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Estimating the Scale Parameter of the Rayleigh Distribution Using Moving Set Size Ranked Set Sampling Technique

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ABSTRACT

The Rayleigh distribution, named after the English Lord Rayleigh, is an important continuous distribution that has many real life applications. In this paper, the estimation of the scale parameter of this distribution is considered, using Moving set size ranked set sampling(MSSRSS), which is the newest version of Ranked set sampling technique(RSS). The suggested estimator is compared to its counterparts using simple random sampling(SRS) and other variations of Ranked set sampling . It turned that the suggested estimator using MSSRSS is more efficient than the corresponding estimator using SRS. Also, the appropriateness of the use this technique in practice is highlighted.

Keywords: Rayleigh distribution, Simple random sampling, Ranked set sampling, Moving extreme ranked set sampling, Moving set size ranked set sampling, Efficiency.

1. Introduction

In simple words**,** Statistics is the science of understanding a population based on a suitable portion(subset) of it. It is the science of choosing a representative sample from the population of interest, collecting, organizing, summarizing the collected information from the elements of the chosen sample, making inference about the population, and providing the accuracy of the inference. The accuracy of an inference is mainly depends on size of the chosen sample as well as on the method used to choose the sample. Simple, stratified, cluster and systematic random samplings are the most well-known sampling techniques. For more details, see, Elementary Survey Sampling by Schaeffer et al. (1996).

Ranked set sampling (RSS) and its main variations, such as Double and Multistage RSS, Median RSS, Moving extreme RSS, Steady state RSS, Moving set size RSS, etc., are newer sampling techniques. These techniques, if applicable, can be more efficient than the usual sampling techniques. For more details about these techniques see Al-Saleh and Matalqah (2025). Section 2 of this paper, contains Details about RSS and its main modifications. A description of the Rayleigh distribution and some of its main properties is given in Section 3. In section 4, the estimation of the scale parameter the Rayleigh distribution using Moving set size ranked set sampling(MSSRSS), is considered. The main conclusions are given in section 5.

2. Ranked Set Sampling Techniques

Ranked Set Sampling technique(RSS) was introduced by McIntyre (1952). For situations, in which ranking the elements w.r.t. the variable of interest by judgment , is cheaper and easier than a precise quantification, this technique has been found to be very useful. The sample obtained using this technique caries more information about the population than SRS. To select a RSS of set size m, we follow the following steps:

1. Choose m random samples of size m each, from the population of interest.
2. Rank the elements in each set by judgment w.r.t. to the variable of interest.
3. Quantify the i^{th} smallest observation from the i^{th} set, $i = 1,2,3,..., m$

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The obtained sample using the above three steps is a RSS of size m. If judgment ranking is as accurate as actual ranking(no ranking error) then the chosen sample consists of m *independent order statistics*: $X_{(1:m)}, X_{(2:m)}, ..., X_{(m:m)}$, where $X_{(i:m)}$ is the i^{th} minimum of the i^{th} sample of size m (the i^{th} order statistic)·

Clearly, to avoid ranking error, m should be small. Increasing the sample size should be done by increasing the number of cycles r, as given in step 4 below, not the set size m.

4. The entire cycle (Step 1, 2 and 3) may be repeated, if necessary, r times to produce a RSS of size $n = rm$.

It was shown by Takahasi and Wakimoto (1968) that the efficiency of the mean of RSS, $\hat{\mu}_{RSS}$, w.r.t. the mean of SRS, $\hat{\mu}_{SRS}$, (as estimators of the population mean μ) is at least 1 and at most $(m+1)/2$:

$$1 \le Eff(\hat{\mu}_{RSS}; \hat{\mu}_{SRS}) = \frac{\text{var}(\hat{\mu}_{SRS})}{\text{var}(\hat{\mu}_{RSS})} \le \frac{m+1}{2}.$$

They showed that highest efficiency of $\frac{m+1}{2}$ is attained if the underlying sampling distribution is uniform and 1 if it is a degenerate one.

Al-Salah and Al-Kadiri (2000) introduced a variation of RSS procedure to increase the efficiency without increasing the set size *m;* the new procedure was coined double ranked set sampling (DRSS). DRSS can be simply obtained by obtaining m RSS of size m each and apply the above procedure to obtain a RSS from them. Multistage RSS, (MSRSS) is a generalization of DRSS introduced by Al-Saleh and Al-Omari (2002). In MSRSS, if the number of stages goes to infinity, then the sample is called a steady-state ranked set sampling (SSRSS); see Al-Saleh (2004). Moving Extreme Ranked Set sampling(MERSS) is another modification of RSS introduced by Al-Odat and Al-Saleh (2001). In this method depends on the extreme values, (maximum or minimum), to obtain the sample. The method is easier to perform than RSS because we only need to be able to rank two extreme units. Median RSS was introduced by Muttlak (1997). Extreme RSS was proposed by Samawi et al. (1996).
Moving set size RSS(MSSRSS), the newest variation of RSS, was introduced by Al-Saleh and Adam (2024). More details about this technique is given in section 4.

3. Rayleigh Distribution

Rayleigh distribution was introduced by Lord Rayleigh in 1880. This distribution has a lot of real life applications, for example, it is widely used to model wind speed, wave heights, sound or light radiation, to check the lifetime of an object depending upon its age. For more details see
https://www.tutorialspoint.com/statistics/rayleigh_distribution.htm

This distribution is closely related to some of other important distributions such as normal, Chi-square and Weibull distributions. For more details, see: Ardianti and Sutarman (2018), Mkolesia1 and Shatalov (2017) and Sanku et al. (2014).

If a r.v. X has the Rayleigh distribution with scale paramete θ, Rayleigh(θ), then the probability density function (pdf) and cumulative distribution function (cdf) of X are:

$$f(x;\theta) = \frac{1}{\theta^2} x e^{-x^2/2\theta^2}, \quad x \ge 0, \theta > 0, \; F(x;\theta) = 1 - e^{-x^2/2\theta^2}.$$

The following properties and facts about this distribution were highlighted by Al-Saleh & Matalqah (2025).

(1) The mean is $\mu = E(X) = \theta\sqrt{\pi/2}$, median $= \theta\sqrt{\ln(4)}$, Mode $= \theta$ and $Var(X) = \sigma^2 = \frac{1}{2}(4-\pi)\theta^2$. Clearly, **Mode <Median<Mean**; thus the its pdf is right skewed.

(2) If Y is U(0,1), then $X = \theta\sqrt{-2\ln(Y)}$ is Rayleigh(θ). Thus, the uniform distribution can be easily used to simulate from Rayleigh.

(3) If X *and* Y are iid $N(0,\theta^2)$ then $R = \sqrt{X^2+Y^2}$ is Rayleigh(θ).

(4) If $X \sim \text{Exp}(\lambda)$ then $Y = \sqrt{X}$ is Rayleigh $\left(\sqrt{\lambda/2}\right)$.

(5) X is Rayleigh(θ) then X^2 *is* $Gamma(1, 2\theta^2)$.

(6) If $X_1, X_2, ..., X_n$ are iid Rayleigh (θ) then it can be shown that the Method-of-Moments Estimator (MME) of θ is

$$\hat{\theta}_{MME} = \overline{X}\sqrt{\tfrac{2}{\pi}} .$$

The maximum likelihood estimator (MLE) of θ is

$$\hat{\theta}_{MLE} = \sqrt{\sum_{i=1}^{n} X_i^2 / 2n} .$$

(See Ardianti and Sutarman (2018)).

Several authors have studied the Rayleigh distribution. More details can be found in Johnson et al. (1994), Abd-Elfattah et al. (2006). Bayesian inference of the parameters of Rayleigh distribution based RSS was considered by Kotb and Raqab (2017). Dey et al. (2016) estimated the parameter of Rayleigh distribution using Bayesian estimation approaches based on different sampling schemes namely, SRS, RSS, modified RSS and median RSS. Tahmasebi et al. (2017) made inferences on the Scale Parameter of bivariate Rayleigh distribution by RSS. Ardianti and Sutarman (2018) estimated the scale parameter of Rayleigh distribution using maximum likelihood estimation and Bayesian method for different loss functions. One new application of Rayleigh distribution was the application in Covid-19; it was done by Almongy et al. (2021) to model Covid-19 in Italy, Mexico, and the Netherlands.

Recently, estimating the scale parameter of the Rayleigh distribution using RSS and MERSS was considered by Al-Saleh and Matalqah (2025). It turned that the suggested estimators are more efficient than the corresponding ones using SRS.

In this paper, a newest variation of RSS, namely Moving Set Size RSS(MSSRSS) is to be considered for estimating the scale parameter of this distribution. The proposed estimators are compared to the corresponding ones using SRS.

4. Estimation the Scale Parameter using Moving Set Size RSS

One of the problems in using RSS is that we should be able to rank the obtained elements with respect to the variable of interest by judgment. This may be difficult for large m ; thus m should be small. Increasing the sample size should be done by increasing the number of cycles not m .

One new modification of RSS, called *"Moving Set Size Ranked Set Sampling*" (MSSRSS), was introduced by Al-Saleh and Adam (2024). This modification helps researchers to get a RSS of larger size using small set size, m.

For a given set size m, MSSRSS is a random sample that consists of m ranked set samples: The first RSS is of size 1, the second RSS is of set size 2,…, and the last RSS is of size m. The total size of the obtained MSSRSS is

$$n = \sum_{i=1}^{m} i = m(m+1)/2.$$

The required number of elements to take from the population to get this sample is

$$\sum_{i=1}^{m} i^2 = m(m+1)(2m+1)/6.$$

On the other hand, if we want to take a RSS of size n, the required number of elements to take from the population is $m^2(m+1)/2.$ Clearly,

$$m^2(m+1)/2 \geq m(m+1)(2m+1)/6.$$

For example, for $m=3$, we quantify 6 units: RSS of set size 1, RSS of set size 2 and RSS of set size 3. The elements of the MSSRSS are:

$$X_{(1:1)}, X_{(1:2)}, X_{(2:2)}, X_{(1:3)}, X_{(2:3)}, X_{(3:3)}.$$

In general, for a given m, the elements of MSSRSS are:

$$X_{(1:1)}, X_{(1:2)}, X_{(2:2)}, X_{(1:3)}, X_{(2:3)}, X_{(3:3)}, ..., X_{(1:m)}, X_{(2:m)}, ..., X_{(m:m)}\ .$$

Which can be displayed as:

$$\left\{ X_{(i:j)} : i=1,...,j,\ j=1,...,m \right\}$$

where, $X_{(i:j)}$ is the ith order statistic of a sample of size j.

Let $X_{(i:j)}, i=1,....,j, j=1,2,....,m$ be the elements of a MSSRSS of $n=m(m+1)/2$ from Rayleigh(θ). Consider the following estimator of θ, which is the usual overall mean(average) of the sample:

$$\hat{\theta}_{MSSRSS} = \frac{2\sum_{j=1}^{m}\sum_{i=1}^{j} X_{(i:j)}}{m(m+1)}\sqrt{\frac{2}{\pi}}.$$

$$E(\hat{\theta}_{MSSRSS}) = 2\sqrt{\frac{2}{\pi}}\left(\frac{1}{m(m+1)}\right)\sum_{j=1}^{m}\sum_{i=1}^{j}E(X_{(i:j)}) == 2\sqrt{\frac{2}{\pi}}\left(\frac{1}{m(m+1)}\right)\sum_{j=1}^{m} j\mu$$
$$= 2\theta\sqrt{\frac{\pi}{2}}\sqrt{\frac{2}{\pi}}\left(\frac{1}{m(m+1)}\right)\frac{m(m+1)}{2} = \theta.$$

Thus , $\hat{\theta}_{MSSRSS}$ is an unbiased estimator of θ.

$$Var\left(\hat{\theta}_{MSSRSS}\right) = \left(\frac{2}{m(m+1)}\right)^2 \frac{2}{\pi}\sum_{j=1}^{m}\sum_{i=1}^{j}\text{var}\left(X_{(i:j)}\right) = \left(\frac{8}{\pi m^2(m+1)^2}\right)\sum_{j=1}^{m}\sum_{i=1}^{j}\text{var}\left(X_{(i:j)}\right).$$

The pdf of $X_{(i:m)}$ is

$$f_{X_{(i:m)}}(x;\theta) = m\binom{m-1}{i-1}\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k \frac{1}{k+m-i+1} f(x;\frac{\theta}{\sqrt{k+m-i+1}}).$$

Thus, the pdf of the i^{th} order statistic of a random sample of size j is:

$$f_{X_{(i:j)}}(x;\theta) = j\binom{j-1}{i-1}\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k \frac{1}{k+j-i+1} f(x;\frac{\theta}{\sqrt{k+j-i+1}}).$$

Therefore,

$$E(X_{(i:j)}) = j\binom{j-1}{i-1}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k \frac{\theta}{\sqrt{(k+j-i+1)^3}},$$

$$E(X^2{}_{(i:j)}) = j\binom{j-1}{i-1}\sqrt{\frac{\pi}{2}}\sum_{k=0}^{i-1}\binom{i-1}{k}(-1)^k \frac{2\theta^2}{(k+j-i+1)^2},$$

$Var(X_{(i:j)}) = E(X^2{}_{(i:j)}) - \left(E(X_{(i:j)})\right)^2$, see Al-Saleh and Matalqah (2025).

Let $X_1, X_2, ..., X_n$ be iid Rayleigh(θ), where , $n = m(m+1)/2$. As mentioned in section 3, $\hat{\theta}_{SRS} = \overline{X}\sqrt{\frac{2}{\pi}}$ is an unbiased estimator of θ with

$$Var(\hat{\theta}_{SRS}) = \frac{4-\pi}{\pi n}\theta^2 = \frac{2(4-\pi)}{\pi m(m+1)}\theta^2. \text{ Thus,}$$

$$Eff(\hat{\theta}_{MSSRSS};\hat{\theta}_{SRS}) = \frac{Var(\hat{\theta}_{SRS})}{Var(\hat{\theta}_{MSSRSS})} = \frac{\frac{2(4-\pi)}{\pi m(m+1)}\theta^2}{\left(\frac{8}{\pi m^2(m+1)^2}\right)\sum_{j=1}^{m}\sum_{i=1}^{j}\text{var}\left(X_{(i:j)}\right)} == \frac{m(m+1)(4-\pi)\theta^2}{4\sum_{j=1}^{m}\sum_{i=1}^{j}\text{var}\left(X_{(i:j)}\right)}.$$

Clearly, $Eff(\hat{\theta}_{MSSRSS};\hat{\theta}_{SRS})$ is free of θ. $Eff(\hat{\theta}_{MSSRSS};\hat{\theta}_{SRS})$ for $m = 2,3,4,5$ are given in Table (4.1).

It can be seen that the efficiency is always larger than one and it is increasing in m. Thus, $\hat{\theta}_{MSSRSS}$ is more efficient than $\hat{\theta}_{SRS}$.

Table 1: $Eff(\hat{\theta}_{MSSRSS};\hat{\theta}_{SRS})$

m	$Var(\hat{\theta}_{SRS})$	$Var\left(\hat{\theta}_{MSSRSS}\right)$	$Eff(\hat{\theta}_{MSSRSS},;\hat{\theta}_{SRS})$
2	0.0912	0.0720	1.267
3	0.0455	0.0306	1.487
4	0.0273	0.0155	1.761
5	0.0182	0.0091	2.000

Al-Saleh and Matalqah (2025) estimated the scale parameter of Rayleigh distribution using RSS and MERSS(Max.). The efficiency $Eff(\hat{\theta}_{RSS};\hat{\theta}_{SRS})$ and $Eff(\hat{\theta}_{MERSS};\hat{\theta}_{SRS})$ are given in Table(4.2). Also, based on $Eff(\hat{\theta}_{RSS};\hat{\theta}_{SRS})$ and $Eff(\hat{\theta}_{MERSS};\hat{\theta}_{SRS})$, $Eff(\hat{\theta}_{RSS};\hat{\theta}_{MERSS})$ is obtained. It can be seen that $\hat{\theta}_{RSS}$ is more efficient than $\hat{\theta}_{SRS}$ and $\hat{\theta}_{MERSS}$ is more efficient than $\hat{\theta}_{SRS}$. Also, $\hat{\theta}_{RSS}$ is more efficient than $\hat{\theta}_{MERSS}$.

Table 2: $Eff(\hat{\theta}_{RSS};\hat{\theta}_{SRS})$, $Eff(\hat{\theta}_{MERSS};\hat{\theta}_{SRS})$, $Eff(\hat{\theta}_{RSS};\hat{\theta}_{MERSS})$

m	$Eff(\hat{\theta}_{RSS};\hat{\theta}_{SRS})$	$Eff(\hat{\theta}_{MERSS};\hat{\theta}_{SRS})$	$Eff(\hat{\theta}_{RSS};\hat{\theta}_{MERSS})$
2	1.458	1.404	1.038
3	1.898	1.764	1.076
4	2.325	2.090	1.112
5	2.744	2.390	1.148

Based on Table 1 and Table 2, it can be see that current suggest estimator based on MSSRSS, $\hat{\theta}_{MSSRSS}$, is little bit less efficient than $\hat{\theta}_{RSS}$ & $\hat{\theta}_{MERSS}$. However, the main advantage of using MSSRSS is that it more applicable because error in ranking could be smaller than that in other variations of RSS. Also, as shown above, the required number of elements to choose to obtain MSSRSS is extremely less than that to obtain RSS.

5. Concluding Remarks

In this paper, the use of a newer variation of RSS technique, Moving set size ranked sampling technique(MSSRSS), is investigated. This technique is used to estimate the scale parameter of the Rayliegh distribution. The suggested estimator obtained using the sample obtained using this method is studied. The properties of the suggested estimator (variance and efficiency) are obtained and compared to the corresponding one using SRS. The results indicate that the suggested estimator based on MSSRSS is more efficient than the corresponding estimator using SRS. The obtained estimator is little bit less efficient than the estimator based on RSS or MERSS. However, this method is easier to use in practice that the other RSS techniques.

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