introduced a diverse method by making some functional changes in the CDF of a family of distributions instead of adding additional parameter. Inspiring the method suggested by [3], in this paper, we proposed a new family by exploiting the functional form of the CDF of DUS transformation. The new family is called AL distributions. As a special case, the Log-Logistic distribution is used to demonstrate the proposed method. The Log-Logistic distribution is a strong competitor of Weibull distribution and thus can be used as an alternative of Weibull distribution. It is in fact a mixture of Gompertz distribution and Gamma distribution with the value of the mean and the variance coincide equal to one. The Log-Logistic distribution as a life testing model has its own standing; it has an increasing failure rate and is viewed as a weighted exponential distribution. We aim to reveal some statistical properties of the proposed model and apply it to the reliability engineering.

Now we define the new proposition starting to CDF, has the following expression

$$F\left(x,\xi\right)=\log\left[1+\left(e-1\right)G^{b}\left(x,\zeta\right)\right], \tag{3}$$

where $e\approx 2.718$, and $b$ is the additional parameter to accommodate the tail behavior. We say that a continuous random variable $X$ follows an alternative distributions ( $X \sim AL(.)$ ) with parameters vector $\xi=\left(b,\zeta\right)$, where $b>0$ is shape parameter.

The PDF can be deduced as follows

$$f\left(x,\xi\right)=\frac{b\left(e-1\right)G^{b-1}\left(x,\zeta\right)g\left(x,\zeta\right)}{1+\left(e-1\right)G^{b}\left(x,\zeta\right)},\ x>0 \tag{4}$$

The inverse or quantile function ( $x=F^{-1}\left(u\right)$ ) of the CDF is obtained in closed form by solving $u=F\left(x\right)$. The quantile function is handy for many features of the proposed family of distribution (PFD) and, therefore, the data generation for simulation study is straight forward. $x=G^{-1}\left\{\left[\left(e-1\right)^{-1}\left(e^{u}-1\right)\right]^{\frac{1}{b}};\xi\right\}$, where $u$ is generated from Uniform distribution with unit interval support.

Before exploring further, it is mentioned for simplicity, from now, we just write $f\left(x\right)=f\left(x,\xi\right)$. The survival function and hazard function are given as

$$S\left(x\right)=1-\log\left[1+\left(e-1\right)G^{b}\right] \text{ and}$$

$$h\left(x\right)=\frac{b\left(e-1\right)gG^{b-1}}{\left\{1+\left(e-1\right)G^{b}\right\}\left\{1-\log\left[1+\left(e-1\right)G^{b}\right]\right\}}. \tag{5}$$

## 2. Derived distribution of the proposed AL family of distribution

In this section, we elaborate further our PFD by taking two lifetime distributions, namely, log-logistic and exponential distributions as a special case. The choice of these two distributions is inspired by their utility and applicability in real life scenario.

### 2.1.Alternative log-logistic distribution

Let we consider the log-logistic distribution having CDF and PDF $G=\left[1+\left(\lambda x\right)^{-\beta}\right]^{-1}$ and $g=\lambda\beta\left(\lambda x\right)^{\beta-1}\left[1+\left(\lambda x\right)^{\beta}\right]^{-2}$ with parameters $\beta$ and $\lambda$, respectively. The alternative log-logistic (AL-L) distribution has the following closed CDF

$$F\left(x\right)=\log\left[1+\left(e-1\right)\left[1+\left(\lambda x\right)^{-\beta}\right]^{-b}\right], \tag{6}$$

and the corresponding PDF of the proposed AL-L distribution is

$$f\left(x\right)=\frac{b\left(e-1\right)\lambda\beta\left(\lambda x\right)^{\beta-1}\left[1+\left(\lambda x\right)^{-\beta}\right]^{-(b-1)}\left[1+\left(\lambda x\right)^{\beta}\right]^{-2}}{1+\left(e-1\right)\left[1+\left(\lambda x\right)^{-\beta}\right]^{-b}}, \tag{7}$$

where, $b>0$, $\beta>0$ are shape parameters and $\lambda$ is a scale parameter. The proposed density in (7) is further explored numerically and graphically in later part of this section. Here, plots of the CDF and PDF of the proposed AL-L model for various combinations of the parameters $b$, $\beta$, $\lambda$ are shown in Figures 1 and 2. Figure 1 shows that the PDF of the proposed AL-L distribution has uni-model shape.

Further, closed form of the qunatile function of the sub model enable us to simplify many features. Specially, the data generation is straighforward.

$$x=\frac{1}{\lambda}\left[\left[\frac{\left(e^{u}-1\right)}{\left(e-1\right)}\right]^{\frac{-1}{b}}-1\right]^{\frac{-1}{\beta}}, \tag{8}$$

where u is generated from uniform with limits 0 and 1. The survival and hazard functions are, respectively given by

$$S\left(x\right)=1-\log\left[1+\left(e-1\right)\left[1+\left(\lambda x\right)^{-\beta}\right]^{-b}\right],$$

$$h(x)=\frac{b(e-1)\lambda\beta(\lambda x)^{\beta-1}\left[1+(\lambda x)^{-\beta}\right]^{-(b-1)}\left[1+(\lambda x)^{\beta}\right]^{-2}}{\left\{1-\log\left[1+(e-1)\left[1+(\lambda x)^{-\beta}\right]^{-b}\right]\right\}\left\{1+(e-1)\left[1+(\lambda x)^{-\beta}\right]^{-b}\right\}}. \tag{9}$$

The hazard function of the AL-L distribution in (9) is presented graphically for different combinations of parametric values and given in Figure 2. Moreover, skewness and kurtosis of the ALL-L distribution for different combinations of parameters are displayed in Figure 3.

Taking a close look on main properties of the proposed family of distributions, we consider the AL-L distribution and compute first five raw moments, standard deviation (*SD*), coefficient of variation (*CV*), coefficient of skewness (*CS*) and coefficient of kurtosis (*CK*) of the AL-L distribution for different combinations of parameters and given in Table 1. The mathematical expressions of *SD, CV, CS* and *CK* are noted down as

$$\sigma^2=u_2'-u^2, CV=\frac{\sigma}{u},\ CS=\frac{u_3'-3uu_2'+2u^3}{\left(u_2'-u^2\right)^{3/2}} \text{ and } CK=\frac{u_4'-4uu_3'+6u^2u_2'-3u^4}{\left(u_2'-u^2\right)^2}.$$

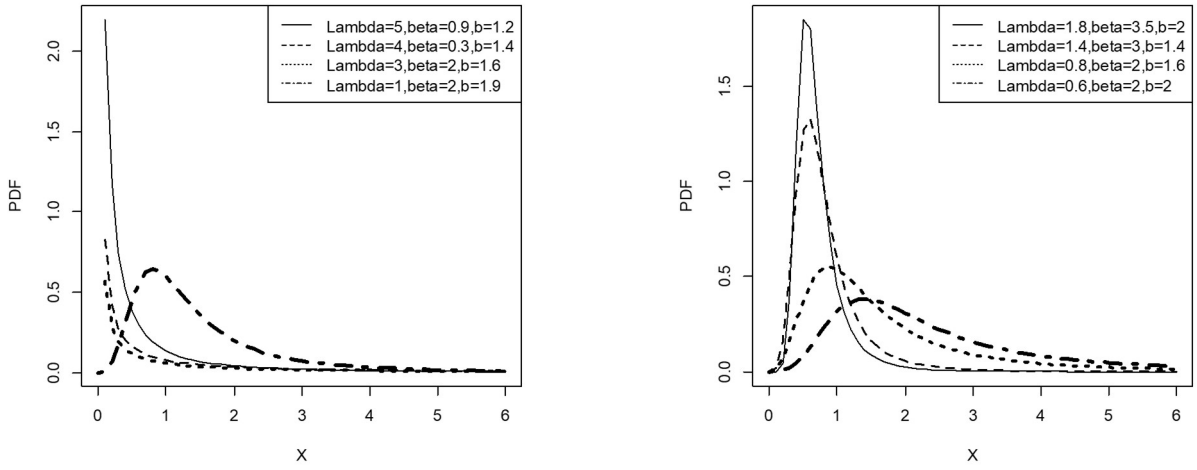



Figure 1: PDF plots of the AL-L distribution for various combinations of parameters.

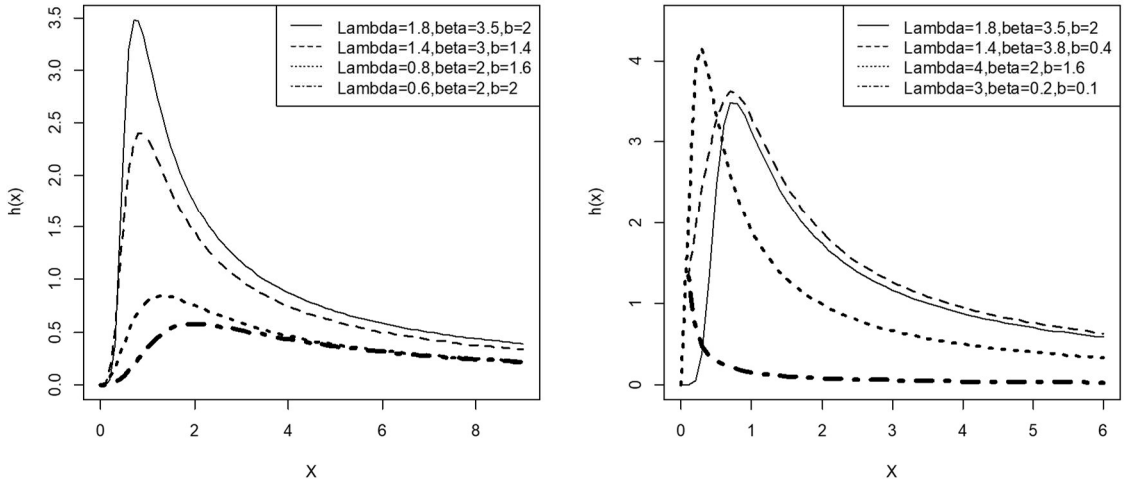



Figure 2: Hazard functions of the AL-L distribution for various combinations of parameters.

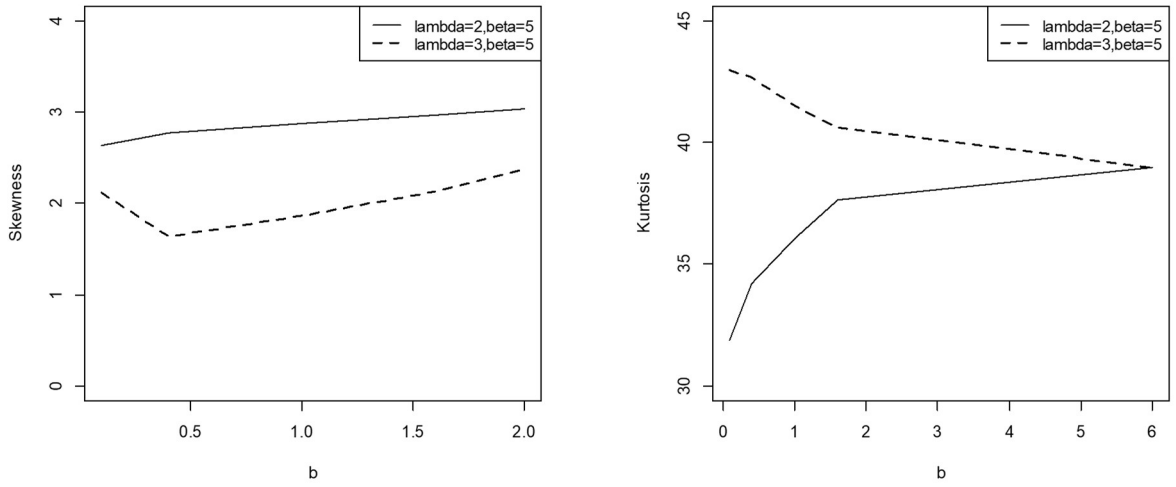



Figure 3: Plots of Skew ness and Kurtosis of the AL-L distribution for various combinations of parameters.

Table 1: Moments, SD, CV, CS and CK of the AL-L distribution for combination of parameters.

| | $\lambda=2,\beta=5,b=2$ | $\lambda=3,\beta=5,b=2$ | $\lambda=3,\beta=5,b=0.2$ | $\lambda=1.3,\beta=5,b=0.2$ |
|---|---|---|---|---|
| $u_1'$ | 0.590103 | 0.393402 | 0.153286 | 0.353736 |
| $u_2'$ | 0.389751 | 0.173223 | 0.039486 | 0.210281 |
| $u_3'$ | 0.304728 | 0.09029 | 0.014333 | 0.176146 |
| $u_4'$ | 0.335963 | 0.066363 | 0.008096 | 0.229618 |

| | | | | |
|---|---|---|---|---|
| $u_5'$ | 14.68539 | 1.933882 | 0.193857 | 1.776372 |
| *SD* | 0.04153 | 0.018458 | 0.01599 | 0.085151 |
| *CV* | 0.070377 | 0.046918 | 0.104312 | 0.24072 |
| *CS* | 3.039136 | 3.039136 | 1.67097 | 1.67097 |
| *CK* | 38.97656 | 38.97656 | 12.58935 | 12.58935 |

**2.2. Alternative Exponential distribution**

Let we consider the exponential distribution having CDF and PDF $G = 1-\exp(-\lambda x)$ and $g = \lambda \exp(-\lambda x)$ with parameter $\lambda$, respectively, as a baseline distribution. The derived CDF and PDF of the proposed distribution is named as Alternative Exponential (AE) distribution and is given as

$$F(x) = \log\left[1+(e-1)\left\{1-\exp(-\lambda x)\right\}^{b}\right], \text{ and} \tag{10}$$

$f(x) = \dfrac{b(e-1)\lambda \exp(-\lambda x)\left\{1-\exp(-\lambda x)\right\}^{b-1}}{1+(e-1)\left\{1-\exp(-\lambda x)\right\}^{b}}$, where $x>0$. The $b>0$ and $\lambda>0$ are shape parameters.

**Estimation of univariate family parameters**

Here in this section, we elaborate the estimation of unknown parameters of the Alternative family of distributions by maximum likelihood method. The maximum likelihood estimates encompass desirable properties and carry simple approximations that efficiently work in finite samples. The resultant normal equations can be elucidated either analytically or numerically. In next sub section, we present the likelihood functions and corresponding normal equations for complete and censored mechanism.

**2.3. Maximum likelihood estimation**

The log-likelihood function denoted by $L(\xi)$, where $\xi = (b, \zeta)$ is a vector of parameters. Considering the random variable $X$ having $n$ observations $x_1, x_2, \ldots x_n$ has the following expression

$$L(\xi) = n\log\left\{b(e-1)\right\} + \sum_{i=1}^{n}\log\left(g(x_i)\right) + (b-1)\sum_{i=1}^{n}\log\left(G(x_i)\right) - \sum_{i=1}^{n}\log\left(1+(e-1)G(x_i)^{b}\right). \tag{11}$$

The MLE $\hat{\xi}$ of $\xi$ can be obtained through maximization. The normal equations after differentiation with respect to $b$ and $\zeta$, respectively, are expressed as

$$\frac{\partial L(\xi)}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left(G(x_i)\right) - \sum_{i=1}^{n}\frac{(e-1)G(x_i)^{b}\log\left(G(x_i)\right)}{1+(e-1)G(x_i)^{b}} = 0, \tag{12}$$

$$\frac{\partial L(\xi)}{\partial \zeta} = \sum_{i=1}^{n}\left[\frac{g'(x_i)}{g(x_i)}\right] + (b-1)\sum_{i=1}^{n}\left[\frac{g(x_i)}{G(x_i)}\right] - \sum_{i=1}^{n}\left[\frac{(e-1)bg(x_i)G(x_i)^{b-1}}{1+(e-1)G(x_i)^{b}}\right] = 0. \tag{13}$$

Setting the score components to zero and solving them simultaneously yields the MLEs of the parameters of the model. Since, the nature of resulting equations is complex; therefore, these equations can solve numerically to obtain the estimates.

### 2.4. Type-II Censoring

We establish the likelihood function of the PFD under Type-II right censoring scheme. Let we have $n$ ordered observation $x_{(1)} < x_{(2)} < ... < x_{(n)}$ and $x_{(t)}$ is the $t^{th}$ order statistic, and it is assumed that $t$ observations out of $n$ observations are censored from the right side. The log-likelihood function under Type-II censoring scheme may be written as

$$L(\xi) = \log\left(\frac{n!}{(n-t)!}\right) + (n-t)\log\left[S\left(x_{(t)}\right)\right] + \sum_{i=1}^{t}\log\left\{g(x_i)\right\}. \tag{14}$$

By putting the PDF and survival function of the PFD in (14), we obtain the following expression.

$$L(\xi) = \log\left(\frac{n!}{(n-t)!}\right) + (n-t)\log\left[1-\log\left[1+(e-1)G\left(x_{(t)}\right)^{b}\right]\right] + \sum_{i=1}^{t}\log\left[\frac{b(e-1)G(x_i)^{b-1}g(x_i)}{1+(e-1)G(x_i)^{b}}\right].$$

The MLEs of the unknown parameters of the PFD under Type-II censoring scheme can be deduced by maximizing the aforementioned log-likelihood function. Alternatively, the normal equations can be constructed by differentiating the log-likelihood function with respect to parameters and equating to zero. Based on experienced, numerical methods are required to find the estimates of the unknown parameters.

### 2.5. Type-I censoring

Let us define an indicator function which takes value 1 in case of the lifetime before terminating a predefined time $x_c$ otherwise 0 in case of termination. We observed the lifetime of $\delta$ observations before terminating the process and the remaining $n-\delta$ observations will be censored. The indicator function is defined as

$I_i = \begin{cases} 1, \text{ if } X_i \le x_c \\ 0, \text{ if } X_i > x_c \end{cases}$, where $\delta = \sum_{i=1}^{n} I_i$ is the number of censored observations.

Under the Type-I censoring scheme, the log-likelihood function is demonstrated by

$$L(\xi) = (n-\delta)\log\left[1-\log\left[1+(e-1)G(x_c)^{b}\right]\right] + \sum_{i=1}^{n} I_i \log\left[\frac{b(e-1)G(x_i)^{b-1}g(x_i)}{1+(e-1)G(x_i)^{b}}\right]. \tag{15}$$

Again the esitmates of the parameters can be obtained by following the instructions which are mentioned in previous section.

## 3. Bayesian estimation

Now, parameters estimation is dealt under Bayesian paradigm. The parameters estimation of the AL-L distribution is discussed, specifically, here in this section. The likelihood function corresponding to (7) can be communicated mathematically as

$$l(b,\beta,\lambda \mid \underline{x}) = \prod_{i=1}^{n}\left\{\frac{b(e-1)\lambda\beta(\lambda x)^{\beta-1}\left[1+(\lambda x)^{-\beta}\right]^{-(b-1)}\left[1+(\lambda x)^{\beta}\right]^{-2}}{1+(e-1)\left[1+(\lambda x)^{-\beta}\right]^{-b}}\right\}. \tag{16}$$

Suppose that the three parameters $b, \beta$ and $\lambda$ in (16) has prior densities $p(b), p(\beta)$ *and* $p(\lambda)$, respectively. Then, the joint posterior densities of $b, \beta$ and $\lambda$ for given data may be derived as

$$f\left(b,\beta,\lambda \mid \underline{x}\right)=\frac{\prod_{i=1}^{n}\left\{\frac{b(e-1)\lambda\beta(\lambda x)^{\beta-1}\left[1+(\lambda x)^{-\beta}\right]^{-(b-1)}\left[1+(\lambda x)^{\beta}\right]^{-2}}{1+(e-1)\left[1+(\lambda x)^{-\beta}\right]^{-b}}\right\}p(b)\,p(\beta)\,p(\lambda)}{\int_{\lambda}\int_{\beta}\int_{b}\prod_{i=1}^{n}\left\{\frac{b(e-1)\lambda\beta(\lambda x)^{\beta-1}\left[1+(\lambda x)^{-\beta}\right]^{-(b-1)}\left[1+(\lambda x)^{\beta}\right]^{-2}}{1+(e-1)\left[1+(\lambda x)^{-\beta}\right]^{-b}}\right\}p(b)\,p(\beta)\,p(\lambda)\,db\,d\beta\,d\lambda}$$

Under the square error loss function, Bayes estimator of $b$ has the following expression

$$\hat{b}_{Bayes}=E\left(b \mid \underline{x}\right)=\frac{\int_{\lambda}\int_{\beta}\int_{b} b^{n+1}(e-1)^{n}\lambda^{\beta n}\beta^{n}\,p(b)\,p(\beta)\,p(\lambda)\prod_{i=1}^{n}\left\{\frac{x^{\beta-1}\left[1+(\lambda x)^{-\beta}\right]^{-(b-1)}\left[1+(\lambda x)^{\beta}\right]^{-2}}{1+(e-1)\left[1+(\lambda x)^{-\beta}\right]^{-b}}\right\}db\,d\beta\,d\lambda}{\int_{\beta}\int_{\alpha} b^{n}(e-1)^{n}\lambda^{\beta n}\beta^{n}\prod_{i=1}^{n}\left\{\frac{x^{\beta-1}\left[1+(\lambda x)^{-\beta}\right]^{-(b-1)}\left[1+(\lambda x)^{\beta}\right]^{-2}}{1+(e-1)\left[1+(\lambda x)^{-\beta}\right]^{-b}}\right\}p(b)\,p(\beta)\,p(\lambda)\,db\,d\beta\,d\lambda}$$

Similarly, the point Bayes estimate of other parameters $\beta$ and $\lambda$ can be calculated on the same lines. Bayes expressions for joint density and Bayes estimator cannot be solved analytically, so we can use suitable numerical methods to find the estimates of the parameters.

## 4. Simulation study

Here we study and present the behavior and accuracy of maximum likelihood estimates and Bayesian estimates of the AL-L distribution using Monte Carlo simulations. The rigorous simulation study is accomplished for sample sizes n = 10, 20, 30, 50, 100, 200, 300. The process is simulated N = 5,000 times each with specified sample size. The mean estimates have been documented with average biases (Bias) of the MLEs and mean squared error (MSE) alongside.

Now, the simulated results of the AL-L distribution for complete data are given in Table 2. The simulated results using type-II and type-I censoring schemes, approximately 10% censoring from right, are presented in Tables 3 & 4. Additionally, the simulation study is performed under Bayesian paradigm and calculate Bayes estimates (BE), standard deviation, lower and upper limits (LCL & UCL) of Bayes estimates for specified parametric values with varying sample sizes. The results are contained in Table 5. The simulated results for complete and censored data, in this section, show the consistency of the maximum likelihood estimates.

Table 2: Simulated results of the AL-L distribution with $\lambda = 1.5$, $\beta = 2$ and $b = 1$ for complete data.

| | Parameter | $n = 10$ | $n = 20$ | $n = 30$ | $n = 50$ | $n = 100$ | $n = 200$ |
|---|---|---|---|---|---|---|---|
| Mean | $\lambda$ | 5.310032 | 3.711223 | 3.25643 | 2.657116 | 1.70104 | 1.512933 |
| | $\beta$ | 9.201235 | 4.364507 | 2.866114 | 2.244649 | 2.060517 | 2.061203 |
| | $b$ | 5.051324 | 3.162487 | 2.592408 | 1.941005 | 1.163321 | 1.004716 |
| Bias | $\lambda$ | 3.810032 | 2.211223 | 1.75643 | 1.157116 | 0.20104 | 0.012933 |
| | $\beta$ | 7.201235 | 2.364507 | 0.866114 | 0.244649 | 0.060517 | 0.061203 |
| | $b$ | 4.051324 | 2.162487 | 1.592408 | 0.941005 | 0.163321 | 0.004716 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| MSE | $\lambda$ | 14.51635 | 4.889508 | 3.085046 | 1.338918 | 0.040417 | 0.000167 |
| | $\beta$ | 51.85778 | 5.590894 | 0.750154 | 0.059853 | 0.003662 | 0.003746 |
| | $b$ | 16.41323 | 4.676348 | 2.535762 | 0.885491 | 0.026674 | 2.22*10^-5 |

Table 3: Simulated results of the AL-L distribution with $\lambda = 1.5$, $\beta = 2$ and $b = 1$ for Type-II right censored (10%) data.

| | Parameter | $n = 10$ | $n = 30$ | $n = 50$ | $n = 100$ |
|---|---|---|---|---|---|
| Mean | $\lambda$ | 4.239841 | 3.507712 | 2.996627 | 2.342985 |
| | $\beta$ | 2.839403 | 1.940915 | 1.731637 | 1.600341 |
| | $b$ | 3.714467 | 2.797329 | 2.318902 | 1.847139 |
| Bias | $\lambda$ | 2.739841 | 2.007712 | 1.496627 | 0.842985 |
| | $\beta$ | 0.839403 | -0.05909 | -0.26836 | -0.39966 |
| | $b$ | 2.714467 | 1.797329 | 1.318902 | 0.847139 |
| MSE | $\lambda$ | 7.50673 | 4.030908 | 2.239893 | 0.710624 |
| | $\beta$ | 0.704598 | 0.003491 | 0.072018 | 0.159727 |
| | $b$ | 7.36833 | 3.230391 | 1.739503 | 0.717645 |

Table 4: Simulated results of the AL-L distribution with $\lambda = 2$, $\beta = 1.5$ and $b = 1$ for Type-I right censored (10%) data ($x_c$=1.58).

| | Parameter | $n = 10$ | $n = 30$ | $n = 50$ | $n = 100$ | $n = 200$ | $n = 300$ |
|---|---|---|---|---|---|---|---|
| Mean | $\lambda$ | 5.705933 | 4.127609 | 3.511428 | 2.625391 | 2.237289 | 2.147489 |
| | $\beta$ | 2.415194 | 1.769203 | 1.623119 | 1.552705 | 1.522534 | 1.517598 |
| | $b$ | 2.640286 | 1.725902 | 1.508481 | 1.213123 | 1.083808 | 1.053533 |
| Bias | $\lambda$ | 3.705933 | 2.127609 | 1.511428 | 0.625391 | 0.237289 | 0.147489 |
| | $\beta$ | 0.915194 | 0.269203 | 0.123119 | 0.052705 | 0.022534 | 0.017598 |
| | $b$ | 1.640286 | 0.725902 | 0.508481 | 0.213123 | 0.083808 | 0.053533 |
| MSE | $\lambda$ | 13.73394 | 4.526721 | 2.284415 | 0.391114 | 0.056306 | 0.021753 |
| | $\beta$ | 0.83758 | 0.07247 | 0.015158 | 0.002778 | 0.000508 | 0.00031 |
| | $b$ | 2.690537 | 0.526933 | 0.258553 | 0.045421 | 0.007024 | 0.002866 |

Table 5: The BE, SD, LCL, UCL and DIC of the AL-L distribution for simulated data.

| $n$ | | $\lambda = 2$ | $\beta = 1.5$ | $b = 1.1$ | $DIC$ |
|---|---|---|---|---|---|
| 20 | BE | 6.4132 | 1.3841 | 5.8801 | 126.981 |
| | SD | 2.2451 | 0.2635 | 2.2748 | |
| | LCL | 1.7379 | 0.9531 | 1.5082 | |
| | UCL | 9.7634 | 1.9598 | 9.7299 | |
| 30 | BE | 5.6270 | 1.0589 | 1.9870 | 138.7025 |
| | SD | 2.3767 | 0.2201 | 0.6838 | |

| | | | | | |
|---|---|---|---|---|---|
| | LCL | 1.5757 | 0.7322 | 0.7598 | |
| | UCL | 9.7481 | 1.5769 | 3.3453 | |
| 50 | BE | 6.6664 | 1.1421 | 3.4744 | 243.725 |
| | SD | 2.2819 | 0.1556 | 1.1082 | |
| | LCL | 1.9913 | 0.8797 | 1.2788 | |
| | UCL | 9.8558 | 1.5160 | 5.5353 | |
| 100 | BE | 2.7007 | 1.5518 | 1.4341 | 357.5196 |
| | SD | 1.6720 | 0.3443 | 0.7597 | |
| | LCL | 0.9622 | 1.0469 | 0.4908 | |
| | UCL | 7.3897 | 2.4229 | 3.4510 | |
| 300 | BE | 1.5575 | 1.6240 | 0.9032 | 1123.831 |
| | SD | 0.3374 | 0.1712 | 0.1872 | |
| | LCL | 1.0669 | 1.3106 | 0.6096 | |
| | UCL | 2.3415 | 1.9846 | 1.3248 | |

## 5. Empirical illustration of the AL-L distribution

In this section, three real data sets have been considered for illustration purpose. The AL-L distribution is compared with most competing lifetime distributions like, alpha power log-logistic distribution by [4], log-logistic distribution by [6], and Weibull distribution by [7].
The data set 1 is IQ scores for 52 non-white males hired by a large insurance company in 1971 and used by [1]. The data set 2 represents the survival times, in weeks, of 33 patients suffering from acute Myelogeneous Leukaemia. These data set have been studied by [2]. The data set 3 show the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England by [5]. For convenience, the data sets are given below

Data set 1: 91, 102, 100, 117, 122, 115, 97, 109, 108, 104, 108, 118, 103, 123, 123, 103, 106,102, 118, 100, 103, 107, 108, 107, 97, 95, 119, 102, 108, 103, 102, 112, 99, 116, 114, 102, 111, 104, 122, 103, 111, 101, 91, 99, 121, 97, 109, 106, 102, 104, 107, 955.

Data set 2:

65,156,100,134,16,108,121,4,39,143,56,26,22,1,1,5,65,56,65,17,16,17, 22,3,4,2,3,8,4,3,30,4,43.

Data set 3: 5.5, 5, 4.9, 6.4, 5.1, 5.2, 5.2, 5, 4.7, 4, 4.5, 4.2, 4.1, 4.56, 5.01, 4.7, 3.13, 3.12, 2.68, 2.77, 2.7, 2.36, 4.38, 5.73, 4.35, 6.81, 1.91, 2.66, 2.61, 1.68, 2.04, 2.08, 2.13, 3.8, 3.73, 3.71, 3.28, 3.9, 4, 3.8, 4.1, 3.9, 4.05, 4, 3.95, 4, 4.5, 4.5, 4.2, 4.55, 4.65, 4.1, 4.25, 4.3, 4.5, 4.7, 5.15, 4.3, 4.5, 4.9, 5, 5.35, 5.15, 5.25, 5.8, 5.85, 5.9, 5.75, 6.25, 6.05, 5.9, 3.6, 4.1, 4.5, 5.3, 4.85, 5.3, 5.45, 5.1, 5.3, 5.2, 5.3, 5.25, 4.75, 4.5, 4.2, 4, 4.15, 4.25, 4.3, 3.75, 3.95, 3.51, 4.13, 5.4, 5, 2.1, 4.6, 3.2, 2.5, 4.1, 3.5, 3.2, 3.3, 4.6, 4.3, 4.3, 4.5, 5.5, 4.6, 4.9, 4.3, 3, 3.4, 3.7, 4.4, 4.9, 4.9, 5.

Adequacy Model Package in R has been used to demonstrate the numerical findings while considering the above three data sets. The most familiar goodness of fit tests such as Akaike Information Criterion (AIC), Corrected Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC), Hannan-Quinn Information Criterion (HQIC), Anderson-Darling (A) and Cramér–von Mises (W*) are used for model comparisons. The smaller values of these statistics

show the better fit of the model. The aforementioned goodness of fit results is given in Tables 6, 7 and 8 for data sets 1, 2 and 3, respectively. Also, the MLEs and corresponding standard errors (SEs) of the AL-L and related models considered in this study are included in Table 9. Based on the documented results, it is shown that the proposed AL-L model outperforms the other competitive models.

**Table 6:** The different goodness of fit tests of the AL-L and other models for data 1.

| Model | *W** | *A* | *AIC* | *CAIC* | *BIC* | *HQIC* |
|---|---|---|---|---|---|---|
| AL-L | 0.452963 | 3.0249 | 435.8184 | 436.3184 | 441.6722 | 438.0626 |
| APQL | 0.641643 | 4.122535 | 451.0972 | 451.5972 | 456.9509 | 453.3414 |
| Log-Logistic | 0.630013 | 4.060168 | 448.1297 | 448.3746 | 452.0322 | 449.6258 |
| Weibull | 2.494228 | 12.78429 | 677.0319 | 677.2768 | 680.9344 | 678.528 |

Table 7: The different goodness of fit tests of the AL-L and other models for data 2.

| Model | *W** | *A* | *AIC* | *CAIC* | *BIC* | *HQIC* |
|---|---|---|---|---|---|---|
| AL-L | 0.08617 | 0.620128 | 314.6157 | 315.4432 | 319.1052 | 316.1262 |
| APQL | 0.641643 | 4.122535 | 451.0972 | 451.5972 | 456.9509 | 453.3414 |
| Log-Logistic | 0.128045 | 0.825984 | 316.0056 | 316.4056 | 318.9986 | 317.0127 |
| Weibull | 0.084869 | 0.602356 | 314.2222 | 314.6222 | 317.2152 | 315.2292 |

Table 8: The different goodness of fit tests of the AL-L and other models for data 3.

| Model | *W** | *A* | *AIC* | *CAIC* | *BIC* | *HQIC* |
|---|---|---|---|---|---|---|
| AL-L | 0.067517 | 0.408254 | 342.4638 | 342.6725 | 350.8011 | 345.8493 |
| APQL | 0.354949 | 2.231374 | 362.6701 | 362.8788 | 371.0075 | 366.0556 |
| Log-Logistic | 0.354956 | 2.231422 | 360.67 | 360.7734 | 366.2282 | 362.927 |
| Weibull | 0.09202 | 0.564363 | 341.414 | 341.5175 | 346.9723 | 343.671 |

Table 9: The maximum likelihood estimates and standard errors of the estimates of the AL-L and other models for all data sets.

| | | *Data* 1 | | *Data* 2 | | *Data* 3 | |
|---|---|---|---|---|---|---|---|
| Model | Parameter | *MLE* | *SE* | *MLE* | *SE* | *MLE* | *SE* |
| AL-L | $\hat{\lambda}$ | 0.015761 | 0.001151 | 0.008595 | 0.003313 | 0.184367 | 0.005427 |
| | $\hat{\beta}$ | 6.654566 | 0.709417 | 3.201192 | 2.14312 | 16.07149 | 3.431072 |
| | $\hat{b}$ | 38.62371 | 23.64596 | 0.204203 | 0.162781 | 0.287908 | 0.079374 |
| APQL | $\hat{\alpha}$ | 0.341157 | 0.298054 | 0.341157 | 0.298054 | 0.980591 | 1.678534 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | $\hat{\lambda}$ | 0.009769 | 0.000355 | 0.009769 | 0.000355 | 0.231617 | 0.028543 |
| | $\hat{\beta}$ | 13.84766 | 1.714096 | 13.84766 | 1.714096 | 7.064276 | 0.553503 |
| Log-Logistic | $\hat{\lambda}$ | 0.009407 | 0.000164 | 0.054048 | 0.015353 | 0.2313 | 0.005131 |
| | $\hat{\beta}$ | 13.50975 | 1.644279 | 1.096308 | 0.153999 | 7.064831 | 0.553508 |
| Weibull | $\hat{\lambda}$ | 0.693875 | 0.122225 | 0.715736 | 0.102147 | 4.959679 | 0.356014 |
| | $\hat{\beta}$ | 40.42888 | 8.816422 | 25.36335 | 5.817342 | 4.712113 | 0.091489 |

To further strengthen the numerical results, histograms of the data and empirical fitting of the PDF are displayed in Figure 4. It is pretty evident that the proposed model closely captures the behavior of all three data sets. Keeping in view the results and fittings, we suggest that the proposed AL-L model has a good fit and recommend for such type of phenomena.

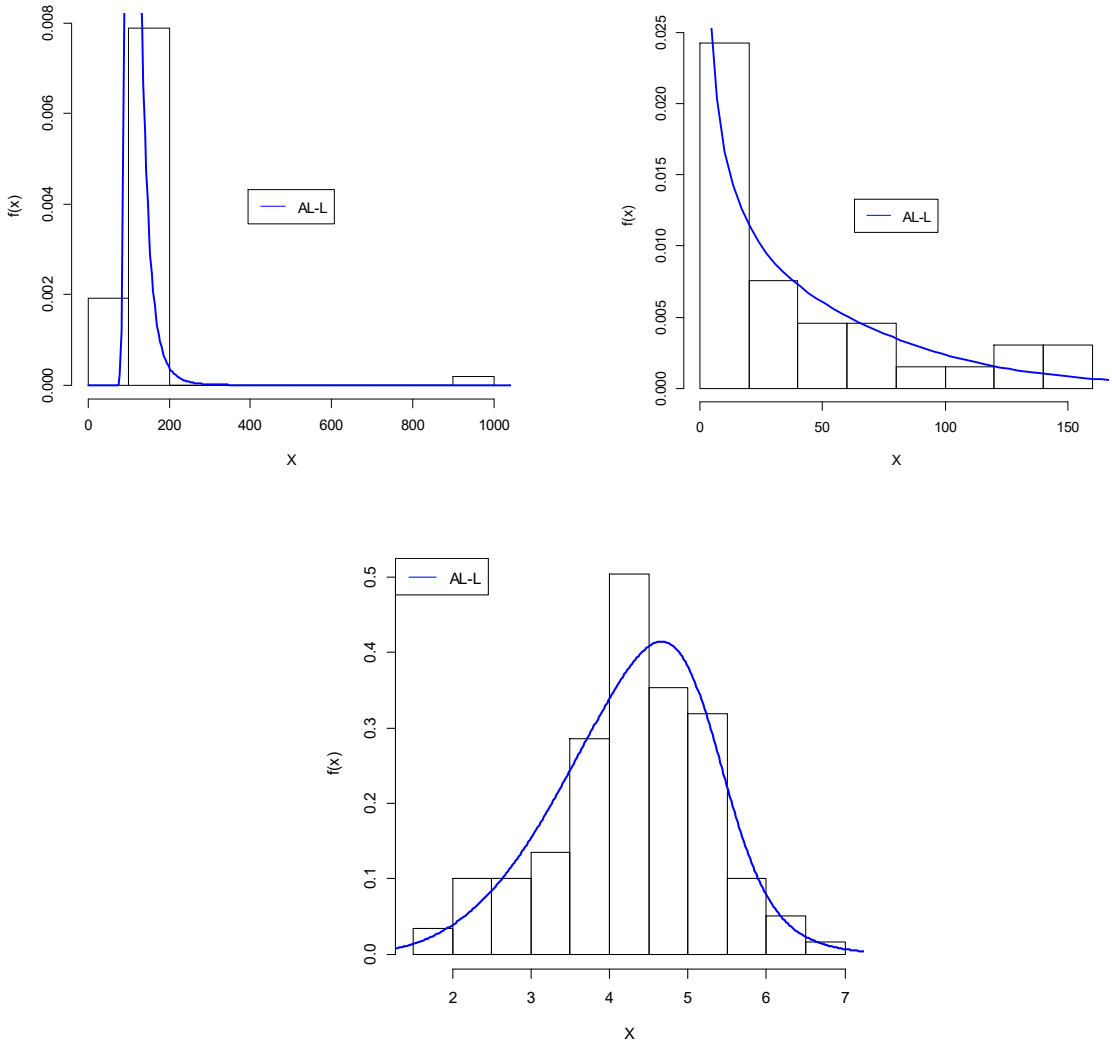



Figure 4: Histogram of data and fitted probability density function of the AL-L distribution for all three data sets.

## 6. Conclusion

An advanced and flexible family of distributions, named as Alternative family (AL) is defined and studied in this article. Numerous important properties of the AL family of distributions are derived. In order to explain the model for statistical inference, the estimation of the parameters is done by the method of maximum likelihood and Bayesian estimation for complete and censored phenomena. The Bayesian estimation of parameters is provided using square error loss function. Further, the applicability of the PFD is illustrated by choosing one sub AL-L distribution by means of real data sets. One of the interesting properties of the proposed model is that its CDF is in closed form. Further, based on the results of different comparative criteria, the proposed model has turned to be an appropriate model for explaining real phenomena. Keeping in view these striking choices of the proposed family of distributions, we hope that it will cover many applications in various filed of studies.

### References

[1]. Cordeiro, G. M., Alizadeh, M., Silva, R. B., & Ramires, T. G. (2017). A new wider family of continuous models: The extended Cordeiro and De Castro family. *Hacettepe Journal of Mathematics and Statistics*, *47*(4), 937-961.
[2]. Feigl, P., & Zelen, M. (1965). Estimation of exponential survival probabilities with concomitant information. *Biometrics*, 826-838.
[3]. Kumar, D., Singh, U., & Singh, S. K. (2015). A method of proposing new distribution and its application to Bladder cancer patients data. *J. Stat. Appl. Pro. Lett*, *2*(3), 235-245.
[4]. Nassar, M., Alzaatreh, A., Abo-Kasem, O., Mead, M., & Mansoor, M. (2018). A new family of generalized distributions based on alpha power transformation with application to cancer data. *Annals of Data Science*, *5*(3), 421-436.
[5]. Smith, R. L., & Naylor, J. (1987). A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, *36*(3), 358-369.
[6]. Shoukri, M. M., Mian, I. U. H., & Tracy, D. S. (1988). Sampling properties of estimators of the log-logistic distribution with application to Canadian precipitation data. *Canadian Journal of Statistics*, *16*(3), 223-236.
[7]. Weibull, W. (1951). A statistical distribution function of wide applicability. *Journal of applied mechanics*, *18*(3), 293-297.