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On the Exponentiated Unit Gompertz Distribution and its Bivariate Extension

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ABSTRACT

In this article, we introduce an extension for the Gompertz distribution, which is called the exponentiated unit Gompertz distribution depending on the proportional hazard model. The density, survival, hazard, and reversed hazard functions. In addition, we introduce the moments, quantiles, mode, order statistics, and joint order statistics. We estimate the unknown parameters of the new distribution by maximum likelihood estimation, and an approximate variance-covariance matrix has been obtained. Some simulations have been performed to see the performance of the MLEs. Depending on three data sets, we compared the fitting of the exponentiated unit Gompertz distribution with some lifetime distributions that are used to describe bounded data. By using the Farlie–Gumbel–Morgenstern coupla, we introduce a new bivariate exponentiated unit Gompertz distribution and study its statistical properties. Maximum likelihood and inference functions for margins approaches are used to estimate the bivariate model's unknown parameters. Finally, we illustrate the importance of the proposed bivariate model using a real data set in the medical field.

Keywords: Unit Gompertz distribution (UG); Exponentiated unit Gompertz distribution (EUG); Moment; Quantile; Log-concave; Log-convex; Maximum Likelihood Estimators; bivariate exponentiated unit Gompertz distribution (BEUG); inference functions for margins (IFM).

1. Introduction

There are various distributions for modeling lifetime data in the statistics literature. However, many, if not all, of these distributions are the result of theoretical considerations, actual implementations, or a combination of the two. For instance, the gamma distribution has better traits and a better fit than the exponential distribution from which it is derived. Weibull and many other distributions are subject to the same reasoning.

Finding distributions with properties that allow statisticians to utilize them to explain the lifetime of various devices is one topic of interest for statisticians. These distributions include those with constant failure rates, such as the exponential distribution, those with increasing failure rates, such as the Gompertz distribution, and those with dropping failure rates, such as the Weibull distribution.

In 1825, Benjamin Gompertz introduced the Gompertz distribution in connection with human mortality and actuarial tables. Since then, considerable attention has been received from demographers and actuaries. This distribution is a generalization of the exponential distribution and has many real-life applications, especially in medical and actuarial studies. It has some nice relations with some of the well-known distributions, such as exponential, double exponential, Weibull, extreme value (Gumbel distribution), or generalized logistic distribution by Willekens (2001). Researchers have tried to make several generalizations of the Gompertz distribution. Notable among them are: Bemmaor (1994) proposed the shifted Gompertz distribution, Roy and

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Adnan (2012) introduced the wrapped generalized Gompertz distribution, El Gohary et al (2013) introduced generalized Gompertz distribution (GGD) and study some of its statistical properties and some nice physical interpretations also, Jafari et al (2014) introduced four parameters beta-Gompertz distribution, El-Damcese et al (2015) proposed the odd generalized exponential Gompertz distribution, Jafari and Tahmasebi (2016) introduced the generalized Gompertz-power series distributions and Benkhelifa (2017) introduced the beta generalized Gompertz distribution, Mazucheli et al (2019) introduced the bounded distribution, called Unit Gompertz distribution, which arises from a logarithmic transformation in the Gompertz distribution and studied some of its properties.

Many authors proposed and studied bivariate distributions, which have wide applications in the fields of reliability, life testing, competing risk models, stress-strength models, sports, medical, engineering, drought, finance, weather, and others. In the last few years, several ways of presenting new bivariate distributions based on different copula functions. Al- Khedhairi and El-Gohary (2008) presented a new class of bivariate Gompertz distributions using shock models. El-Sherpieny et al (2013) introduced a new bivariate generalized Gompertz distribution by using the method of Marshall and Olkin (1967), El-Damcese et al (2015) proposed a bivariate exponentiated generalized Weibull-Gompertz distribution. Moreover, Muhammed and Mandouh (2024) inroduced the bivariate unite Gompertz distribution.

We are motivated to introduce the exponentiated unit Gompertz distribution by using a proportional hazard model and deriving some properties of this distribution because it is capable of modeling the bathtub shaped hazard rate, and it can be viewed as a suitable model for fitting the skewed data which may not be properly fitted by other common distributions and can also be used in a variety of problems in different areas such as environmental studies and industrial reliability and survival analysis. In addition to presenting bivariate exponentiated unit Gompertz distribution (BEUG) depending on Farlie–Gumbel–Morgenstern (FGM) coupla.

The rest of the paper is organized as follows. In Section 2, we introduce the exponentiated unit Gompertz distribution and discuss the basic properties of this distribution. In Section 3, maximum likelihood estimators of the unknown parameters along with the expected Fisher information matrix are obtained. Monte Carlo simulations are conducted in Section 4 to investigate the performance of the maximum likelihood estimators and the asymptotic confidence intervals of the parameters. The analysis of two real data sets has been presented in Section 5. In Section 6 we obtain the bivariate exponentiated unit Gompertz distribution. In Section 7 we discuss some different properties of the bivariate exponentiated unit Gompertz distribution. Parameter estimation methods are discussed in Section 8 . Application of one real data set is discussed in Section 9. Finally, the conclusion of some remarks for the EUG and BEUG distributions is addressed in Section 10.

2. Exponentiated Unit Gompertz Distribution (EUG)

In this section, we describe the new bounded distribution, which arises from the unit Gompertz distribution by using a proportional hazard model.

Let Y be a non-negative random variable with a unit Gompertz distribution, then its probability density function is given by:

$$f_{UG}(y)= \alpha\beta y^{-(\beta+1)} e^{-\alpha(y^{-\beta}-1)} \tag{2.1}$$

where $0 < y < 1, \beta > 0$, and $\alpha > 0$ are shape and scale parameters, respectively.

The corresponding probability function and hazard function are given, respectively, by:

$$F_{UG}(y) = e^{-\alpha(y^{-\beta}-1)} \quad (2.2)$$

$$h_{UG}(y) = \frac{\alpha\beta y^{-(\beta+1)}\, 1-e^{-\alpha(y^{-\beta}-1)}}{1-e^{-\alpha(y^{-\beta}-1)}} \quad (2.3)$$

Cox (1972) defined the proportional hazard formula as follows

$$\bar{F}_{\theta}(x) = (\bar{F}_{b}(x))^{\theta} \quad (2.4)$$

By using (2.4), we obtained the pdf and cdf for the exponentiated unit Gompertz Distribution (EUG), respectively :

$$f_{EUG}(x) = \alpha\beta\theta x^{-(\beta+1)}\, e^{-\alpha(x^{-\beta}-1)} \left(1 - e^{-\alpha(x^{-\beta}-1)}\right)^{\theta-1} \quad (2.5)$$

and

$$F_{EUG}(x) = 1 - \left(1 - e^{-\alpha(x^{-\beta}-1)}\right)^{\theta} \quad (2.6)$$

where $0 < x < 1, \beta > 0, \theta > 0,\ \alpha > 0$, such that β, θ are shape parameters, and α scale parameter

lemma1:The density of the EUG distribution is

1- Log concave if $\theta \geq 1$
2- Log convex if $\theta < 1$ and $x < x_0$ and log-concave if $x > x_0$, where the x_0 is the solution of the equation (2.7) with respect to x;

Proof :

The logarithm of the density function of EUG is

$$\log f(x) = \log(\alpha) + \log(\beta) + \log(\theta) - \alpha\left(x^{-\beta} - 1\right) - (\beta+1)\log(x) + (\theta-1)\log\left[1 - e^{-\alpha(x^{-\beta}-1)}\right] \quad (2.7)$$

Differentiating (2.5) twice with respect to x, then we get

$$(\theta-1)\,\vartheta_{(x;\alpha,\beta)} - \frac{\alpha\beta(\beta+1)}{x^{\beta+1}}\ , \quad (2.8)$$

Where

$$\vartheta_{(x;\alpha,\beta)} = \alpha\beta\, x^{-(\beta+1)}\, e^{-\alpha(x^{-\beta}-1)} \left(1 - e^{-\alpha(x^{-\beta}-1)}\right)^{-2} .[1 - e^{-\alpha(x^{-\beta}-1)}(\beta+1)x^{-1} - \left(1 - e^{-\alpha(x^{-\beta}-1)}\right)x^{-(\beta+1)} + \alpha\beta e^{-\alpha(x^{-\beta}-1)}x^{-(\beta+1)}]$$

Then we have the following cases:

1- When $\theta \geq 1$, from (2.8), we get
$\frac{d^2}{dx^2} logf(x) < 0$ then the density of EUG is log-concave if $x > x_0$
2- When $\theta < 1$, from (2.8) we get
$\frac{d^2}{dx^2} logf(x) > 0$ then the density of EUG is log-convex if $x < x_0$, where x_0 is the solution of Equation (2.5) for x.

Figures 1,2 show the various curves for the pdf and cdf of the EUG distribution with various values of α, θ, and β. Figure 1 indicates that the EUG distribution can be increasing, unimodal, decreasing, bathtub-shaped, and positively right-skewed.

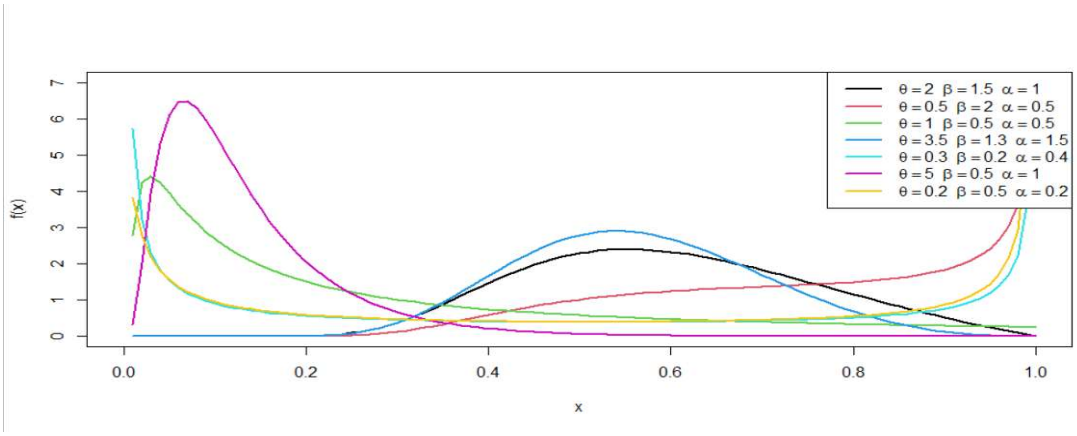


Figure 1:. Probability density function for the EUG distribution for different values of the parameters.

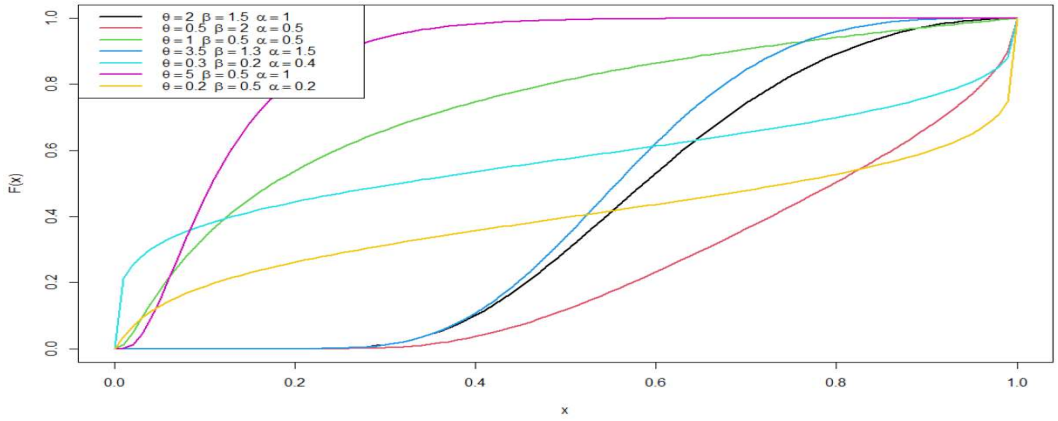


Figure 2: The cumulative density function for the EUG distribution for different values of the parameters.

2.1 Some Reliability Measures for The EUG Distribution

We study Reliability, hazard, and reversed hazard functions, which make the distribution suitable for fitting various lifetime phenomena.

2.1.1 Reliability Function for EUG Distribution

The survival function for the failure time X for the EUG Distribution is defined as:

$$R(x) = p(X > x) = \left(1 - e^{-\alpha(x^{-\beta}-1)}\right)^{\theta} \quad (2.9)$$

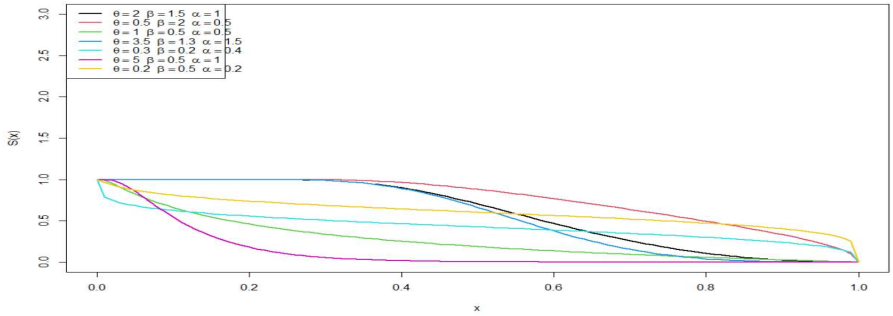


Figure 3: The reliability function for the EUG distribution for different values of the parameters.

Figure 3 shows the various curves for the reliability of EUG distribution with various values of α, θ, and β for fixed $x, which$ means the probability of survival up to time x

2.1.2 Hazard Rate Function for EUG Distribution

The hazard function (or failure rate) is important for lifetime modeling in reliability studies, given as

$$h(x) = \lim_{\Delta x \to 0} \frac{p(X<x+\Delta x | X>x)}{\Delta x} = \frac{f(x)}{R(x)}$$

That is $h(x)\Delta x$ represents the conditional probability that an individual of age x will fail within the interval $(x, x+\Delta x)$. The hazard rate for EUG is defined as :

$$h(x) = \frac{\alpha\beta\theta x^{-(\beta+1)} e^{-\alpha(x^{-\beta}-1)}}{1- e^{-\alpha(x^{-\beta}-1)}} \quad (2.10)$$

The nature of the failure rate (FR) will be discussed by the following [Glaser et al. (1980)]

Lemma 2.

- if $\eta'(x) > 0$ for all $x > 0$, then FR is an increasing FR (IFR).
- if $\eta'(x) < 0$ for all $x > 0$, then FR is decreasing FR (DFR).

Suppose there exists $x_0 > 0$ such that

$\eta'(x) < 0$ for all $x \in (0, x_0)$, $\eta'(x_0) = 0$ and $\eta'(x) > 0$ for all $x > x_0$,

i- If $\lim_{x\to 0^+} f(x) = \infty$, then bathtub-shaped FR (BFR).

ii- If $\lim_{x\to 0^+} f(x) = 0$, then IFR.

Suppose there exists $x_0 > 0$ such that

$\eta'(x) > 0$ for all $x \in (0, x_0)$, $\eta'(x_0) = 0$ and $\eta'(x) < 0$ for all $x > x_0$,

i. If $\lim_{x\to 0^+} f(x) = 0$, then upside-down bathtub-shaped FR (UFR).

ii. ii. $\lim_{x\to 0^+} f(x) = \infty$, then DFR.

Theorem1:

The FR of the EUG distribution, which is given by (2.10), is

1- Increasing FR if $\theta \geq 1$.

2- Bathtub-shaped FR with a change point x_0 , defined in (2.9), if $\theta < 1$.

Proof:

$$\eta(x) = \frac{\alpha\beta(\theta-1)x^{-(\beta+1)} e^{-\alpha\left(x^{-\beta}-1\right)}}{1-e^{-\alpha\left(x^{-\beta}-1\right)}} - \alpha\beta x^{-(\beta+1)} + \frac{\beta+1}{x} \quad (2.11)$$

Differentiating (2.11) for x, we get

$$\eta'(x) = \alpha\beta(\theta-1)\,\psi_{(x;\theta,\alpha,\beta)} - (\beta+1)x^{-2} \quad (2.12)$$

Where $\psi_{(x;\theta,\alpha,\beta)}$

$$= x^{-(\beta+1)} e^{-\alpha\left(x^{-\beta}-1\right)} x^{-(\beta+2)} \alpha\beta(\beta+1)\left(1-e^{-\alpha\left(x^{-\beta}-1\right)}\right)^{-2}$$

$$.\left[\left(1-e^{-\alpha\left(x^{-\beta}-1\right)}\right)\left(\alpha\beta x^{-(\beta+1)} - (\beta+1)x^{-1}\right) + \alpha\beta x^{-(\beta+1)} e^{-\alpha\left(x^{-\beta}-1\right)}\right]$$

By using Lemma 2,

we get the following cases:

1- When $\theta \geq 1$, we find from (2.10) that $\eta'(x) > 0$, for all $x > 0$, then the FR of EUG distribution is IFR.

2- When $\theta < 1$, we find from (2.10) that $\eta'(x) < 0$, and $\lim_{x\to 0^+} f(x) = \infty$, then the FR of EUG distribution is BFR.

2.1.3 Reversed Hazard Rate Function for EUG Distribution

The reversed hazard function is defined as

$$r(x) = \lim_{\Delta x \to 0} \frac{p(X<x-\Delta x | X>x)}{\Delta x} = \frac{f(x)}{F(x)}$$

That is $r(x)\Delta\, x$ defines the conditional probability of a failure of a unit in $(x - \Delta\, x, x)$ given that the failure had occurred in$[0, x]$.

$$r(x) = \frac{\alpha\beta\theta x^{-(\beta+1)}\, e^{-\alpha(x^{-\beta}-1)} \left(1- e^{-\alpha(x^{-\beta}-1)}\right)^{\theta-1}}{1- \left(1- e^{-\alpha(x^{-\beta}-1)}\right)^{\theta}} \tag{2.13}$$

It is easy to verify that X is a decreasing reversed hazard rate (DRHR) if and only if $F\ (x)$ is log-concave.

Respectively. Figures 4 and 5 show the various curves for the hazard and reversed hazard rate functions of the EUG distribution with various values of α, θ, and β.

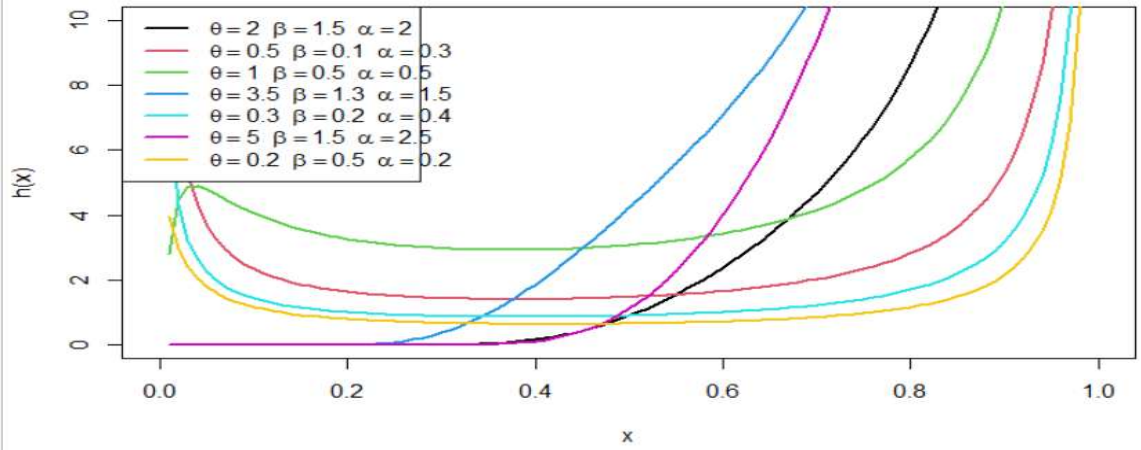


Figure 4: The plot of the hazard rate function for different values of the parameters.

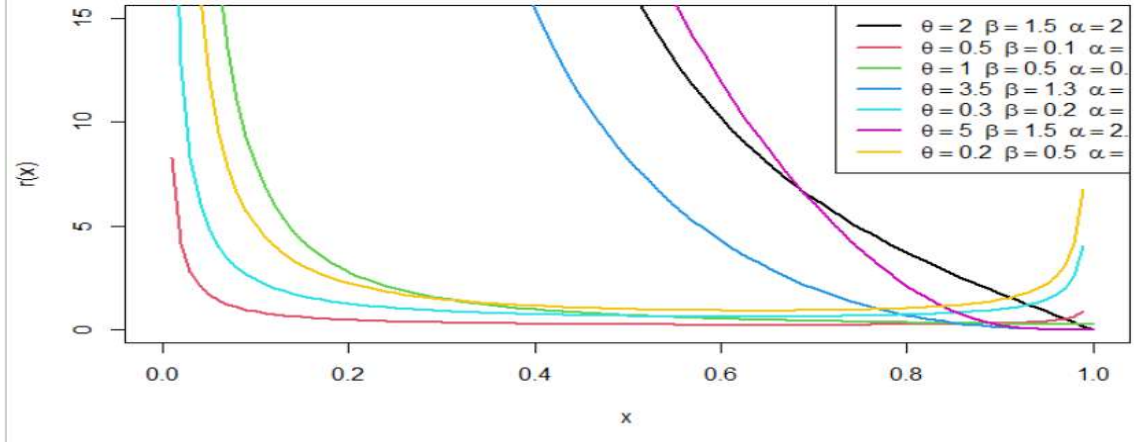


Figure 5: The plot of the reversed hazard rate function for different values of the parameters.

2.2 Statistical Properties for The EUG Distribution

This subsection is devoted to some statistical properties of the EUG distribution, such as mode, median, quantile function, and the moments about the origin.

2.2.1 Mode

The first derivative with respect to x is given by.

$$\alpha\beta\, x^{-(\beta+1)} - \frac{(\beta+1)}{x} - \frac{\alpha\beta\, x^{-(\beta+1)}(\theta-1)\, e^{-\alpha\left(x^{-\beta}-1\right)}}{1-e^{-\alpha\left(x^{-\beta}-1\right)}} = 0 \quad (2.14)$$

Since this is a nonlinear equation, there is no analytic solution in x. As a result, we must solve it numerically using mathematical software.

The mode of UG distribution can be obtained from (2.14) by setting $\theta = 1$ as

$$x_{mod_{UG}} = \left(\frac{\alpha\beta}{1+\beta}\right)^{\frac{1}{\beta}} \quad (2.15)$$

2.2.2 Quantile Function for the EUG Distribution

The EUG distribution's quantile is as follows:

$$x_u = \left(\frac{\alpha}{\alpha - \ln\left[1-(1-u)^{\theta^{-1}}\right]}\right)^{\beta^{-1}};\ 0 < u < 1 \quad (2.16)$$

The quantile function has a specific case called the median. when u =0.5

$$x_{0.5} = \left(\frac{\alpha}{\alpha - \ln\left[1-(1-0.5)^{\theta^{-1}}\right]}\right)^{\beta^{-1}} \quad (2.17)$$

And the interquartile range (IQR) is given as

$$IQR = \left(\frac{\alpha}{\alpha - ln\left[1-(1-0.75)^{\theta^{-1}}\right]}\right)^{\beta^{-1}} - \left(\frac{\alpha}{\alpha - ln\left[1-(1-0.25)^{\theta^{-1}}\right]}\right)^{\beta^{-1}} \quad (2.18)$$

2.2.3 Moments for the EUG Distribution

If the random variable X has a EUG distribution, then, the r^{th} moment around zero can be obtained as:

$$\mu'_r(x) = \alpha\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+2j}\,\Gamma(\theta)\Gamma\left(1-\frac{r}{\beta}\right)\Gamma(1+j)}{i!j!(\alpha i+\alpha)^{j+1}\Gamma(\theta-i)\Gamma\left(1-\frac{r}{\beta}-j\right)} \quad r = 1,2,\ldots \quad (2.19)$$

Where $\frac{r}{\beta} < 1, \theta > i, j < 1 - \frac{r}{\beta}$.

Proof.

$$\mu'_r(x) = E(X^r) = \int_0^1 x^r f(x)dx = \alpha\beta\theta \int_0^1 x^{-\beta-1+r} e^{-\alpha(x^{-\beta}-1)} \left(1 - e^{-\alpha(x^{-\beta}-1)}\right)^{\theta-1} dx \quad (2.20)$$

Since $0 < \left(1 - e^{-\alpha(x^{-\beta}-1)}\right)^{\theta-1} < 1$ for $x > 0$, then by using the binomial series expansion of $\left(1 - e^{-\alpha(x^{-\beta}-1)}\right)^{\theta-1}$ given by

$$\left(1 - e^{-\alpha(x^{-\beta}-1)}\right)^{\theta-1} = \sum_{i=0}^{\infty} \frac{(-1)^i\,\Gamma(\theta)}{i!\Gamma(\theta-i)}\, e^{-\alpha(x^{-\beta}-1)i} \quad (2.21)$$

Substituting from (2.21) into (2.20), we get

$$\mu'_r(x) = \alpha\beta\theta \sum_{i=0}^{\infty} \frac{(-1)^i\,\Gamma(\theta)}{i!\,\Gamma(\theta-i)} \int_0^1 x^{-\beta-1+r} e^{-\alpha(x^{-\beta}-1)} e^{-\alpha(x^{-\beta}-1)i}\, dx$$

Using transformation$y = x^{-\beta} - 1$, one gets

$$\mu'_r(x) = \alpha\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+2j}\,\Gamma(\theta)\Gamma(1-\frac{r}{\beta})}{i!j!(\alpha i+\alpha)^{j+1}\Gamma(\theta-i)\Gamma(1-\frac{r}{\beta}-j)} \int_0^{\infty} [(\alpha i+\alpha)y]^j e^{-\alpha(x^{-\beta}-1)} e^{-(\alpha i+\alpha)y}$$
$$.(\alpha i+\alpha)dy$$

$$\mu'_r(x) = \alpha\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+2j}\,\Gamma(\theta)\Gamma\left(1-\frac{r}{\beta}\right)\Gamma(1+j)}{i!j!(\alpha i+\ )^{j+1}\Gamma(\theta-i)\Gamma\left(1-\frac{r}{\beta}-j\right)} \quad r = 1,2,\ldots$$

Where $\frac{r}{\beta} < 1, \theta > i, j < 1 - \frac{r}{\beta}$.

2.3 Distributions and Moments of Order Statistics from the EUG Distribution

Order statistics have been extensively applied in many fields of statistics, such as reliability and life testing. Let $X_1, X_2, X_3, \ldots X_n$ be independent and identically distributed random variables with their corresponding continuous distribution function $F(.)$. Let $X_{1:n}, X_{2:n}, X_{3:n}, \ldots X_{n:n}$ be the corresponding ordered random sample from a population of size n. The pdf of k^{th} the order statistic for the EUG distribution is defined as

$$f_{k:n}(x) = C_{n,k}\alpha\beta\theta x^{-(\beta+1)}\,[1 - a_1(x;\alpha,\beta)][a_1(x;\alpha,\beta)]^{\theta(1+n-k)-1} .\left[1 - \left(a_1(x;\alpha,\beta)\right)^{\theta-1}\right]^{k-1} \quad (2.22)$$

Where $C_{n,k} = \frac{n!}{k!(n-k)!}; a_1(x;\alpha,\beta) = \left(1 - e^{-\alpha(x^{-\beta}-1)}\right)$

In particular, the pdf of the smallest order statistics is obtained by substituting $k = 1$ in (2.22) as follows

$$f_{1:n}(x) = n\alpha\beta\theta {x_{(1)}}^{-(\beta+1)}\,[1 - a_1(x_{(1)};\alpha,\beta)][a_1(x_{(1)};\alpha,\beta)]^{\theta n-} \quad (2.23)$$

Also, the pdf of the largest order statistics is obtained by substituting $k = n$ in (2.22) as follows

$$f_{n:n}(x) = \alpha\beta\theta {x_{(n)}}^{-(\beta+1)}\left[1 - a_1(x_{(n)};\alpha,\beta)\right]\left[a_1(x_{(n)};\alpha,\beta)\right]^{\theta-1}\left[1 - \left(a_1(x_{(n)};\alpha,\beta)\right)^{\theta}\right]^{n-1} \quad (2.24)$$

The joint pdf of $x_{(r)}\ and\ x_{(s)}$, $1 \le r < s \le n$, for a sample of size n is defined as

$$f(x,y) = C_{n,r,s}\left[1 - [\,a_1(x;\alpha,\beta)]^{\theta}\right]^{r-1}[a_1(y;\alpha,\beta)]^{\theta(n-s)}\,[1 - a_1(x;\alpha,\beta)]\,[a_1(x;\alpha,\beta)]^{\theta-1} .[1 - a_1(y;\alpha,\beta)]\ [a_1(y;\alpha,\beta)]^{\theta-1}\left\{\left[1 - \ [a_1(y;\alpha,\beta)]^{\theta}\right] - \left[1 - \ [a_1(x;\alpha,\beta)]^{\theta}\right]\right\}^{s-r-1} \quad (2.25)$$

Where $x = x_{(r)}, y = x_{(s)}, C_{n,r,s} = \frac{n!}{r!(s-r-1)!(n-s)!}; a_1(x;\alpha,\beta) = \left(1 - e^{-\alpha(x^{-\beta}-1)}\right);$ $a_1(y;\alpha,\beta) = \left(1 - e^{-\alpha(y^{-\beta}-1)}\right); \psi(x;y;\alpha,\beta,\theta) = (\alpha\beta\theta)^2 y^{-(\beta+1)}x^{-(\beta+1)}$

2.3.1 Moments of Order Statistics for EUG Distribution

Moments and product moments of order statistics around zero will be discussed in the following subsection.

The r^{th}moment about zero ${\mu^r}_{k:n}$ of k^{th} order statistic $X_{(k)}$is given as:

$${\mu^r}_{k:n} = C_{n,k}\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+j+2l}\,\Gamma(k)\Gamma(\theta(1+n-k+i)-i)\Gamma\left(r-\frac{1}{\beta}\right)\Gamma(l+1)}{i!j!l!\Gamma(k-i)\Gamma(\theta(1+n-k+i)-i-j)\Gamma\left(r-\frac{1}{\beta}-l\right)(j+1)(\alpha j+\ )^{l}} \quad (2.26)$$

Proof.

Start with ${\mu^r}_{k:n} = E(X^r) = C_{n,k}\int_0^1 x^r[F(x)]^{k-1}[1 - F(x)]^{n-k}f(x)\,dx$

$${\mu^r}_{k:n} = C_{n,k}\alpha\beta\theta\sum_{i=0}^{\infty}\frac{(-1)^i\,\Gamma(k)}{i!\,\Gamma(k-i)}\int_0^1 x^{-(\beta+1)+r}\,e^{-\alpha(x^{-\beta}-1)}[a_1(x;\alpha,\beta)]^{\theta(1+n-k)-1+i\theta-i}\,dx$$

$${\mu^r}_{k:n} = C_{n,k}\alpha\beta\theta\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\,\Gamma(\theta(1+n-k+i)-i)\Gamma(k)}{i!j!\Gamma(\theta(1+n-k+i)-i-j)\Gamma(k-i)}\int_0^1 x^{-(\beta+1)+r}\,e^{-\alpha(x^{-\beta}-1)[j+1]}\,dx$$

Using transformation $y = x^{-\beta} - 1$, one gets

$$\mu^{r}{}_{k:n} = C_{n,k}\alpha\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}\,\Gamma(\theta(1+n-k+i)-i)\Gamma(k)}{i!j!\Gamma(\theta(1+n-k+i)-i-j)\Gamma(k-i)}$$
$$. \int_0^{\infty}(1+y)^{\left(\frac{-1}{\beta}+r\right)-1}\, e^{-(\alpha j+\alpha)y}\; dy$$
$$\mu^{r}{}_{k:n} = C_{n,k}\alpha\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+j+2l}\,\Gamma(k)\Gamma(\theta(1+n-k+i)-i)\Gamma\left(r-\frac{1}{\beta}\right)}{i!j!l!\Gamma(k-i)\Gamma(\theta(1+n-k+i)-i-j)\Gamma\left(r-\frac{1}{\beta}-l\right)(\alpha j+\alpha)^{l+1}}$$
$$. \int_0^{\infty}[(\alpha j+\alpha)y]^{l}\, e^{-(\alpha j+\alpha)y}\; (\alpha j+\alpha)dy$$
$$\mu^{r}{}_{k:n} = C_{n,k}\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+j+2l}\,\Gamma(k)\Gamma(\theta(1+n-k+i)-i)\Gamma\left(r-\frac{1}{\beta}\right)\Gamma(l+1)}{i!j!l!\Gamma(k-i)\Gamma(\theta(1+n-k+i)-i-j)\Gamma\left(r-\frac{1}{\beta}-l\right)(j+1)(\alpha j+\alpha)^{l}}$$

Where $r < \frac{1}{\beta}, \theta(1+n-k+i)-i > j, l < r-\frac{1}{\beta}$.

The r^{th}moment about zero $\mu^{r}{}_{k:n}$ of k^{th} order statistic $X_{(1)}$is

$$\mu^{r}{}_{1:n} = n\theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+j+2l}\,\Gamma(\theta(n+i)-i)\Gamma\left(r-\frac{1}{\beta}\right)\Gamma(l+1)}{i!j!l!\Gamma(1-i)\Gamma(\theta(n+i)-i-j)\Gamma\left(r-\frac{1}{\beta}-l\right)(j+1)(\alpha j+\alpha)^{l}}$$

The r^{th}moment about zero $\mu^{r}{}_{k:n}$ of k^{th} order statistic $X_{(n)}$is

$$\mu^{r}{}_{n:n} = \theta \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty} \frac{(-1)^{i+j+2l}\,\Gamma(n)\Gamma(\theta(1+i)-i)\Gamma\left(r-\frac{1}{\beta}\right)\Gamma(l+1)}{i!j!l!\Gamma(n-i)\Gamma(\theta(1+i)-i-j)\Gamma\left(r-\frac{1}{\beta}-l\right)(j+1)(\alpha j+\alpha)^{l}}\ .$$

2.3.2 Product Moments of Joint Order Statistics for EUG Distribution

The k^{th} and L^{th} product moments about zero $\mu^{(K,L)}{}_{r:n}$ of $X_{(r)}, and\ X_{(s)}$ is given as:

$$\mu^{(K,L)}{}_{r:n} = \psi(x;y;\alpha,\beta,\theta)\ C_{n,r,s}$$
$$. \int_0^1\int_0^1 x^k y^l \left[1-\ [a_1(x;\alpha,\beta)]^{\theta}\ \right]^{r-1}\left[\ a_1(y;\alpha,\beta)^{\theta}\ \right]^{n-s}$$
$$. \left\{\left[1-\ [a_1(x;\alpha,\beta)]^{\theta}\right]-\left[1-\ [a_1(y;\alpha,\beta)]^{\theta}\right]\right\}^{s-r-1}$$
$$.\ [1-a_1(x;\alpha,\beta)]\ [1-a_1(y;\alpha,\beta)][a_1(x;\alpha,\beta)]^{\theta-1}\ [a_1(x;\alpha,\beta)]^{\theta-1}dxdy$$

$$\mu^{(K,L)}{}_{r:n} = (\beta\alpha\theta)^2\ C_{s-r-1,m}C_{n,r,s}\sum_{m=0}^{\infty}(-1)^m$$
$$. \int_0^1\int_0^1 x^k y^l \left[1-\ [a_1(x;\alpha,\beta)]^{\theta}\ \right]^{r-1}\left[\ a_1(y;\alpha,\beta)^{\theta}\ \right]^{n-s}$$
$$. \left[1-\ [a_1(y;\alpha,\beta)]^{\theta}\right]^{m} x^{-(\beta+1)}y^{-(\beta+1)}$$
$$.[1-a_1(x;\alpha,\beta)]\ [1-a_1(y;\alpha,\beta)][a_1(y;\alpha,\beta)]^{\theta-1}\ [a_1(x;\alpha,\beta)]^{\theta-1}dxdy$$

$$\mu^{(K,L)}{}_{r:n} = (\beta\alpha\theta)^2\ C_{s-r-1,m}C_{n,r,s}\sum_{m=0}^{\infty}(-1)^{m+i}$$
$$. \int_0^1 x^{-(\beta+1)+k}\left[1-\ [a_1(x;\alpha,\beta)]^{\theta}\ \right]^{r-1}$$
$$. \left[1-\ [a_1(x;\alpha,\beta)]^{\theta}\right]^{s-r-1-m} e^{-\alpha(x^{-\beta}-1)}\ [a_1(x;\alpha,\beta)]^{\theta-1}dx$$
$$. \int_0^1 y^{-(\beta+1)+l}\left[\ [a_1(y;\alpha,\beta)]^{\theta}\ \right]^{n-s}\left[1-[\ a_1(y;\alpha,\beta)]^{\theta}\right]^{m}\ [a_1(y;\alpha,\beta)]^{\theta-1}dy$$

$$\mu^{(K,L)}{}_{r:n} = \theta^2\ C_{s-r-1,m}C_{n,r,s}\sum_{m,i,j,u,q,v,l,t,w,z,p,h,c}^{\infty}(-1)^{m+i+j+u+q+v+2s+t+w+z+p+h+2c}$$
$$. \frac{\varphi\Gamma(\theta i+1)\Gamma(s-r-m)\Gamma(\theta i+1)\Gamma(l+1)}{i!j!u!q!v!s!t!w!z!p!h!c!\delta\Gamma(k+1-s)}\Gamma(s+1)\Gamma(c+1). \qquad (2.27)$$

Where $\varphi = \Gamma(r)\Gamma(\theta+1)\Gamma(\theta)\Gamma(k+1)\Gamma(r)\Gamma(s-r-m)\Gamma(\theta+1)\Gamma(\theta)$

$\delta = \Gamma(r-i)\Gamma(\theta i+1-j)\Gamma(s-r-m-u)\Gamma(\theta+1-q)\Gamma(\theta-v)\Gamma(r-i)\Gamma(\theta i+1-j)$

$$. \Gamma(s-\Gamma(s-r-m-u)\Gamma(\theta+1-q)\Gamma(\theta-v)\Gamma(l+1-c)$$

3. Maximum Likelihood Estimation for the EUG Distribution

The maximum likelihood estimators for the EUG's unknown parameters $\alpha, \beta,$ and θ will be derived in this section. The parameters' asymptotic confidence intervals will be provided as well. Assuming that $x_1, x_2, \ldots, x_n$ is a random sample of size n from EUG, the log-likelihood function is

$$logL = n[\text{Log}(\alpha) + \text{Log}(\beta) + \text{Log}(\theta)] - \alpha \sum_{i=1}^{n}[(x_i)^{-\beta} - 1] + (\theta - 1)\sum_{i=1}^{n} \text{Log}[1 - e^{-\alpha((x_i)^{-\beta}-1)}] - (\beta + 1)\sum_{i=1}^{n} \text{Log}[(x_i)^{-\beta}] \tag{3.1}$$

The first partial derivatives of the log of the likelihood L to $\alpha, \beta,$ and θ are obtained as follows.

$$\frac{\partial logl}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}[(x_i)^{-\beta} - 1] + (\theta - 1)\sum_{i=1}^{n} \frac{e^{-\alpha[(x_i)^{-\beta}-1]}((x_i)^{-\beta}-1)}{1-e^{-\alpha[(x_i)^{-\beta}-1]}} \tag{3.2}$$

$$\frac{\partial logl}{\partial \beta} = \frac{n}{\beta} - \sum_{i=1}^{n} Log(x_i) + \alpha \sum_{i=1}^{n} Log(x_i)(x_i)^{-\beta} - (\theta - 1)\sum_{i=1}^{n} \frac{e^{-\alpha[(x_i)^{-\beta}-1]}\alpha Log(x_i)(x_i)^{-\beta}}{1-e^{-\alpha[(x_i)^{-\beta}-1]}} \tag{3.3}$$

$$\frac{\partial logl}{\partial \alpha\theta} = \frac{n}{\theta} + \sum_{i=1}^{n} Log[1 - e^{-\alpha[(x_i)^{-\beta}-1]}] \tag{3.4}$$

It appears that the system of three non-linear equations (3.1), (3.2), and (3.3) has no analytical solution. As a result, these will be solved numerically with the R software.

3.1 Asymptotic Confidence Intervals

The asymptotic confidence intervals for these parameters are calculated in this subsection. Inverting the observed information matrix, given by, yields the asymptotic variance-covariance matrix of the MLEs. The most straightforward large-sample method is to assume that the MLE($\hat{\alpha}, \hat{\beta}$ and $\hat{\theta}$) are approximately multivariate normal with mean $\alpha, \beta,$ and θ.

$$I^{-1}(\hat{\alpha}, \hat{\beta}, \hat{\theta}) = -E\begin{bmatrix} \frac{\partial^2 LnL}{\partial \alpha^2} & \frac{\partial^2 LnL}{\partial \alpha \partial \beta} & \frac{\partial^2 LnL}{\partial \alpha \partial \theta} \\ \frac{\partial^2 LnL}{\partial \beta \partial \alpha} & \frac{\partial^2 LnL}{\partial \beta^2} & \frac{\partial^2 LnL}{\partial \beta \partial \theta} \\ \frac{\partial^2 LnL}{\partial \theta \partial \alpha} & \frac{\partial^2 LnL}{\partial \theta \partial \beta} & \frac{\partial^2 LnL}{\partial \theta^2} \end{bmatrix}^{-1} = \begin{bmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{bmatrix}.$$

The following are the components of the observed information matrix:

$$\frac{\partial^2 L}{\partial \alpha^2} = -\frac{n}{\alpha^2} + (\theta - 1)\sum_{i=1}^{n} \frac{e^{-2\alpha\left(x_i^{-\beta}-1\right)}(1-x_i^{-\beta})(x_i^{-\beta}-1)}{\left(1-e^{-\alpha\left(x_i^{-\beta}-1\right)}\right)^2} + \frac{e^{-\alpha\left(x_i^{-\beta}-1\right)}(1-x_i^{-\beta})(x_i^{-\beta}-1)}{1-e^{-\alpha\left(x_i^{-\beta}-1\right)}}, \tag{3.5}$$

$$\frac{\partial^2 L}{\partial \alpha \partial \beta} = \sum_{i=1}^{n} Log(x_i)(x_i)^{-\beta} - (\theta - 1)\sum_{i=1}^{n}\left[\frac{e^{-\alpha\left[(x_i)^{-\beta}-1\right]}Log(x_i)(x_i)^{-\beta}}{1-e^{-\alpha\left[(x_i)^{-\beta}-1\right]}} + \frac{e^{-2\alpha\left(x_i^{-\beta}-1\right)}\alpha Log(x_i)(x_i)^{-\beta}(1-x_i^{-\beta}))}{\left(1-e^{-\alpha\left(x_i^{-\beta}-1\right)}\right)^2} + \frac{e^{-\alpha\left(x_i^{-\beta}-1\right)}\alpha Log(x_i)(x_i)^{-\beta}(1-x_i^{-\beta})}{1-e^{-\alpha\left(x_i^{-\beta}-1\right)}}\right], \tag{3.6}$$

$$\frac{\partial^2 LnL}{\partial\alpha\partial\theta} = -\sum_{i=1}^{n} \frac{e^{-\alpha[(x_i)^{-\beta}-1]}(1-x_i^{-\beta})}{1-e^{-\alpha\left(x_i^{-\beta}-1\right)}}\ , \tag{3.7}$$

$$\frac{\partial^2 LnL}{\partial\beta\partial\theta} = -\sum_{i=1}^{n} \frac{e^{-\alpha\left[(x_i)^{-\beta}-1\right]}\,\alpha Log(x_i)(x_i)^{-\beta}}{1-e^{-\alpha\left(x_i^{-\beta}-1\right)}}\ , \tag{3.8}$$

$$\frac{\partial^2 LnL}{\partial\theta^2} = -\frac{n}{\theta^2}\ , \tag{3.9}$$

And

$$\frac{\partial^2 LnL}{\partial\beta^2} = -\frac{n}{\beta^2} - \alpha\sum_{i=1}^{n} Log(x_i^2)\,(x_i)^{-\beta} - (\theta-1)\sum_{i=1}^{n}[\frac{e^{-2\alpha\left(x_i^{-\beta}-1\right)}\,\alpha^2 Log(x_i^2)(x_i)^{-2\beta}}{\left(1-e^{-\alpha\left(x_i^{-\beta}-1\right)}\right)^2} +$$

$$\frac{e^{-\alpha\left(x_i^{-\beta}-1\right)}\,\alpha^2 Log(x_i^2)(x_i)^{-2\beta}}{1-e^{-\alpha\left(x_i^{-\beta}-1\right)}} - \frac{e^{-\alpha\left(x_i^{-\beta}-1\right)}\,\alpha\, Log(x_i^2)(x_i)^{-\beta}}{1-e^{-\alpha\left(x_i^{-\beta}-1\right)}}]. \tag{3.10}$$

According to particular regularity conditions, the two-sided 100(1− γ)%,$0 < \gamma > 1$, asymptotic confidence intervals for the parameters α, β, and θ can be obtained as:

$$\hat{\alpha}\ \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{\alpha\alpha}}\ ,\ \hat{\beta}\ \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{\beta\beta}}\ ,\ \hat{\theta}\ \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{\theta\theta}}. \tag{3.11}$$

Where $Z_{\frac{\gamma}{2}}$ is the upper$(\frac{\gamma}{2})$ the percentile of the standard normal distribution.

In 2019, Kumar and Kumar provided the coverage probability (CP) for parameters α, β, and θ as follows.

$$CP_\alpha = \left[\left|\frac{\hat{\alpha}-\alpha}{\sqrt{\hat{V}_{\alpha\alpha}}}\right| \le Z_{\frac{\gamma}{2}}\right]\ , CP_\beta = \left[\left|\frac{\hat{\beta}-\beta}{\sqrt{\hat{V}_{\beta\beta}}}\right| \le Z_{\frac{\gamma}{2}}\right], CP_\theta = \left[\left|\frac{\hat{\theta}-\theta}{\sqrt{\hat{V}_{\theta\theta}}}\right| \le Z_{\frac{\gamma}{2}}\right]. \tag{3.12}$$

3.2 Bootstrap Confidence Intervals

The bootstrap confidence intervals are approximations of ordinary confidence intervals; however, they are better in general. There are two types of parametric bootstrap methods:-

1- The percentile bootstrap method (Boot-p) was proposed by Efron(1993) and is rather simple, and constructs confidence intervals directly from the percentiles of the bootstrap distribution of the estimated parameters. It is given by the following steps:

I. A complete sample is generated from the original data $T = (t_1, t_2, \ldots, t_n)$ and the MLE $\hat{\Theta} = \left(\hat{\alpha}, \hat{\beta}, \hat{\theta}\right)'$ of the parameter $\Theta = (\alpha, \beta, \theta)'$ is computed.

II. Again, an independent complete bootstrap sample $T^* = (t_1^*, t_2^*, \ldots, t_n^*)$ is generated by using θ.

III. Now, compute the bootstrap MLE. $\hat{\Theta}^*$ of parameter Θ based on T^*, as in step-1.

IV. Repeat steps 2–3, B times, representing B bootstrap MLEs $\hat{\Theta}^*$'s based on B different bootstrap samples, $i = 1,2,\ldots B$.

V. Arrange all $\hat{\Theta}^*$'s in an ascending order to obtain the bootstrap sample i.e. $\hat{\Theta}^*{}_1 \le \hat{\Theta}^*{}_2 \le \ldots \le \hat{\Theta}^*{}_n$. An approximate 100(1 − ω)% boot-p confidence interval for Θ is obtained by
($\hat{\Theta}^*{}_{[(\frac{\omega}{2})\times B]}, \hat{\Theta}^*{}_{[(1-\frac{\omega}{2})\times B]}$). Where, $\frac{\omega}{2}$ is the quantity that helps to determine the bootstrap point.

2- The Bootstrap-t approach (Boot-t) was proposed by Hall (1988). It is given by the following steps:

I. Steps 1 and 2 of the boot-p and boot-t methods are the same.

II. Compute the bootstrap-t statistic. $T^* = \frac{\hat{\theta}_b{}^* - \hat{\theta}}{\sqrt{V(\hat{\theta}_b{}^*)}}$ where $b = 1,2,\dots,B$.

III. To obtain a set of bootstrap statistics $T_i{}^*$; $i = 1,2,\dots,B$ repeat steps 2–3, B times.

IV. IV. Let $T_1{}^* \leq T_2{}^* \leq \dots \leq T_n{}^*$ be the ordered values of $T_i{}^*: i = 1,2,\dots,B$.

V. Now, the approximate 100(1 − ω)% boot-t confidence interval for parameter θ is obtained by ($\hat{\theta} - \hat{T}_{[(1-\frac{\omega}{2})\times B]}\sqrt{V(\hat{\theta})}, \hat{\theta} - \hat{T}_{[(\frac{\omega}{2})\times B]}\sqrt{V(\hat{\theta})}$)

4. Simulation Study

Through Monte Carlo simulation, we evaluate the effectiveness of the maximum likelihood estimators as well as the asymptotic confidence intervals of the parameters that index the Exponentiated unit-Gompertz distribution presented in the preceding sections. The sample size was set at n = 20, 50,100 and 200, while the parameters were set at $\alpha = 0.02,0.06 \text{and } 0.5, \beta = 0.6,0.7, and\ 0.8$, and $\theta = 0.5,0.6,$ and 0.3, respectively. The number of replications (M = 1,000) is generated for each combination. all the computations are performed using the statistical package R. Table 1 describes the simulation results for MLE, which were achieved using the following steps:

I. Specify real values for parameters $\alpha, \beta,$ and θ as (0.02,0.06,0.5),(0.6,0.7,0.8), and (0.5,0.6,0.3)

II. Specify the sample size n. as $n = 20,50,100,200$

III. Generated complete samples of size n from the EUG distribution by using the formula $x_u = \left(\frac{\alpha}{\alpha - \ln\left[1-(1-u)^{\theta^{-1}}\right]}\right)^{\beta^{-1}}$

IV. Obtain the maximum likelihood estimates (MLEs).

V. Obtain the mean square error (MSE) = $mean\left(\hat{\theta} - \Theta\right)^2$ where $\hat{\theta}$ is the estimated value of Θ, bias = $\hat{\theta} - \Theta$, confidence intervals (CIs), average interval lengths (AILs) =Upper.CL- Lower. CL, coverage probability (CP) = $\frac{\sum_{j=1}^{M} I(j,\theta)}{M}$ where M is the number of replications and $(j, M) = \begin{cases} 1, \theta \in CL \\ 0, \theta \notin CL \end{cases}$, and bootstrap, for different sample sizes.

VI. Repeat steps 1-5 1000 times.

Obtain a 95 % confidence level to examine the effectiveness of maximum likelihood estimators and their asymptotic confidence intervals. From Tables 1, 2, and 3, the following conclusions can be drawn:

Table1: MLEs, MSE, AsyCI-Upper, AsyCI-Lower, AIL, Bias, Boot -P, Boot –t of EUG distribution parameters for $(\alpha, \beta, \theta) = (0.02,0.6,0.5)$

N	$\alpha = 0.02$	$\beta = 0.6$	$\theta = 0.5$

		MLEs	MSE	AsyCI-Upper	Asy CI-Lower	AIL	Bias	Boot−P	Boot−t
20	$\hat{\alpha}$	0.244	1.7896	-5.7552	6.2444	11.999	0.2246	0.00576 1.95830	-0.015 0.4330
	$\hat{\beta}$	0.725	0.1236	0.0631	1.3861	1.3231	0.1246	1.057425 .2233	1.92436. 5136
	$\hat{\theta}$	0.6437	0.2137	-0.2965	1.5839	1.8804	0.1436	0.816362 9.262	0.60823. 0142
50	$\hat{\alpha}$	0.1222	0.0577	-0.5611	0.8055	0.3029	0.024	0.005851 .33684	-0.015 0.4288
	$\hat{\beta}$	0.618	0.0559	-0.050	1.1867	0.8069	0.0411	1.1906 5.1717	1.886 6.232
	$\hat{\theta}$	0.659	0.1340	0.0268	1.3461	0.8201	0.085	0.83181 17.4200	0.515 2.876
100	$\hat{\alpha}$	0.0326	0.0008	-0.0338	0.0991	0.1328	0.0126	0.00584 1.8851	-0.014 0.4095
	$\hat{\beta}$	0.5823	0.0128	0.359	0.8061	0.447	0.0174	1.065 5.168	1.926 6.113
	$\hat{\theta}$	0.521	0.0116	0.298	0.7430	0.4471	0.0205	0.6175 26.710	0.5249 2.8885
200	$\hat{\alpha}$	0.023	0.0004	-0.01332	0.0584	0.0717	0.0025	0.0058 1.5952	-0.151 0.402
	$\hat{\beta}$	0.671	0.0222	0.48125	0.8611	0.3798	0.0711	1.1738 5.2412	2.0008 6.192
	$\hat{\theta}$	0.515	0.0101	0.3512	0.6801	0.3289	0.0156	0.611362 2.2563	0.521 2.968

Table2: MLEs, MSE, Asy CI-Upper, Asy CI-Lower, AIL, Bias, Boot -P, Boot –t of EUG distribution parameters for $(\alpha, \beta, \theta) = (0.06,0.7,0.6)$

N	$\alpha = 0.06$	$\beta = 0.7$	$\theta = 0.6$

		MLEs	MSE	AsyCI-Upper	AsyCI-Lower	AIL	Bias	Boot−P	Boot−t
20	$\hat{\alpha}$	0.2794	0.2305	-3.18495	3.7438	6.9288	0.2194	0.00599 1.40357	-0.022 0.3814
	$\hat{\beta}$	0.8498	0.2305	0.03932	1.6605	1.6211	0.1498	1.2135 5.1609	2.0306 7.0557
	$\hat{\theta}$	0.7330	0.2690	-0.30711	1.7732	2.080	0.1330	0.819722 .4323	0.57552. 8685
50	$\hat{\alpha}$	0.1056	0.0156	-0.20492	0.4162	0.6211	0.0456	0.0056 1.79996	-0.021 0.4348
	$\hat{\beta}$	0.7484	0.0594	0.29989	1.1969	0.8970	0.0484	1.107591 5.1743	1.90976. 6648
	$\hat{\theta}$	0.6377	0.0481	0.22587	1.0495	0.8236	0.0377	0.8229 21.848	0.6185 3.0205
100	$\hat{\alpha}$	0.0687	0.0026	-0.05283	0.1902	0.2431	0.0087	0.00592 1.58042	-0.016 0.4187
	$\hat{\beta}$	0.7267	0.0219	0.43894	1.0146	0.5756	0.0267	1.1396 5.0024	1.91146. 4682
	$\hat{\theta}$	0.6056	0.0226	0.3464	0.8647	0.5182	0.0056	0.8053 22.428	0.6109 2.9077
200	$\hat{\alpha}$	0.0649	0.0010	-0.01036	0.1401	0.1505	0.0048	0.00607 1.5625	-0.016 0.453
	$\hat{\beta}$	0.7092 3	0.0064	0.51421	0.9042	0.3900	0.0092	1.180255 .15378	1.895 6.689
	$\hat{\theta}$	0.0648	0.0010	-0.01035	0.1401	0.1505	0.0025	0.8140 24.8640	0.613 2.998

Table3: MLEs, MSE, Asy CI-Upper, Asy CI-Lower, AIL, Bias, Boot -P, Boot –t of EUG distribution parameters for $(\alpha, \beta, \theta) = (0.5,0.8,0.3)$.

N	$\alpha = 0.5$	$\beta = 0.8$	$\theta = 0.3$

		MLEs	MSE	AsyCI-Upper	AsyCI-Lower	AIL	Bias	Boot -P	Boot -t
20	$\hat{\alpha}$	0.9489	2.8512	-10.98	12.884	23.87	0.4489	0.00585 2.01038	-0.015 0.3876
	$\hat{\beta}$	1.3754	1.1372	-0.32	3.0705	3.3903	0.5753	1.0370 5.1628	1.9087 6.1884
	$\hat{\theta}$	0.2508	0.0047	0.095	0.4070	0.3123	0.0491	0.82109 25.357	0.6267 2.7938
50	$\hat{\alpha}$	0.1837	0.0084	-0.199	0.5613	0.7672	0.3162	0.00567 1.45547	-0.019 0.3755
	$\hat{\beta}$	1.5233	0.0827	0.5196	2.527	2.007	0.7233	1.23089 5.21360	2.0206 6.4310
	$\hat{\theta}$	0.3278	0.0026	0.1912	0.4644	0.2732	0.0278	0.828641 9.9200	0.6239 2.9343
100	$\hat{\alpha}$	0.2959	10^{-16}	-0.101	0.6931	0.7943	0.2041	0.005891 .53510	-0.017 0.4608
	$\hat{\beta}$	0.9333	10^{-18}	0.4265	1.4401	1.0136 0	0.1333	1.20139 5.17333	1.8860 6.4484
	$\hat{\theta}$	0.3613	10^{-12}	0.2584	0.4643	0.2059	0.0613	0.83001 25.2001	0.6128 2.8962
200	$\hat{\alpha}$	0.4200	10^{-15}	-0.161	1.0011	1.1621	0.0799	0.0056 1.8129	-0.016 0.3679
	$\hat{\beta}$	0.9677	10^{-12}	0.3602	1.5753	1.2150	0.1677	1.1712 5.1033	1.91286. 0696
	$\hat{\theta}$	0.3066	10^{-20}	0.2251	0.3882	0.1631	0.0066	0.7943 29.573	0.6016 2.7664

5. Real Data Analysis

We compared the fitting of the EUG distribution to unit Gompertz, Kumaraswamy, and Topp-Leone distributions, which are used to describe bounded data, using three data sets to demonstrate the EUG model's potential. Mazucheli et al. (2019) showed the pdf for the unit Gompertz as below in Equation (2.1), Poondi Kumaraswamy (1980) introduced the pdf for the Kumaraswamy distribution as below in Equation (5.1), and in 1955 Topp and Leone presented a pdf for the Topp-Leone distribution as below in Equation (5.2).

$$f(x;\alpha,\beta) = \alpha\beta x^{\alpha-1}(1-x^{\alpha})^{\beta-1} \quad (5.1)$$

$$f(x;\beta) = 2\beta(1-x)(2x-x^{2})^{\beta-1} \quad (5.2)$$

The first data set represents 20 observations of the maximum flood level (in millions of cubic feet per second) for the Susquehanna River at Harrisburg, Pennsylvania, and is reported in Dumonceaux and Antle (1973). The second data set related to 30 measurements of tensile strength of polyester fibers, Quesenberry and Hales (1980). The third data set is about the ordered failure of 20 components, which is reported in Nigm et al(2013). All sets of data were represented in Table 4).

Table4: Flood level, tensile strength, and failure of components data sets.

Data set (1)

0.26, 0.27, 0.30, 0.32, 0.32, 0.34, 0.38, 0.38, 0.39, 0.40, 0.41, 0.42, 0.42, 0.42, 0.45, 0.48, 0.49, 0.61, 0.65, 0.74
Data set (2)
0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926
Data set (3)
0.0009, 0.004, 0.0142, 0.0221, 0.0261, 0.0418, 0.0473, 0.0834, 0.1091, 0.1252, 0.1404, 0.1498, 0.175, 0.2031, 0.2099, 0.2168, 0.2918, 0.3465, 0.4035, 0.6143

The unknown parameters of these distributions will be estimated using the maximum-likelihood method for each data set, as well as the estimate (MLE), the goodness-of-fit measures, the Kolmogorov-Smirnov statistic (KS) (the distance between the empirical CDFs and the fitted CDFs), Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC). The statistics with smaller values and larger p-values are said to be a better fit. The results for the data sets are reported in Tables (5,6,7).

Table5: Maximum Likelihood Estimates and Goodness-of-fit Measures (p-values) and Likelihood-based Statistics to the First Data Set.

Distribution	data set (1)									
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	P-value	k-s	logL	AIC	BIC	AICC	HQIC
EUG	0.059	3.1556	1.562	0.8633	0.1343	16.509	-30.02	-24.03	-32.82	-26.4
Unit Gompertz	0.015	4.11	–	0.7431	0.1522	16.366	-30.73	-26.74	-33.13	-28.3
Kumurswamy	3.377	11.998	–	0.3001	0.2175	12.973	-23.94	-19.95	-26.34	-21.5
Topp-Leone	–	2.243	–	0.019	0.3411	7.3813	-13.76	-11.77	-15.56	-12.6

Table6: Maximum Likelihood Estimates and Goodness-of-fit Measures (p-values) and Likelihood-based Statistics to the Second Data Set.

Distribution	data set (2)									
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	P-value	k-s	Log L	AIC	BIC	AICC	HQIC
EUG	1.6333	0.332	1.1644	0.9939	0.0726	4.0297	-5.0	2.144	-8.26	-0.72
UnitGompertz	1.038	0.421	–	0.9931	0.0734	3.9488	-5.8	-1.095	-8.50	-3.001
Kumaraswamy	0.9626	1.6082	–	0.9987	0.0649	3.311	-4.6	0.1803	-7.22	-1.725
Topp-Leone	–	1.1094	–	0.998	0.066	2.903	-4.8	-2.41	-6.67	-3.359

Table7.: Maximum Likelihood Estimates and Goodness-of-fit Measures (p-values) and Likelihood-based Statistics to the Third Data Set.

Distribution	data set (3)

	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	P-value	k-s	Log L	AIC	BIC	AICC	HQIC
EUG	29.4	0.024	3.21	0.937	0.1125	17.159	-31.32	-25.33	-34.12	-27.7
UnitGompertz	0.77	0.278	–	0.709	0.149	14.763	-27.52	-23.53	-29.92	-25.14
Kumaraswamy	0.76	3.434	–	0.97	0.1026	17.205	-32.40	-28.41	-34.80	-30.02
Topp-Leone	–	0.511	–	0.4481	0.1849	15.62	-30.23	-28.24	-32.03	-29.04

Figure (6) shows the fitted lines for the cdfs and pdfs for the given data set and corresponding distributions. The figures also indicate that the EUG distribution provides a better fit than the other distributions, at least for three data sets.

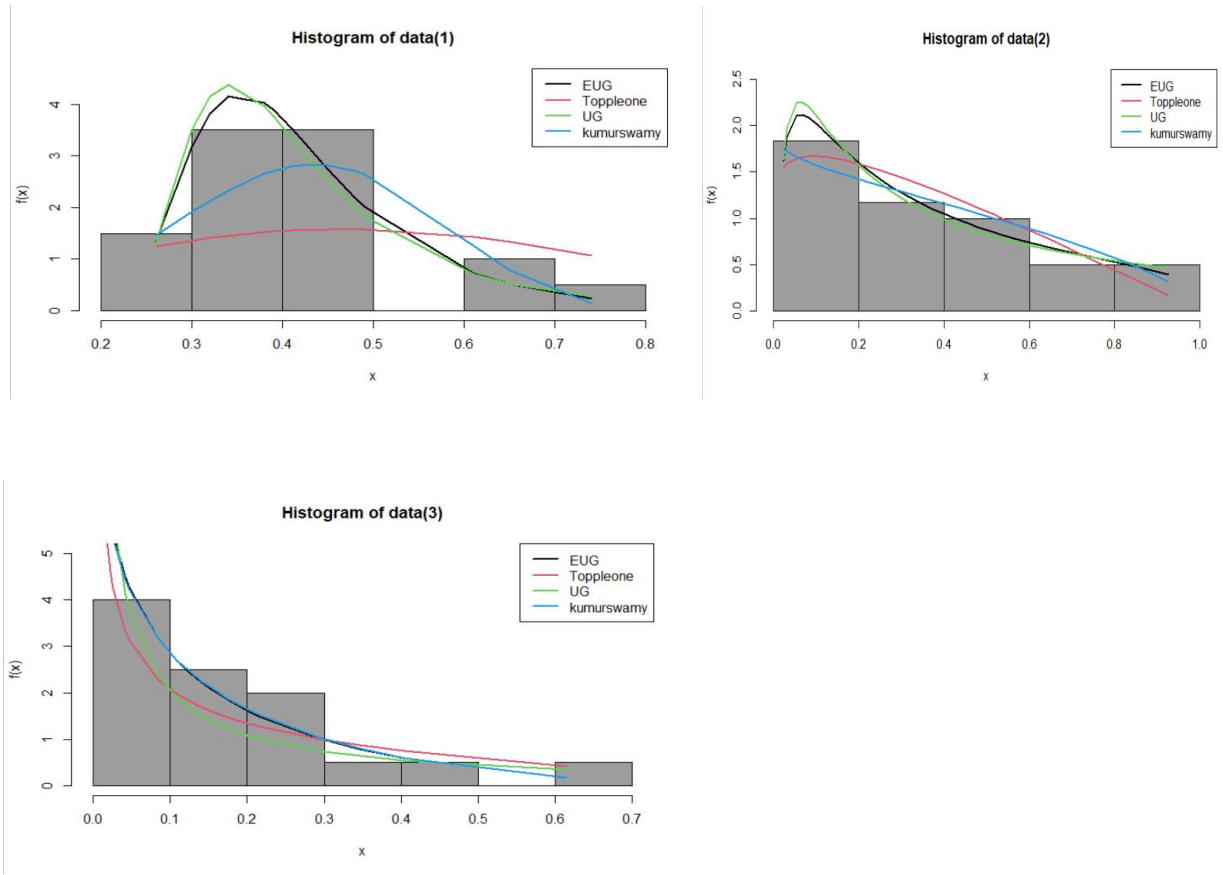


Figure 6: Histogram of the fitted distributions for three data sets

6. Bivariate Exponentiated Unit –Gompertz (BEUG) Distribution

In 2009, Balakrishnan and Lai showed that copulas are functions that connect one-dimensional marginal distribution functions to multivariate distributions. Copulas are multivariate

distributions with uniform marginals on the interval (0, 1). With distribution functions $F_1\ (y_1)$ and $F_2\ (y_2)$, Sklar (1973) considered the two random variables Y_1 and Y_2. The following cdf and pdf for the bivariate copula are as follows.

$$F(y_1, y_2) = C\big(F_1\ (y_1), F_2\ (y_2)\big) \tag{6.1}$$

And

$$f(y_1, y_2) = f_1\ (y_1) f_2\ (y_2) c\big(F_1\ (y_1), F_2\ (y_2)\big) \tag{6.2}$$

Farlie–Gumbel–Morgenstern (FGM) is one of the most well-known parametric copula families Gumbel (1960) presented the family. The following are the joint cdf and joint pdf for the FGM copula respectively.

$$C\big(F_1\ (y_1), F_2\ (y_2)\big) = F_1\ (y_1) F_2\ (y_2)\Big(1 + \gamma\big(1 - F_1\ (y_1)\big)\big(1 - F_2\ (y_2)\big)\Big) \tag{6.3}$$

And

$$c\big(F_1\ (y_1), F_2\ (y_2)\big) = \Big(1 + \gamma\big(1 - 2F_1\ (y_1)\big)\big(1 - 2F_2\ (y_2)\big)\Big); -1 < \gamma < 1 \tag{6.4}$$

Where γ corresponds to a coupla parameter.

According to Sklar, the joint pdf and cdf of the bivariate exponentiated unit – Gompertz (BEUG) distribution is as follows:

$$\begin{aligned} f(x,y) = {} & \alpha_1\beta_1\theta_1\,\alpha_2\beta_2\theta_2\ x^{-(\beta_1+1)}\,[1 - \xi(x;\ \alpha_1, \beta_1)][\xi(x;\ \alpha_1, \beta_1]^{\ \theta_1 - 1}\ y^{-(\beta_2+1)} \\ & .\Big\{1 + \gamma\Big[\Big(1 - 2\big(1 - (\xi(x;\ \alpha_1, \beta_1)\ )^{\theta_1}\big)\Big)\Big(1 - 2\Big(1 - \big(\xi(y;\ \alpha_2, \beta_2)\big)^{\theta_2}\Big)\Big)\Big]\Big\} \\ & .[1 - \xi(y;\ \alpha_2, \beta_2)]\ [\xi(y;\ \alpha_2, \beta_2)]^{\theta_2 - 1}. \end{aligned} \tag{6.5}$$

$$; 0 < x <\ 1, 0 < y < 1, \beta > 0, \theta > 0, \alpha\ >\ 0, -1 < \gamma < 1$$

Where: $\xi(x;\ \alpha_1, \beta_1) = 1 -\ e^{-\alpha_1(x^{-\beta_1} - 1)}, \xi(y;\ \alpha_2, \beta_2) = 1 -\ e^{-\alpha_2(y^{-\beta_2} - 1)}$

And

$$\begin{aligned} F(y_1, y_2) = {} & [1 - (\ \xi(x;\ \alpha_1, \beta_1))^{\theta_1}]\ [1 - (\ \xi(y;\ \alpha_2, \beta_2))^{\theta_2}] \\ & .\big\{1 + \gamma\big[(\ \xi(x;\ \alpha_1, \beta_1))^{\theta_1}(\ \xi(y;\ \alpha_2, \beta_2))^{\theta_2}\big]\big\} \end{aligned} \tag{6.6}$$

7. Properties of the BEUG Distribution

Marginal Distributions, product moments, moment generating function, conditional distribution, generating random variables, and survival function are some of the statistical properties of the BEUG distribution that are discussed in this section.

7.1 Marginal Distributions

The joint pdf of the BEUG distribution given in Equation (6.5) has EUG distribution marginals. The marginal density function for X is as follows

$$f(x) = \int_{all\ y} f(x,y)\ dy = f_{EUG}(x) = \alpha_1\beta_1\theta_1 x^{-(\beta_1+1)}\ e^{-\alpha_1(x^{-\beta_1} - 1)}\ [\xi(x;\ \alpha_1, \beta_1)]^{\theta_1 - 1} \tag{7.1}$$

and Y given by

$$f(y) = \int_{all\ x} f(x,y)\ dx = \alpha_2\beta_2\theta_2 y^{-(\beta_2+1)}\ e^{-\alpha_2(y^{-\beta_2} - 1)}\ [\xi(y;\ \alpha_2, \beta_2)]^{\theta_2 - 1} \tag{7.2}$$

which are EUG distribution as shown in Equations (7.1,7.2)

7.2 Conditional Distribution

The Conditional probability distributions for X and Y are given by

$$f(x|\ y) = \frac{f(x,y)}{f\ (y)} =\ \alpha_1\beta_1\theta_1 x^{-(\beta_1+1)}\ e^{-\alpha_1(x^{-\beta_1} - 1)}\ [\xi(x;\ \alpha_1, \beta_1)]^{\theta_1 - 1} \tag{7.3}$$

And

$$f(y|\,x)=\frac{f(x,y)}{f\,(x)}=\alpha_2\beta_2\theta_2 y^{-(\beta_2+1)}\,e^{-\alpha_2(y^{-\beta_2}-1)}\,[\xi(y;\,\alpha_2,\beta_2)]^{\theta_2-1} \tag{7.4}$$

7.3 Product Moments, Moment Generating Function

Let (X,Y) denote a random variable with the joint pdf of the BEUG distribution as in (6.5). Then its r^{th} and s^{th}joint moments around zero, denoted by μ'_{rs}can be expressed as follows:

$$\mu'_{rs}=E(X^rY^s)=\int_0^1\int_0^1 x^r y^s\;f(x,y)dy\,dx$$

$$=(\alpha\theta)^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\;\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\;\frac{(-1)^{i+2j}\,\Gamma(\theta)\Gamma\left(1-\frac{r}{\beta}\right)\Gamma(1+j)}{i!j!(\alpha i+\alpha)^{j+1}\Gamma(\theta-i)\Gamma\left(1-\frac{r}{\beta}-j\right)}\frac{(-1)^{n+2m}\,\Gamma(\theta)\Gamma\left(1-\frac{s}{\beta}\right)\Gamma(1+m)}{n!m!(\alpha n+\alpha)^{m+1}\Gamma(\theta-i)\Gamma\left(1-\frac{s}{\beta}-m\right)}$$

$$.\,[1+\gamma+4\gamma k_1k_2-2\gamma k_2-2\gamma k_1] \tag{7.5}$$

Where $k_1=\sum_{q=0}^{\infty}\sum_{u=0}^{\infty}\;\frac{(-1)^{q+2u}\,\Gamma(\theta)\Gamma\left(1-\frac{r}{\beta}\right)\Gamma(1+u)}{q!u!(\alpha q+\alpha)^{u+1}\Gamma(\theta-q)\Gamma\left(1-\frac{r}{\beta}-u\right)}-\sum_{h=0}^{\infty}\sum_{v=0}^{\infty}\;\frac{(-1)^{h+2v}\,\Gamma(2\theta)\Gamma\left(1-\frac{r}{\beta}\right)\Gamma(1+v)}{h!v!(\alpha h+v)^{v+1}\Gamma(2\theta-h)\Gamma\left(1-\frac{r}{\beta}-v\right)}$

And

$$k_2=\sum_{q=0}^{\infty}\sum_{u=0}^{\infty}\;\frac{(-1)^{q+2u}\,\Gamma(\theta)\Gamma\left(1-\frac{s}{\beta}\right)\Gamma(1+u)}{q!u!(\alpha q+\alpha)^{u+1}\Gamma(\theta-q)\Gamma\left(1-\frac{s}{\beta}-u\right)}-\sum_{h=0}^{\infty}\sum_{v=0}^{\infty}\;\frac{(-1)^{h+2v}\,\Gamma(2\theta)\Gamma\left(1-\frac{s}{\beta}\right)\Gamma(1+v)}{h!v!(\alpha h+v)^{v+1}\Gamma(2\theta-h)\Gamma\left(1-\frac{s}{\beta}-v\right)}$$

Consequently the joint moment generating function of the BEUG distribution is given as:

$$M_{X,Y}(t_1,t_2)=E(e^{t_1X}e^{t_2Y})=\sum_{d_1=0}^{\infty}\sum_{d_2=0}^{\infty}\;\frac{t_1^{d_1}t_2^{d_2}}{d_1!d_2!}\mu'_r\,\mu'_s\,. \tag{7.6}$$

7.4 Joint Survival Function

Osmetti and Chiodini (2011) studied the joint survival function for bivariate distributions using the copula function, which is more straightforward to write as a copula of its marginal survival functions, where X and Y are random variables with survival values $1-F(x)$ and $1-F(y)$, respectively. The joint survival function for the copula is expressed as follows:

$$\bar{F}(y_1,y_2)=C\big(\bar{F}_1\,(y_1),\bar{F}_2\,(y_2)\big) \tag{7.7}$$

The expression of the joint survival function for the FGM copula is as follows.

$$\bar{F}(y_1,y_2)=\bar{F}_1\,(y_1)\bar{F}_2\,(y_2)\big[1+\gamma(1-\bar{F}_1\,(y_1)\big(1-\bar{F}_2\,(y_2)\big)\big],$$

Then, the joint survival function of the BEUG distribution is given as

$$\bar{F}(x,y)=[\xi(x;\,\alpha_1,\beta_1)]^{\theta_1}[\xi(y;\,\alpha_2,\beta_2)]^{\theta_2}$$
$$.\big\{1+\gamma\big[\big(1-[\xi(x;\,\alpha_1,\beta_1)]^{\theta_1}\big)\big(1-[\xi(y;\,\alpha_2,\beta_2)]^{\theta_2}\big)\big]\big\} \tag{7.8}$$

8. Parameter Estimation Methods

Maximum likelihood estimation (MLE) and Semiparametric Method [SP] are two estimation methods used to estimate the unknown parameters of the BEUG distribution in this section.

8.1 Maximum Likelihood Estimation (MLE)

In this subsection, we use the maximum likelihood technique to estimate the BEUG distribution's unknown parameters. Assume $(x_1,y_1),(x_1,\,y_2),\ldots,(x_n,\,y_n)$ are all samples of size n from the BEUG distribution. Kim et al.(2007) showed further information on this strategy. The BEUG distribution's likelihood function is as follows:

$$lnl(\Theta)=n[Log(\alpha_1)+Log(\beta_1)+Log(\theta_1)+Log(\alpha_2)+Log(\beta_2)+Log(\theta_2)]$$

$$+(\theta_1-1)\sum_{i=1}^{n} Log[1-e^{-\alpha_1((x_i)^{-\beta_1}-1)}]-\alpha_1\sum_{i=1}^{n}[(x_i)^{-\beta_1}-1]$$
$$+(\theta_2-1)\sum_{i=1}^{n} Log\left[1-e^{-\alpha_2((y_i)^{-\beta_2}-1)}\right]-(\beta_1+1)\sum_{i=1}^{n} Log[(x_i)^{-\beta_1}]$$
$$-\alpha_2\sum_{i=1}^{n}[(y_i)^{-\beta_2}-1]-(\beta_2+1)\sum_{i=1}^{n} Log[(y_i)^{-\beta_2}]$$
$$+\sum_{i=1}^{n} Log[1+\eta(-1+2(1-e^{-\alpha_1((x_i)^{-\beta_1}-1)})^{\theta_1})(-1+2(1-e^{-\alpha_2((y_i)^{-\beta_2}-1)})^{\theta_2})]. \quad (8.1)$$

where $\Theta=(\alpha_1,\beta_1,\theta_1,\alpha_2,\beta_2,\theta_2,\gamma)$

where $A(x_i;\alpha_1,\beta_1)=1-e^{-\alpha_1((x_i)^{-\beta_1}-1)}$

All parameter estimates are derived by differentiating the log-likelihood function in (8.1) for each parameter individually and equating to zero, as shown below.

$$\frac{\partial lnl(\Theta)}{\partial\alpha_1}=\frac{n}{\alpha_1}-(\theta_1-1)\sum_{i=1}^{n}\frac{[1-A(x_i;\alpha_1,\beta_1)]((x_i)^{-\beta_1}-1)}{A(x_i;\alpha_1,\beta_1)}-\sum_{i=1}^{n}[(x_i)^{-\beta_1}-1]+$$
$$\sum_{i=1}^{n}\frac{2\eta\theta1(-1+2[A(y_i;\alpha_2,\beta_2)]^{\theta2})[A(x_i;\alpha_1,\beta_1)]^{\theta1-1}e^{-\alpha_1((x_i)^{-\beta_1}-1)}[(x_i)^{-\beta_1}-1]}{[1+\eta(-1+2(A(x_i;\alpha_1,\beta_1))^{\theta_1})(-1+2(A(y_i;\alpha_2,\beta_2))^{\theta_2})]} \quad (8.2)$$

$$\frac{\partial lnl(\Theta)}{\partial\alpha_2}=\frac{n}{\alpha_2}-(\theta_2-1)\sum_{i=1}^{n}\frac{e^{-\alpha_2\left((x_i)^{-\beta_2}-1\right)}((y_i)^{-\beta_2}-1)}{A(y_i;\alpha_2,\beta_2)}-\sum_{i=1}^{n}[(x_i)^{-\beta_2}-1]+$$
$$\sum_{i=1}^{n}\frac{2\eta\theta_1(-1+2(1-e^{-\alpha_2\left((y_i)^{-\beta_2}-1\right)})^{\theta2})(1-e^{-\alpha_1((x_i)^{-\beta_1}-1)})^{\theta_1-1}e^{-\alpha_1((y_i)^{-\beta_1}-1)}[(y_i)^{-\beta_1}-1]}{[1+\eta(-1+2(1-e^{-\alpha_1((y_i)^{-\beta_1}-1)})^{\theta_1})(-1+2(A(y_i;\alpha_2,\beta_2))^{\theta_2})]} \quad (8.3)$$

$$\frac{\partial lnl(\Theta)}{\partial\beta_1}=\frac{n}{\beta_1}+\alpha_1\sum_{i=1}^{n}[(x_i)^{-\beta_1}]\log(x_i)-(\theta_1-1)\sum_{i=1}^{n}\frac{\alpha_1(x_i)^{-\beta_1}log(x_i)(1-A(x_i;\alpha_1,\beta_1))}{A(x_i;\alpha_1,\beta_1)}$$
$$+(\beta_1+1)\sum_{i=1}^{n}\frac{(x_i)^{-\beta_1}Log(x_i)}{(x_i)^{-\beta_1}}+\sum_{i=1}^{n}Log[(x_i)^{-\beta_1}]$$
$$+\sum_{i=1}^{n}\frac{2\eta\theta_1\alpha_1(A(x_i;\alpha_1,\beta_1))^{\theta1-1}[1-A(x_i;\alpha_1,\beta1]\sum_{i=1}^{n}[(x_i)^{-\beta_1}]log(x_i)(-1+2(A(y_i;\alpha_2,\beta_2))^{\theta_2})}{1+\eta(-1+2(A(x_i;\alpha_1,\beta_1))^{\theta_1})(-1+2(A(y_i;\alpha_2,\beta_2)^{\theta_2})}, \quad (8.4)$$

$$\frac{\partial lnl(\Theta)}{\partial\beta_2}=\frac{n}{\beta_2}+\alpha_2\sum_{i=1}^{n}[(y_i)^{-\beta_2}]log(y_i)-(\theta_2-1)\sum_{i=1}^{n}\frac{[1-A(y_i;\alpha_2,\beta_2]\alpha_2(y_i)^{-\beta_2}log(y_i)}{A(x_i;\alpha_1,\beta_1)}+$$
$$(\beta_2+1), \quad (8.5)$$

$$\frac{\partial lnl(\Theta)}{\partial\theta_1}=\frac{n}{\theta_1}+\sum_{i=1}^{n}Log[A(x_i;\alpha_1,\beta_1)]+$$
$$\sum_{i=1}^{n}\frac{\eta(-1+2(A(x_i;\alpha_1,\beta_1))^{\theta_1})\log(-1+2(A(x_i;\alpha_1,\beta_1))(-1+2(A(y_i;\alpha_2,\beta_2)^{\theta_2})}{[1+\eta(-1+2(A(x_i;\alpha_1,\beta_1))^{\theta_1})(-1+2(A(y_i;\alpha_2,\beta_2)^{\theta_2})]}, \quad (8.6)$$

$$\frac{\partial lnl(\Theta)}{\partial\theta_2}=\frac{n}{\theta_2}+\sum_{i=1}^{n}Log[1-e^{-\alpha_2((y_i)^{-\beta_2}-1)}]+$$
$$\sum_{i=1}^{n}\frac{\eta(-1+2(A(y_i;\alpha_2,\beta_2))^{\theta_2})\log(-1+2(A(y_i;\alpha_2,\beta_2)(-1+2(A(x_i;\alpha_1,\beta_1))^{\theta_1})}{[1+\eta(-1+2(A(x_i;\alpha_1,\beta_1))^{\theta_1})(-1+2(A(y_i;\alpha_2,\beta_2))^{\theta_2})]}, \quad (8.7)$$

$$\frac{\partial lnl(\Theta)}{\partial\gamma}=\sum_{i=1}^{n}\frac{(-1+2(A(x_i;\alpha_1,\beta_1))^{\theta_1})(-1+2(A(y_i;\alpha_2,\beta_2))^{\theta_2})}{[1+\eta(-1+2(A(x_i;\alpha_1,\beta_1))^{\theta_1})(-1+2(A(y_i;\alpha_2,\beta_2))^{\theta_2})]}, \quad (8.8)$$

$$\frac{\partial lnl(\Theta)}{\partial\alpha}|\alpha_i=\hat{\alpha}_i=0,\frac{\partial lnl(\Theta)}{\partial\beta}\Big|\beta_i=\hat{\beta}_i=0,\frac{\partial lnl(\Theta)}{\partial\theta}\Big|\theta_i=\hat{\theta}_i=0,\frac{\partial lnl(\Theta)}{\partial\gamma}|\gamma=\hat{\gamma}=0\ ,i=1,2$$

The MLE for estimators $(\hat{\alpha}_1, \hat{\alpha}_2, \hat{\beta}_1, \hat{\beta}_2, \hat{\theta}_1, \hat{\theta}_1, \hat{\gamma})$ can be determined by solving the likelihood equations using the R package

8.2 Estimation by Inference Functions for Margins (IFM)

In 2005, Joe introduced another estimating method for more information on inference functions for margins (IFM). It is a two-step parametric estimation approach. To begin, maximize the log-likelihood function of each marginal to estimate the marginal of the BEUG distribution. Then, the second step involves maximizing the log-likelihood function of the copula density using the ML estimates of the marginal $\hat{F}_1\ (t_1)\ and\ \hat{F}_2\ (t_2)$ from the first step to estimate the copula parameter.

$$logL = n[\text{Log}(\alpha_j) + \text{Log}(\beta_j) + \text{Log}(\theta_j)] - \alpha_j \sum_{i=1}^{n} [t_{ji}{}^{-\beta} - 1]$$
$$+(\theta_j - 1)\sum_{i=1}^{n} Log[1 - e^{-\alpha_j (x_i{}^{-\beta_j - 1})}] - (\beta_j + 1)\sum_{i=1}^{n} Log(t_{ji}{}^{-\beta_j})\,; j = 1,2 \quad (8.9)$$

$$\frac{\partial logl}{\partial \alpha_j} = \frac{n}{\alpha_j} - \sum_{i=1}^{n}\left[(x_i)^{-\beta_j} - 1\right] + \left(\theta_j - 1\right)\sum_{i=1}^{n} \frac{e^{-\alpha_j (x_i{}^{-\beta_j - 1})}\left((x_i)^{-\beta_j - 1}\right)}{1 - e^{-\alpha_j\left[(x_i)^{-\beta_j - 1}\right]}}, \quad (8.10)$$

$$\frac{\partial logl}{\partial \beta_j} = \frac{n}{\beta_j} - \sum_{i=1}^{n} Log(t_{ji}) + \alpha_j \sum_{i=1}^{n} Log(t_{ji})(t_{ji})^{-\beta_j}$$
$$-(\theta_j - 1)\sum_{i=1}^{n} \frac{e^{-\alpha_j[(t_{ji})^{-\beta_j - 1}]}\alpha_j Log(t_{ji})(t_{ji})^{-\beta_j}}{1 - e^{-\alpha_j[(t_{ji})^{-\beta_j - 1}]}}\ , \quad (8.11)$$

$$\frac{\partial logl}{\partial \theta_j} = \frac{n}{\theta_j} + \sum_{i=1}^{n} Log[1 - e^{-\alpha_j\left[(t_{ji})^{-\beta_j - 1}\right]}, \quad (8.12)$$

$$logL_\gamma = \sum_{i=1}^{n} \log\{1 + \gamma[(1 - 2(1 - (B_1(t_{ji}; \hat{\alpha}_1, \hat{\beta}_1))^{\hat{\theta}_1}))(1 - 2(1 - (B_1(t_{ji}; \hat{\alpha}_2, \hat{\beta}_2))^{\hat{\theta}_2}))]\} \quad (8.13)$$

$$\frac{\partial logl}{\partial \gamma} = \sum_{i=1}^{n} \frac{(1-2(1-(B_1(t_{ji};\hat{\alpha}_1,\hat{\beta}_1))^{\hat{\theta}_1}))(1-2(1-(B_1(t_{ji};\hat{\alpha}_2,\hat{\beta}_2))^{\hat{\theta}_2}))}{\{1+\gamma[(1-2(1-(B_1(t_{ji};\hat{\alpha}_1,\hat{\beta}_1))^{\hat{\theta}_1}))(1-2(1-(B_1(t_{ji};\hat{\alpha}_2,\hat{\beta}_2))^{\hat{\theta}_2}))]\}} \quad (8.14)$$

Where $B_1(t_{ji}; \hat{\alpha}_1, \hat{\beta}_1) = 1 - e^{-\hat{\alpha}_1 (t_{ji}{}^{-\hat{\beta}_1 - 1})}$

The estimators of the parameters are handled numerically through statistical software.

9. Data Analysis

In this section, we analyze the medical data of kidney patients. Let X refer to the first recurrence time and Y to the second recurrence time, but we will divide this data by 1000 to be suitable for the nature of our distributions. The data for 30 patients are reported in Table 8. We study the parameter to estimate the appropriate distribution for each data set, where the correlation between the two variables (bivariate data) is low, as shown in Table 9. Through this, we arrive at an appropriate model that specializes in studying weak relationships and their existence effect, and efficacy. We compare the proposed bivariate distributions with different bivariate distributions, which are used in this data set, like the bivariate FGM Weibull distribution, which was discussed by Almetwaly et al (2020), and FGM Bivariate Generalized Exponential (FGMBGE), which was discussed by Elaal and Jarwan (2017).

Table 8: Kidney infection data for 30 patients.

X	0.008	0.023	0.022	0.447	0.030	0.024	0.007	0.511	0.053	0.015	0.007	0.013	0.039	0.132	0.034
Y	0.016	0.013	0.028	0.318	0.012	0.245	0.009	0.030	0.196	0.154	0.333	0.008	0.038	0.070	0.025
X	0.141	0.096	0.149	0.536	0.017	0.185	0.292	0.022	0.015	0.152	0.402	0.012	0.113	0.002	0.130
Y	0.004	0.117	0.114	0.159	0.108	0.362	0.024	0.066	0.046	0.040	0.201	0.156	0.030	0.025	0.026

First, we make sure that the correlation between T_1, T_2 is weak if the value is between (-1/3:1/3) as shown in Table 9. Based on that, we make certain that all univariate distributions are fitted to this data and estimate the goodness-of-fit test, one-sample by Kolmogorov-Smirnov test, Akaike information criterion (AIC), Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC) in Table 10. In Table 11, it is observed that the BEUG model provides a better fit than the other tested models (FGM Weibull, FGMBGE) because it has the smallest value of L and AIC. The BEUG distribution is a good alternative to several bivariate lifetime distributions for modeling non-negative real-valued data in the application. In Table 12, it is observed that the MLE method provides a better fit than the IFM method because it has the smallest value of AIC for the parameters of the BEUG distribution.

Table 9: The Correlation Coefficient and Test of Correlation for Real Data

	Correlation	p-value
Pearson	0.05105553	0.7887

Table 10:The Maximum Likelihood Estimation Parameters of Distributions, p-value,k-s, LL, AIC, BIC, CAIC, HQIC

Distribution X	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\theta}_1$	P-value	k-s	LL	AIC	BIC	CAIC	HQIC
EUG	0.3217	0.448	1.3726	0.871	0.109	38.14	-73.2	-66.08	-76.4	-68.93
Generalized Exponential	0.66	–	6.19	0.34	0.17	35.2	-68	-63.6	-71.0	-71.04
Weibull	0.751	0.100	–	0.549	0.14	35.74	-69.4	-64.6	-72.0	-66.58
Distribution Y	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}_2$	P-value	k-s	LL	AIC	BIC	CAIC	HQIC
EUG	0.7374	0.3679	2.4635	0.3138	0.1755	40	-78	-71.41	-81.8	-74.27
Generalized Exponential	0.922	–	9.56	0.46	0.15	39	-76	-72.0	-79.4	-73.91
Weibull	0.9320	0.095	–	0.215	0.19	39	-76	-72.1	-80	-74.1

Table 11: The Estimates of Parameters of Bivariate Distributions

Distributions	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\theta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}_2$	γ	P-value	L-L	AIC
EUG	0.385	0.2455	0.454	0.0249	0.750	0.31	0.8978	0.586	-86.6	.181
Generalized Exponential	0.409	0.8054		0.4064	0.689		0. 6040	0.58	-95.7	.196
Weibull	1.035	0.1386		0.8060	0.059		1.0196	0.59	-92.7	.190

Table 12. The Estimates of Parameters of Bivariate Distribution under MLE, IFM

Distribution	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\theta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}_2$	γ	P-value	k-s	AIC
EUG MLE	0.385	0.2455	0.454	0.0249	0.750	0.314	0.897	0.58	0.2	181.2
EUG IFM	0.322	0.448	1.373	0.7374	0.3679	2.464	-0.5	0.58	0.2	218.7

10. Conclusion

We suggest a new three-parameter model in this paper, called the exponentiated unit Gompertz distribution, which can depend on the proportional hazard function. The distribution of EUG is motivated by its wide use in life testing and provides more flexibility for evaluating lifetime data. Survival function, hazard function, reversed hazard, quantile, and moments are given. We estimate the parameters of the EUG distribution by the MLE method. We use three data sets to compare the new model to some finite distributions and observe that the new model is better than the unit Gompertz, Kumaraswamy, and Topp-Leone distributions. We don't stop here, we study statistical properties of the bivariate exponentiated unit Gompertz distribution, which is introduced depending on FGM coupla, such as marginal, conditional, moments, and estimate parameters of the distribution by two methods, MLE, and IFM. And we apply medical data to make sure the BEUG distribution is better than other distributions that are being compared.

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