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Generalized Topp- Leone Distribution as a Contribution to a Semi-Parametric Family of Distributions

Hiba Zeyada Muhammed
Mathematical Statistics Department
Cairo University, Egypt

Ghidaa A. Hamouda
Faculty of Commerce
University of Sadat City (USC), Egypt

ABSTRACT

In this paper, the generalized Topp-Leone distribution (T-L) is proposed as a member of a semi parametric family of distributions. Quantile, mode, moments, stochastic ordering, order statistics and moments of both order statistics and joint order statistics are discussed. The maximum likelihood estimation method is considered. To evaluate the study's results, simulation issues are provided and two real data sets are reanalyzed. Moreover, a bivariate extension of this distribution named bivariate generalized Topp-Leone (BGT-L) is introduced. The proposed bivariate version is of type Farlie–Gumbel–Morgenstern (FGM) copula. The BGT-L distribution has generalized Topp-Leone marginals. The joint cumulative distribution function, the joint survival function, and the joint probability density function are also derived in compact forms. Maximum likelihood and inference functions for margins methods are used to estimate the bivariate model's unknown parameters. Finally, we illustrate the importance of the proposed bivariate model using a real data set in the medical field.

Keywords: Topp-Leone distribution; generalized Topp-Leone; quantile function, mode, moments; maximum likelihood estimation; bivariate generalized Topp-Leone distribution; FGM copula.

1. Introduction

Olkin (2007) classified distributions into: parametric, non-parametric, and semi-parametric. parametric distributions, such as the exponential, Weibull, gamma, and lognormal distributions, are families of distributions indexed by a real number or several real numbers. The non-parametric families are characterized by properties with physically relevant interpretations; these distribution families have primarily been studied in the context of reliability theory. The notation of components and structures, which may be mechanical, electrical, hydraulic, or biological systems, has thus been established as part of the theory for these families.

Semiparametric families of distributions are characterized by the presence of a parameter that is also a distribution function. These families have a real-valued parameter; one method for using a semiparametric model is to use a distribution function as the first parameter. The underlying distribution is the name given to this distribution function. In effect, selecting an underlying distribution leads to the selection of a parametric model, but only for families that have the semiparametric model's structure. Semiparametric families can be studied for two reasons. Since the standard families of gamma distributions and Weibull distributions can be thought of as deriving from the exponential distribution through semiparametric families that added a second parameter, it provides a new understanding of standard parametric families of distributions. Using the same method, it is possible to find a three parameters family that includes both the gamma and Weibull families as special cases in a class of bivariate semiparametric families of distributions. As a result, it offers methods for expanding families to increase data fitting stability. It is important to see that the main criteria for the semiparametric families are that the underlying distribution is

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- Hiba Zeyada Muhammed (corresponding author) is affiliated with Mathematical Statistics Department, Faculty of Graduate Studies for Statistical Research, Cairo University, Egypt
Email: hiba_stat@cu.edu.eg

a member of the parametric family. And the second criteria, is that once the semiparametric family is used to add a parameter, its reuse may reparametrize the family, but it should fail to again add a new parameter. This is a kind of stability property.

There are many types of semiparametric families such as: Univariate Power Parameter Family (PP), Univariate Frailty (hazard) Parameter Family (FP), Univariate Resilience Parameter Family (RP), Univariate Hazard Power Parameter Family (HPP), and Univariate Reversed Hazard Power Parameter Family (RPP). Muhammed (2023) introduced the bivariate extension for these semi-parametric families of distribution based on Marshall-Olkin technique. In this paper the Frailty (hazard) parameter family (FP) of distributions will be considered in both univariate and bivariate version.

Topp and Leone (1955) presented the Topp-Leone distribution as a probability distribution with bounded support that can be used to model life-time phenomena it's a simple bounded J-shaped distribution that has sparked interest among statisticians as a potential replacement for the beta distribution. It's also a good model for life testing and reliability studies because the hazard function is shaped like a bathtub. Many contributions continued with studies of many authors, such as; . Nadarajah and Kotz (2003), studied the distribution and derived all of its moments and gave an expression for its characteristic function. Van Dorp and Kotz (2004) used a generalized version of the Topp–Leone distribution to model some financial data. Kotz and Van Dorp (2004) defined a variant of the generalized Topp–Leone distribution called the reflected generalized Topp–Leone distribution and studied its properties. Ghitany et al. (2005) studied some reliability measures of the distribution such as the hazard rate, mean residual life, reversed hazard rate, expected inactivity time and their stochastic orderings.

Currently, there is a lot of interest in making statistical distributions more flexible. The amount of data available for analysis is growing increasingly faster, requiring new probabilistic distributions to better describe each phenomenon. Many generalized distributions have been developed over the past decades for modeling data in several areas such as biological studies, environmental sciences, economics, engineering, finance and medical sciences. Recently, there has been an increased interest in defining new classes of families of univariate continuous distributions by introducing additional shape parameters to the baseline model. By various methods, new parameters can be introduced to expand families of distributions for added flexibility or to construct covariate models. Introduction of a scale parameter leads to the accelerated life model, and taking powers of the survival function introduces a parameter that leads to the proportional hazards model

The Generalized Topp- Leone (GT-L) distribution is defined with two shape parameters using proportional hazards model. Many statistical properties of this distribution are discussed. The point and interval estimation for the unknown parameters are given. Moreover, a bivariate extension of the GT-L distribution is introduced based on Farlie–Gumbel–Morgenstern (FGM) copula and its properties and parameters estimation are also discussed.

The paper is organized as follows: In Section 2 the univariate Frailty (hazard) parameter family of distribution is introduced, then generalized Topp-Leone distribution according to univariate Frailty (hazard) parameter Family is introduced in Section 3, with some basic properties of this distribution. The distribution is estimated by MLE in Section 4. Two real data sets are reanalyzed in Section 5. A simulation study is discussed in Section 6. A bivariate generalized Topp -Leone is introduced in Section 7. Finally, we conclude the paper in Section 8.

2. Univariate Frailty (hazard) Parameter Family (UFP)

Let $\bar{F}_B$ be a baseline with survival function with cumulative hazard function $H(x) = -log\bar{F}_B$. Suppose that $\bar{F}_\alpha(x)$ is defined in terms of $\bar{F}_B(x)$ by

$$\bar{F}_\alpha(x) = [\bar{F}_B(x)]^\alpha \qquad \alpha > 0 \tag{1}$$

In this case α is called a frailty (hazard) parameter. The corresponding pdf and hazard function is given respectively, as

$$f_\alpha(x) = \alpha\,[\bar{F}_B(x)]^{\alpha-1} f_B(x)\,,\;\; h_\alpha(x) = \alpha h_B(x) \tag{2}$$

Where f_B and h_B are a baseline pdf and hazard functions respectively.

There exist number of distributions that are produced by adding a frailty (hazard) parameter. But the Topp-Leone (TL) distribution with finite support has been proposed by Topp and Leone as an attractive model in reliability studies. TL distribution has J-shaped in its density function and a bathtub shaped in its hazard function. Due to the importance of TL distribution, various authors have conducted relevant studies.

The probability density function (pdf) of the TL distribution is given by:

$$f_B\,(x) = 2\beta\;\;(1-x)[x(2-x)]^{\beta-1}\;\; 0 < x < 1\,, \beta > 0 \tag{3}$$

The cumulative distribution function (cdf) and survival function related to (3) is given by

$$F_B\,(x) = [x(2-x)]^\beta \tag{4}$$

$$\bar{F}_B\,(x) = 1 - [x(2-x)]^\beta \tag{5}$$

According to (2) the probability density function (pdf) is,

$$f_\alpha(x) = 2\beta\alpha\,[\,1-[x(2-x)]^\beta]^{\alpha-1}\,(1-x)[x(2-x)]^{\beta-1}\,;\, 0 < x < 1\;\;, \beta, \alpha > 0 \tag{6}$$

Equation (6) refers to the Generalized Topp-Leone distribution denoted by (GT-L) (β, α)

Figure (1) illustrated the behavior of the GT-L distribution at $\beta = 1$ and for some various values of α.

The survival function of the GT-L distribution is given by,

$$\bar{F}_\alpha(x) = [\,1-[x(2-x)]^\beta]^\alpha\,. \tag{7}$$

With graph given in Figure (2) for various values of the parameter α

Cumulative function (cdf) of GT-L distribution is given, as

$$F_\alpha(x) = 1 - \,[\,1-[x(2-x)]^\beta]^\alpha\,. \tag{8}$$

And its graph is illustrated in Figure (3) for some various values of α.

Hazard function

Let X be a non-negative continuous random variable, usually representing the time to failure of a unit or a system, with pdf $f(x)$ and cdf $F(x)$. It is well known that an important measure of ageing is the hazard rate (HR) and use to characterize a life phenomenon. It is given as:

$$h\,(x) = \lim_{\Delta x \to 0} \left(\frac{p(X<x+\Delta x|X>x}{\Delta x}\right) = \frac{f(x)}{\bar{F}(x)}$$

Where $\bar{F}(x) = 1 - F(x)$ is the survival function of X. That is $h\,(x)\Delta x$ represents the conditional probability that an individual of age x will fail within the interval $(x, x + \Delta x)$. For the GT-L, the hazard function is easily calculated to be

$$h_\alpha(x) = \alpha\;\frac{2\beta\;\;(1-x)[x(2-x)]^{\beta-1}}{1-[x(2-x)]^\beta} \qquad \alpha\;> 0, \beta > 0 \tag{9}$$

$$h_\alpha(x) = \alpha\;\;h_B(x) \qquad .$$

Figure (4) shows the graph of the hazard function given in (9) with different parameter values.

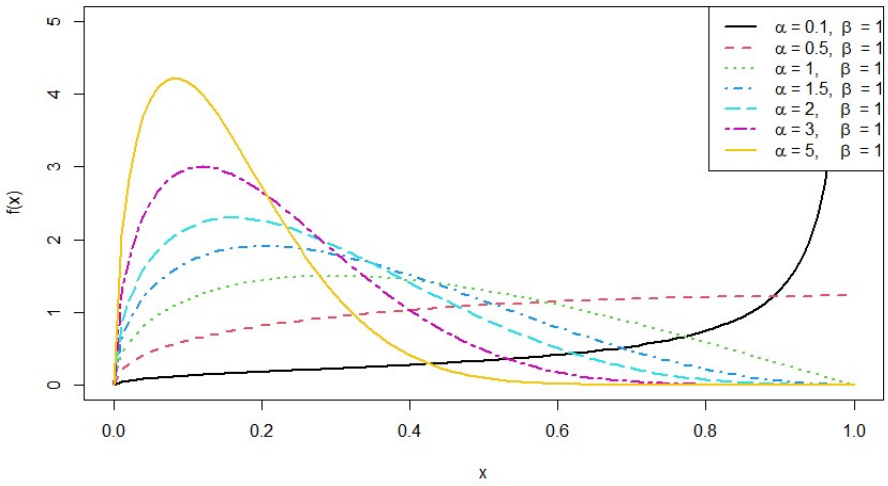


Figure 1: pdf of GT-L distribution for some values of α.

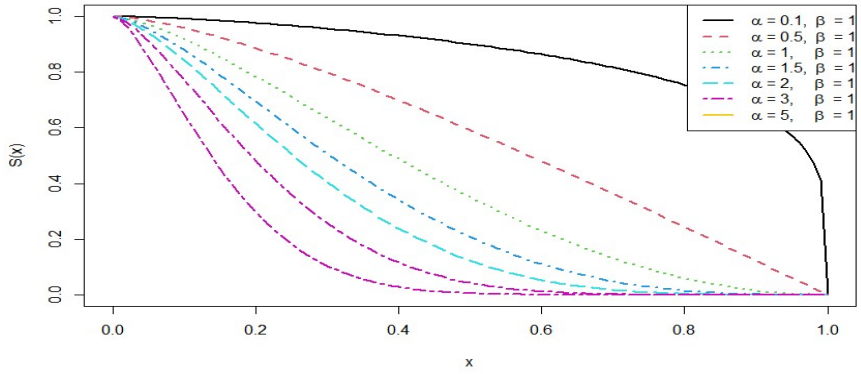


Figure 2: Survival function of the GT-L distribution for different values of α.

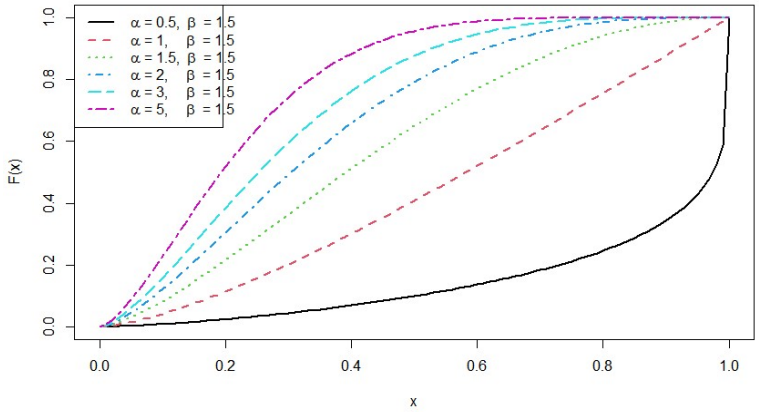


Figure 3: cdf of the GT-L distribution for different values of α.

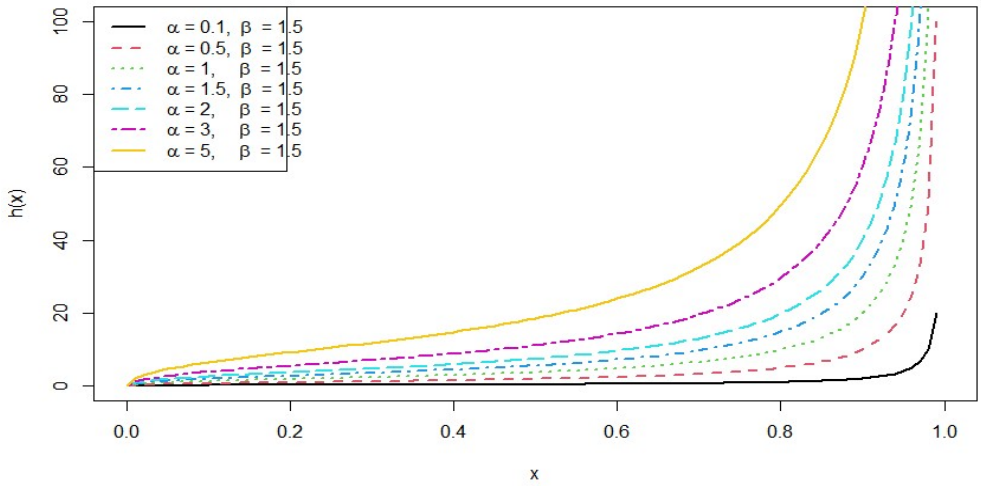


Figure 4: Hazard function of GT-L distribution for different values of α.

Reversed hazard function

Of a non-negative continuous random variable X with pdf $f(x)$ and cdf $F(x)$ at time x is defined as

$$r(x) = \lim_{\Delta x \to 0} \left(\frac{p(X > x - \Delta x | X \leq x}{\Delta x} \right) = \frac{f(x)}{F(x)}.$$

That is $r(x)\Delta x$ defines the conditional probability of a failure of a unit in $(x - \Delta x, x)$ given that the failure had occurred in $[0, x]$. The Reversed hazard function is a dual property of the hazard function.

$$r_\alpha(x) = \frac{2\beta\alpha\,[\,1-[x(2-x)]^\beta]^{\alpha-1}\,(1-x)[x(2-x)]^{\beta-1}}{1-\,[\,1-[x(2-x)]^\beta]^\alpha}, \quad \alpha > 0, \beta > 0. \qquad (10)$$

And its graph is given in Figure (5) with different parameter values

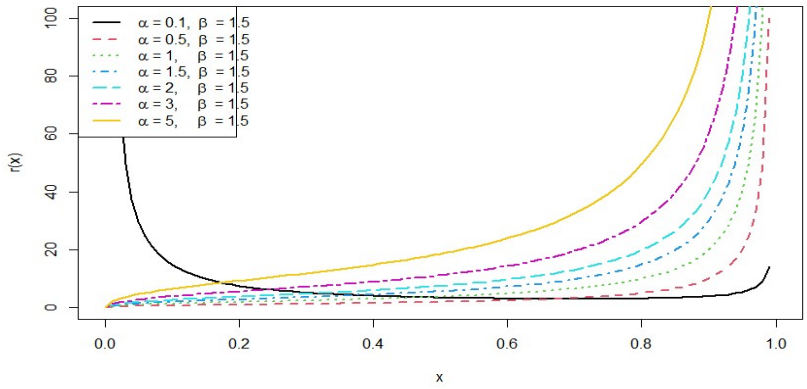


Figure 5: Reversed hazard function for GT-L distribution for different values of α.

Stochastic ordering

The stochastic ordering of positive continuous random variable is an important tool for judging their comparative behavior. A continuous random variable X is said to be smaller than acontinous random variable Y in the

(i) Stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all x

(ii) Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x

(iii) Likelihood ratio order ($X \leq_{lr} Y$) if $\frac{f_X(x)}{f_Y(x)}$ decrease in x

The following stochastic ordering relationships due to Shaked and Shanthikumer (1994) are well known for establishing stochastic ordering of continuous distribution.

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y. \quad (11)$$

Suppose random variable X and Y are distributed as GT-L distribution $(\alpha_1, \beta_1)$and GT-L (α_2, β_2) respectively, then, the following results hold true.

(i) if $\alpha_1 = \alpha_2, , \beta_1 < \beta_2$, $then\ X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$

(ii) if $\alpha_1 < \alpha_2, , \beta_1 = \beta_2$, $then\ X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y$

Proof.

Start with $\frac{f_X(x)}{f_Y(x)} = \frac{2\alpha_1\beta_1\,[\,1-[x(2-x)]^{,\beta_1}]^{\alpha_1-1}\,(1-x)[x(2-x)]^{\beta_1-1}}{2\alpha_2\beta_2\,[\,1-[x(2-x)]^{\beta_2}]^{\alpha_2-1}\,(1-x)[x(2-x)]^{\beta_2-1}}$

If $\alpha_1 = \alpha_2, \beta_1 < \beta_2$ We obtain, $\frac{f_X(x)}{f_Y(x)} = \frac{\beta_1\,[\,1-[x(2-x)]^{,\beta_1}]^{\alpha_1-1}\,[x(2-x)]^{\beta_1-\beta_2}}{\beta_2\,[\,1-[x(2-x)]^{\beta_2}]^{\alpha_1-1}}$,

$$\frac{log}{}\frac{f_X(x)}{f_Y(x)} = \log(\beta_1) + (\alpha - 1)\log\big(1 - [x(2-x)]^{,\beta_1}\big) + (\beta_1 - 1)\log(2x - x^2) - \log(\beta_2)$$
$$-(\alpha - 1)\log\,[1 - [x(2-x)]^{\beta_2}] - (\beta_2 - 1)log(2x - x^2).$$

We obtain $\frac{d}{dx}\left(\frac{logf_X(x)}{f_Y(x)}\right) < 0$ for all $x \geq 0$. Thus $\frac{d}{dx}\left(\frac{logf_X(x)}{f_Y(x)}\right)$ is decreasing in x and hence $X \leq_{lr} Y$.

Moreover; X is said to be smaller than Y in other different orderings as stochastic order (denoted by $X \leq_{st} Y$), hazard rate order (denoted by $X \leq_{hr} Y$), and reversed hazard rate. These stochastic orders are related to each other and the remaining statements follow from the implications (11).

3. Statistical Properties for Generalized Topp-Leone distribution

This section is devoted to study some statistical properties of GT-L distribution such as quantile function, median, first quartile, third quartile, mode, moments about origin, moment generating function, order statistics, and moments of joint order statistics.

3.1 Quantile for GT-L Distribution

The quantile function is useful measure for describing the distribution of a random variable.it plays a key role when simulating random numbers and can also be used to compute the median and skewness of the distribution of a random variable. If U has a uniform $U(0,1)$ distribution, the quantile can be obtained by inverting (8) as follows

$$x_u = 1 - \sqrt{1 - [1 - (1-u)^{1/\alpha}]^{1/\beta}} \quad 0 < u < 1 \quad (12)$$

Where $F^{-1}(u) = x_u = Q_x$. In particular, the first quartile, say Q1, the median, say Q2, and the third quartile, say Q3 are obtained by setting u=0.25,0.5,0.75, respectively, in (12) as follows:

$$x_q = 1 - \sqrt{1 - [1 - 0.25^{1/\alpha}]^{1/\beta}}, \quad 0 < u < 1 \tag{13}$$

$$x_q = 1 - \sqrt{1 - [1 - 0.5^{1/\alpha}]^{1/\beta}}, \quad 0 < u < 1 \tag{14}$$

$$x_q = 1 - \sqrt{1 - [1 - 0.75^{1/\alpha}]^{1/\beta}}, \quad 0 < u < 1 \tag{15}$$

Moreover, the mode of the GT-L distribution is given as follows

$$x_{mod} = 1 - \sqrt{1 - \left(\frac{\beta-1}{\beta\alpha-1}\right)^{1/\beta}} \quad . \tag{16}$$

3.2 Moments for GT-L Distribution

The r^{th}moment about zero μ'_r is given by the following theorem.

Therom1. The r^{th}moment about zero μ'_r is given by

$$\mu'_r = \beta \sum_{j=0}^{\infty} \frac{c(r,j)(-1)^{\frac{j}{\beta}}}{j - r + \beta} \quad , \text{for } \beta \neq 1, and\ r = 1, \dots, n$$

Where $c(r,0) = 2^r$, $c(r,1) = r2^{r-1}, \dots \prod_{i=1}^{j-1}(r - j - i\ , j) \geq 2$ (17)

Proof.

Start with $\mu'_r = \boldsymbol{E}(X^r) = \int_0^1 x^r\ dF\ (x)$.

Using power series expansion identity given in Prudnikov et al. (1986), for $\theta > 0$

$$(\mathbf{1} - \boldsymbol{x})^{\boldsymbol{\theta}} = \sum_{\boldsymbol{i}=\mathbf{0}}^{\infty} \frac{(\mathbf{-1})^{\boldsymbol{i}}\ \Gamma(\boldsymbol{\theta}+\mathbf{1})}{\boldsymbol{i}!\Gamma(\boldsymbol{\theta}+\mathbf{1}-\boldsymbol{i})}\ x^i \qquad |\ x\ | < 1 \tag{18}$$

$$(\mathbf{1} - \boldsymbol{x})^{\boldsymbol{\theta}-\mathbf{1}} = \sum_{\boldsymbol{i}=\mathbf{0}}^{\infty} \frac{(\mathbf{-1})^{\boldsymbol{i}}\ \Gamma(\boldsymbol{\theta})}{\boldsymbol{i}!\Gamma(\boldsymbol{\theta}-\boldsymbol{i})}\ x^i \qquad |\ x\ | < 1 \tag{19}$$

$$\mu'_r = E(X^r) = \sum_{i,j,k=0}^{\infty} \frac{(-1)^i\ \Gamma(2)}{i!\Gamma(2-i)}\ \frac{(-1)^j\ \Gamma\left(\frac{i}{2}+1\right)}{j!\Gamma\left(\frac{i}{2}+1-j\right)}\ \frac{(-1)^k\ \Gamma\left(\frac{j}{\beta}+1\right)}{k!\Gamma\left(\frac{j}{\beta}+1-k\right)} B(1, \frac{k+\alpha}{\alpha}) \tag{20}$$

Such that $\beta > \frac{j}{k-1}$, $i > 2(j-1)$.

3.3 Moment Generating Function for GT-L Distribution

The moment generating function of GT-L distribution can be easily obtained from the r^{th}moment using Taylor expansion through this equation

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!}\ E(X^r) \tag{21}$$

By Substitution from (20), we get,

$$M_X(t) = \sum_{r,i,j,k=0}^{\infty} \frac{t^r}{r!}\ \frac{(-1)^i\ \Gamma(r+1)}{i!\Gamma(r+1-i)}\ \frac{(-1)^j\ \Gamma\left(\frac{i}{2}+1\right)}{j!\Gamma\left(\frac{i}{2}+1-j\right)}\ \frac{(-1)^k\ \Gamma\left(\frac{j}{\beta}+1\right)}{k!\Gamma\left(\frac{j}{\beta}+1-k\right)} B\left(1, \frac{k+\alpha}{\alpha}\right) \tag{22}$$

Such that $\beta > \frac{j}{k-1}$, $i > 2(j-1)$ and $r > i - 1$.

3.4 Order statistics of GT-L distribution

Let $X_1, X_2, X_3, \dots X_n$ be independent and identically distributed random variables drawn from GT-L distribution. Let $X_{(k)}; k = 1,2,3, \dots, n$ be the k^{th} order statistics, then the pdf of $X_{(k)}$ is defined as

$$f_{x_{(k)}(x) = C_{n,k}[F(x)]^{k-1}[1-F(x)]^{n-k}\ f(x)}$$

Where $x = x_{(k)}$, $C_{n,k} = \frac{n!}{k!(n-k)!}$.

So, the pdf of the k^{th} order statistics of GT-L distribution is given as

$$f_{x_{(k)}}(\mathrm{x}) = 2\alpha\beta C_{n,k}\left(1 - x_{(k)}\right)[x_{(k)}(2 - x_{(k)})]^{\beta-1}[1 - [\left(x_{(k)}\left(2 - x_{(k)}\right)\right]^{\beta}]^{\alpha-1}$$
$$.\left[\,[\,1 - \left[x_{(k)}\left(2 - x_{(k)}\right)\right]^{\beta}]^{\alpha}\right]^{k-1}\left[(1 - \left[\,1 - \left[x_{(k)}\left(2 - x_{(k)}\right)\right]^{\beta}\right]^{\alpha})\right]^{n-k}.$$

Special cases for $X_{(1)}$ and $X_{(n)}$ are respectively considered as

$$f_{X_{(1)}(x)= 2\alpha\beta n\left(1-x_{(1)}\right)[x_{(1)}(2-x_{(1)})]^{\beta-1}[1-[\left(x_{(1)}\left(2-x_{(1)}\right)\right]^{\beta}]^{n-2+\alpha}} \quad , \qquad 0< x_{(1)}<1 \tag{23}$$

and

$$f_{X_{(n)}(x) = 2\alpha\beta n\left(1-x_{(n)}\right)[x_{(n)}(2-x_{(n)})]^{\beta-1}[1-[\left(x_{(n)}\left(2-x_{(n)}\right)\right]^{\beta}]^{\alpha-1}}$$
$$.[1 - \{1 - [\left(x_{(n)}\left(2 - x_{(n)}\right)\right]^{\beta}\}]^{\alpha-1} \;, 0 < x_{(n)} < 1 \tag{24}$$

3.5 Moments of Order Statistics for GT-L Distribution

The k^{th}moment about zero ${\mu^{k}}_{r:n}$ of r^{th} order statistic $X_{(r)}$is given through the following theorem

Theorem 2. The k^{th}moment about zero ${\mu^{k}}_{r:n}$ of r^{th} order statistic $X_{(r)}$is given as

$${\mu^{k}}_{r:n} = C_{n,k}\sum_{i,j,m=0}^{\infty} \frac{(-1)^{i+j+m}\,\Gamma(n+1)}{i!j!m!\Gamma(n+1-i)} \frac{\Gamma\left(\frac{i}{2}+1\right)}{\Gamma\left(\frac{i}{2}+1-j\right)} \frac{\Gamma\left(\frac{j}{\beta}+1\right)}{\Gamma\left(\frac{j}{\beta}+1-m\right)} \frac{\Gamma(\frac{nm}{\alpha}+n)}{\Gamma(\frac{m}{\alpha}+n+1)}. \tag{25}$$

Proof.

Start with ${\mu^{k}}_{r:n} = E(X^k) = C_{n,r}\int_0^1 x^k[F(x)]^{r-1}[1 - F(x)]^{n-r}\,dF(x)$

Use the transformation $U = F(X)$ such that $x = F^{-1}(u) = 1 - \sqrt{1 - [1 - (1-u)^{1/\alpha}]^{1/\beta}}$

So,

$${\mu^{k}}_{r:n} = C_{n,r}\sum_{i,j,m=0}^{\infty} \frac{(-1)^{i}\,\Gamma(n+1)}{i!\Gamma(n+1-i)} \frac{(-1)^{j}\,\Gamma\left(\frac{i}{2}+1\right)}{j!\Gamma\left(\frac{i}{2}+1-j\right)} \frac{(-1)^{m}\,\Gamma\left(\frac{j}{\beta}+1\right)}{m!\Gamma\left(\frac{j}{\beta}+1-m\right)} B(\frac{m}{\alpha} + n - r + 1, r) \tag{26}$$

Such that $i < n+1,\ j < \frac{i}{2}+1,\ 2(j-1), m < \frac{j}{\beta}+1, \beta > \frac{j}{m} - 1.$

The k^{th}moment of $X_{(1)}$ is given as

$${\mu^{k}}_{1:n} = C_{n,k}\sum_{i,j,m=0}^{\infty} \frac{(-1)^{i}\,\Gamma(n+1)}{i!\Gamma(n+1-i)} \frac{(-1)^{j}\,\Gamma\left(\frac{i}{2}+1\right)}{j!\Gamma\left(\frac{i}{2}+1-j\right)} \frac{(-1)^{m}\,\Gamma\left(\frac{j}{\beta}+1\right)}{m!\Gamma\left(\frac{j}{\beta}+1-m\right)} \frac{\Gamma(\frac{m}{\alpha}+n)}{\Gamma(\frac{m}{\alpha}+n+1)}. \tag{27}$$

$${\mu^{k}}_{n:n} = C_{n,k}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^{i}\,\Gamma(n+1)}{i!\Gamma(n+1-i)} \frac{(-1)^{j}\,\Gamma\left(\frac{i}{2}+1\right)}{j!\Gamma\left(\frac{i}{2}+1-j\right)} \frac{(-1)^{m}\,\Gamma\left(\frac{j}{\beta}+1\right)}{m!\Gamma\left(\frac{j}{\beta}+1-m\right)} \frac{\Gamma(\frac{m}{\alpha}+1)}{\Gamma(\frac{m}{\alpha}+n+1)} \tag{28}$$

3.6 Joint Order Statistics for GT-L Distribution

The joint pdf of $X_{(r)}$ and $X_{(s)}$, $1 \le r < s \le n$, for a sample of size n

$$f(x_{(r)}, x_{(s)}) = C_{n,r,s}\left[F(x_{(r)})\right]^{r-1}\left[1 - F\left(x_{(s)}\right)\right]^{n-s}\left[F\left(x_{(s)}\right) - F\left(x_{(r)}\right)\right]^{s-r-1} f\left(x_{(r)}\right)f\left(x_{(s)}\right)$$

$$where\ C_{n,r,s} = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}$$

$$f(x_{(r)}, x_{(s)}) = 4C_{n,r,s}\,\alpha^2\beta^2\,{x_{(r)}}^{\beta-1}{x_{(s)}}^{\beta-1}\left(2 - x_{(r)}\right)^{\beta-1}\left(2 - x_{(s)}\right)^{\beta-1}\left(1 - x_{(r)}\right)\left(1 - x_{(s)}\right)$$

$$.\left\{1-\left[1-\left[x_{(r)}\left(2-x_{(r)}\right)\right]^{\beta}\right]^{\alpha}\right\}^{r-1}\left\{\left[1-\left[x_{(s)}\left(2-x_{(s)}\right)\right]^{\beta}\right]^{\alpha}\right\}^{n-s}$$

$$.\left\{1-\left[1-\left[x_{(s)}\left(2-x_{(s)}\right)\right]^{\beta}\right]^{\alpha}-\left(1-\left[1-\left[x_{(r)}\left(2-x_{(r)}\right)\right]^{\beta}\right]^{\alpha}\right)\right\}^{s-r-1}$$

$0 < x_{(r)} < x_{(s)} < 1$ (29)

3.7 Moments of Joint Order Statistics for GT-L Distribution

Therom3. The k^{th} and L^{th} moment about zero ${\mu^{(k,L)}}_{r:n}$ of $X_{(r)}$ and $X_{(s)}$ is given by

$${\mu^{(k,L)}}_{r:n} = C_{n,r,s}\sum_{j_3}^{s-r-1}\sum_{j_2}^{\infty}\sum_{j_1}^{\infty} c(k,j_1)\, c(L,j_2)\frac{\binom{s-r-1}{j_3}(-1)^{\frac{j_1+j_2}{\beta}+j_3}}{\frac{j_1-k}{\beta}+j_3+r}$$

$$. B\left(\frac{j_1+j_2-k-L}{\beta}+s, n-s+1\right), for\ k=1,2,\dots,n, L=1,2,\dots,n. \quad (30)$$

Where $c(k,j_1), c(L,j_2)$ as given in (25)

Proof.

Start with

$${\mu^{(k,L)}}_{r:n} = C_{n,r,s}\int_0^1\int_0^1 \{x_{(s)}\}^L\{x_{(r)}\}^k\left[F(x_{(r)})\right]^{r-1}\left[1-F\left(x_{(s)}\right)\right]^{n-s}\left[F\left(x_{(s)}\right)\right.$$

$$\left.-F\left(x_{(r)}\right)\right]^{s-r-1}\, du\, dv$$

Use the transformations $U = F\left(X_{(r)}\right)$ and $V = F\left(X_{(s)}\right)$, gets

$x_{(s)} = F^{-1}(v) = 1-\sqrt{1-\left[1-(1-v)^{1/\alpha}\right]^{1/\beta}}$ and $x_{(r)} = F^{-1}(u) = 1-\sqrt{1-\left[1-(1-u)^{1/\alpha}\right]^{1/\beta}}$

$${\mu^{(k,L)}}_{r:n} = 4\alpha^2\,\beta^2 C_{n,r,s}\int_0^1\int_0^1 [u]^{r-1}[1-v]^{n-s}[v-u]^{s-r-1}\left\{1\right.$$

$$\left.-\left\{1-\left[1-(1-u)^{1/\alpha}\right]^{1/\beta}\right\}.5\right\}^k\left\{1-\left\{1-\left[1-(1-v)^{1/\alpha}\right]^{1/\beta}\right\}.5\right\}^L du\, dv$$

$$=\binom{s-r-1}{z}\sum_{z=0}^{\infty}\ \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^i\,\Gamma(k+1)}{i!\Gamma(k+1-i)}\frac{(-1)^j\,\Gamma(\frac{i}{2}+1)}{j!\Gamma(\frac{i}{2}+1-j)}\frac{(-1)^m\,\Gamma(\frac{j}{\beta}+1)}{m!\Gamma(\frac{j}{\beta}+1-m)}\int_0^1(1-u)^{(\frac{t}{\alpha})}\ u^{r-1}\,du$$

$$=\binom{s-r-1}{z}\sum_{z=0}^{\infty}\ \sum_{\boldsymbol{i=0}}^{\infty}\sum_{\boldsymbol{j=0}}^{\infty}\sum_{\boldsymbol{m=0}}^{\infty}\frac{(-1)^i\,\Gamma(\boldsymbol{k+1})}{\boldsymbol{i}!\Gamma(\boldsymbol{k+1-i})}\frac{(-1)^j\,\Gamma\left(\frac{i}{2}+\boldsymbol{1}\right)}{\boldsymbol{j}!\Gamma\left(\frac{i}{2}+\boldsymbol{1}-\boldsymbol{j}\right)}\frac{(-1)^m\,\Gamma\left(\frac{j}{\beta}+\boldsymbol{1}\right)}{\boldsymbol{m}!\Gamma\left(\frac{J}{\beta}+\boldsymbol{1}-\boldsymbol{m}\right)}\ B\left(\frac{t}{\alpha}+1,r\right)\int_0^1[1-$$

$$v]^{n-s}[v]^{s-r-1-z}\left\{1-\left\{1-\left[1-(1-v)^{1/\alpha}\right]^{1/\beta}\right\}.5\right\}^L\ dv$$

$$=\binom{s-r-1}{z}\sum_{z=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-\boldsymbol{1})^i\,\Gamma(\boldsymbol{L+1})}{\boldsymbol{i}!\,\Gamma(\boldsymbol{L+1-i})}\frac{(-\boldsymbol{1})^j\,\Gamma\left(\frac{i}{2}+\boldsymbol{1}\right)}{\boldsymbol{j}!\,\Gamma\left(\frac{i}{2}+\boldsymbol{1}-\boldsymbol{j}\right)}\frac{(-\boldsymbol{1})^m\,\Gamma\left(\frac{j}{\beta}+\boldsymbol{1}\right)}{\boldsymbol{m}!\,\Gamma\left(\frac{J}{\beta}+\boldsymbol{1}-\boldsymbol{m}\right)}\frac{(-1)^{\mathrm{i}}\,\Gamma(\mathrm{k}+1)}{\mathrm{i}!\,\Gamma(\mathrm{k}+1-\mathrm{i})}$$

$$\boldsymbol{B}\left(\frac{t}{\alpha}+1,r\right)B\left(\frac{m}{\alpha}-s+n+1,s-r-z\right). \quad (31)$$

4. MLE for the Parameters of GT-L Distribution

We consider the estimation of the unknown parameters of the GT-L distribution by maximum likelihood estimation method for the complete samples. The maximum likelihood estimates (MLEs) enjoy desirable properties that can be used when constructing confidence intervals and deliver simple approximations that work well in finite samples. Let $x_1, x_2, x_3, \dots, x_n$ be independent and identically distributed observed random sample of size n from the GT-L distribution. The likelihood function is given by.

$$Lx;\alpha,\beta) = \prod_{i=1}^{n} 2\beta\alpha\,[\,1-[x_i(2-x_i)]^{\beta}]^{\alpha-1}(1-x_i)[x_i(2-x_i)]^{\beta-1}$$

After differentiating the $l(x;\alpha,\beta) = \log L(x;\alpha,\beta)$ and equating to zero the MLEs for α,β can be expressed as follows.

$$\hat{\alpha} = \frac{-n}{\sum_{i=1}^{n}\log 1-[x_i(2-x_i)]^{\hat{\beta}}} \tag{32}$$

$$\hat{\beta} = n\,\frac{1-[x_i(2-x_i)]^{\hat{\beta}}}{(\hat{\alpha}-1)\sum_{i=1}^{n} log(2x_i-x_i^{2})[x_i(2-x_i)]^{\hat{\beta}}-\sum_{i=1}^{n}\log\,(2x_i-x_i^{2})[1-x_i(2-x_i)]^{\hat{\beta}}} \tag{33}$$

4.1 Approximate Confidence Intervals

The most common method to set confidence bounds for the parameters is to use the asymptotic normal distribution of the ML estimators. In relation to the asymptotic variance covariance matrix of the ML estimators of the parameters, Fisher information matrix $I(\hat{\alpha},\hat{\beta})$ is obtained. The asymptotic variance- covariance matrix for the estimators $\hat{\alpha}\ and\ \hat{\beta}$ can be obtained by inverting Fisher information matrix. The elements of the asymptotic variance covariance matrix can be approximated by Cohen`s (1965) as follows;

$$I^{-1}(\hat{\alpha},\hat{\beta}) = -E\begin{bmatrix} \dfrac{\partial^2 LnL}{\partial\alpha^2} & \dfrac{\partial^2 LnL}{\partial\alpha\partial\beta} \\ \dfrac{\partial^2 LnL}{\partial\alpha\partial\beta} & \dfrac{\partial^2 LnL}{\partial\beta^2} \end{bmatrix}^{-1} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

The elements of Fisher information matrix are obtained as follows

$$I_{11} = \frac{\partial^2 LnL}{\partial\alpha^2} = \frac{-n}{\hat{\alpha}^2} \tag{34}$$

$$I_{22} = \frac{\partial^2 LnL}{\partial\beta^2} = \frac{-n-(\hat{\alpha}-1)(2x_i-x_i^{2})\log\,(2x_i-x_i^{2})\sum_{i=1}^{n}\log\,(2x_i-x_i^{2})}{(\,[1-x_i(2-x_i)]^{\widehat{\beta)}}\,^{2}} \tag{35}$$

$$I_{21} = I_{12} = \frac{\partial^2 LnL}{\partial\alpha\partial\beta} = \frac{\partial^2}{\partial\beta\partial\alpha} = \frac{-\sum_{i=1}^{n}(2x_i-x_i^{2})^{\beta}\log\,(2x_i-x_i^{2})}{1-[x_i(2-x_i)]^{\hat{\beta}}} \tag{36}$$

A numerical technique using R package are used to obtain the variance-covariance matrix
The large sample $(1-\gamma)100\%$ confidence interval for α,β are given by

$$(\hat{\alpha}_L\,,\hat{\alpha}_U) = \hat{\alpha}\ \pm Z_{\frac{\gamma}{2}}\sqrt{V(\hat{\alpha})} \tag{37}$$

$$(\hat{\beta}_L\,,\hat{\beta}_U) = \hat{\beta}\ \pm Z_{\frac{\gamma}{2}}\sqrt{V(\hat{\beta})} \tag{38}$$

Where $Z_{\frac{\gamma}{2}}$ is the standard normal random variable and $(1-\gamma)$ confidence coefficient.

5. Application to Real Data

In this section, The GT-L distribution is compared with the Topp-Leone distribution that is given in Equation (1), and the unit Gompertz (UG) distribution which is given below

$$f_{UG}(y;\alpha,\beta)= \alpha\beta y^{-(\beta+1)} e^{-\alpha(y^{-\beta}-1)} \ , \quad 0 < y < 1, \beta > 0 \text{ , and } \alpha > 0$$

All the competitive models will be fitted to two real data sets, and the unknown parameters of the GT-L distribution and all the distributions mentioned will be estimated by the maximum-likelihood method (MLE), the values of the Kolmogorov-Smirnov (KS) statistic (the distance between the empirical CDFs and the fitted CDFs), Akaike information criterion (AIC), Bayesian information criterion (BIC), and the consistent Akaike information criterion (CAIC) are calculated for this distributions to observe which distribution is better. The numerical results are obtained by R program, and listed in Tables 2,3

Data Set 1: This data is about ordered failure of 20 components, Nigm et al. (2003)

The actual data are:

0.0009, 0.004, 0.0142, 0.0221, 0.0261, 0.0418, 0.0473, 0.0834, 0.1091, 0.1252, 0.1404, 0.1498, 0.175, 0.2031, 0.2099, 0.2168, 0.2918, 0.3465, 0.4035, 0.614.

Figure 6 shows empirical and fitted cdf of GT-L distribution for data set1. Moreover, Figure 7 shows histogram of the fitted distributions for data set 1

Data Set 2: Failure data

The second data are taken from Murthy et al. (2004) which represents the time between failures of 30 repairable items and is reported in Table 1.

Table1: The Failure Time of 30 Items in Data Set 2

0.143	0.011	0.071	0.0221	0.077	0.263	0.149	0.346	0.246	0.059
0.074	0.123	0.094	0.040	0.174	0.473	0.223	0.045	0.070	0.106
0.146	0.030	0.182	0.237	0.063	0.123	0.124	0.197	0.186	0.117

Figure (8) showed the fitted lines for the pdfs for the given data sets and corresponding distributions. The figures also indicate that the GT-L distribution provides a better fit than the other distributions at least for data set 2.

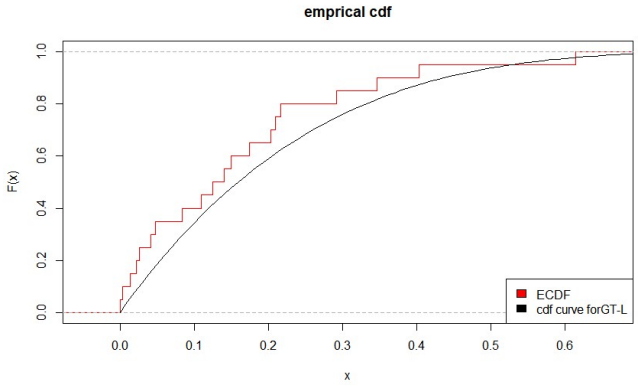


Figure 6: Empirical, fitted cdf of GT-L distribution for data set

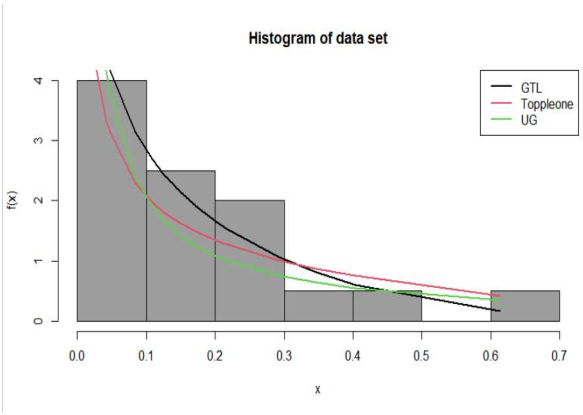


Figure 7. Histogram of the Fitted Distributions for Data Set 1

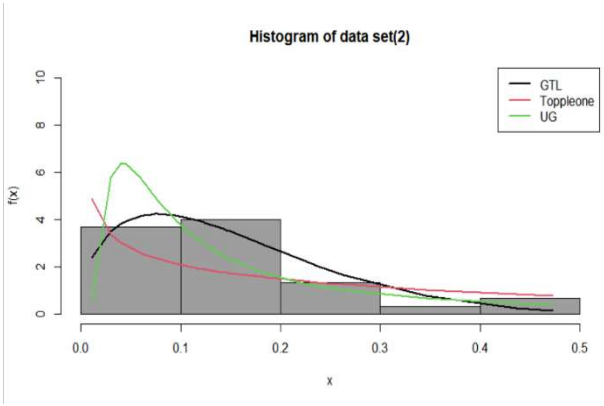


Figure 8. Histogram of the Fitted Distributions for Data Set 2

Table 2: Maximum Likelihood Estimates and Goodness-of-fit Measures (p-values) and Likelihood-Based Statistics for GT-L Distribution Depending on Data Set 1 of the Ordered Failure of 20 Components.

Distributions										
GT-L	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	P-value	k-s	LL	-AIC	-BIC	-AICC	-HQIC
	_	0.739	1.859	0.9757	0.1003	17.234	32.47	28.48	34.87	30.079
T-L	_	0.511	–	0.4481	0.1848	15.617	30.23	28.24	32.03	29.038
UG	0.775	0.278	_	0.7091	0.1494	14.763	27.53	23.53	29.93	25.136

Table 3: Maximum Likelihood Estimates and Goodness-of-fit Measures (p-values) and Likelihood-Based Statistics for GT-L Distribution Depending on Data Set 2 of the Failure Time of 30 Items.

Distributions										
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	P-value	k-s	LL	-AIC	-BIC	-AICC	-HQIC
GT-L	_	1.452	4.772	0.9854	0.083	29.059	56.12	51.32	58.72	53.223
T-L	_	0.653	–	0.0089	0.3003	18.629	36.26	33.86	38.126	34.811
UG	0.115	0.912	_	0.60971	0.1388	25.17	48.35	43.54	50.945	45.449

6. Simulation Study

A simulation study was carried out to check the performance of the accuracy of point and interval estimates for several cases. We generate N = 1000 random samples of size n =15, 35, and 100 from GT-L distribution for specific values of the parameters $\alpha, \beta = .5,1,1.5, .5, 2.5$and 3 respectively The ML estimates of the parameter are obtained in complete. The biases of the estimated parameters, coverage probability (CP), asymptotic confidence interval (ACI), average interval length (AIL) and mean square errors (MSEs) are computed. The numerical results are obtained by R program and listed in Table 4.

Table 4. Estimates values, bias, asymptotic confidence interval, and average interval length

N	Real value	Estimate	MSE	ACI	CP	AIL	bias
15	$\alpha = 1.5$	2.156	1.543	4.380 (-0.0662)	96.4	4.4464	0.6122
	$\beta = 2.5$	3.137	0.9362	4.959 (1.316)	95.2	3.6425	0.5963
	$\alpha = 1$	1.366	0.4390	2.5426 (0.1909)	95.7	2.3517	0.3826
	$\beta = 3$	3.878	1.7804	6.3741 (1.3824)	95.2	4.9916	0.9039
	$\alpha = .5$	0.6339	0.0486	1.0609 (0.2070)	96.1	0.85389542	0.131
	$\beta = .5$	0.7044	0.1023	1.2821 (0.1266)	94.7	1.1555262	0.198
35	$\alpha = 1.5$	1.848	0.2057	0.8992 (2.7970)	96.7	1.8977	0.3309
	$\beta = 2.5$	2.8719	0.2474	1.7800 (3.9639)	97.4	2.1839	0.3898
	$\alpha = 1$	1.1959	0.0714	0.6389 (1.7529)	95.7	1.1140	0.20990
	$\beta = 3$	3.5313	0.5415983	2.0452 (5.0173)	96.3	2.9721	0.54986
	$\alpha = .5$	0.57667	0.0129	0.34072 (0.8126)	96.3	0.47187	0.08055
	$\beta = .5$	0.61889	0.0243	0.2943 (0.94347)	96.5	0.64917	0.11729
50	$\alpha = 1.5$	1.77278	0.1099	1.0347 (2.5108)	96.7	1.4761	0.27278
	$\beta = 2.5$	2.8493	0.1960	1.9392 (3.7594)	97.2	1.8202	0.3493
	$\alpha = 1$	1.1770	0.0465	0.72557 (1.6285)	96.9	0.90289	0.17701
	$\beta = 3$	3.4763	0.3369	2.2542 (4.6984)	96.8	2.4442	0.47631
	$\alpha = .5$	0.57107	0.0083	0.37782 (0.76432)	96.6	0.38651	0.07107
	$\beta = .5$	0.60472	0.0159	0.3409 (0.8686)	96.1	0.52769	0.10472
100	$\alpha = 1.5$	1.75319	0.05736	1.2453 (2.2611)	96.2	1.01573	0.25319
	$\beta = 2.5$	2.82078	0.0943	2.1855 (3.45604)	96.9	1.2705	0.32077
	$\alpha = 1$	1.1462	0.0189	0.84202 (1.4503)	98	0.60831	0.14617
	$\beta = 3$	3.4194	0.1582	2.5697 (4.2692)	97.4	1.6995	0.4194
	$\alpha = .5$	0.5615	0.00389	0.42876 (0.69422)	97.7	0.26546	0.06148
	$\beta = .5$	0.58961	0.00712	0.40862 (0.7706)	97.3	0.36197	

							0.08961

7. Bivariate Generalized Topp-Leone (BGT-L) Distribution

In this section, the univariate GT-L distribution is used for creating the bivariate GT-L distribution by using Farlie–Gumbel–Morgenstern (FGM) copula.

Suppose X and Y distributed as GT-L distribution with (β_1, θ_1) and (β_2, θ_2) respectivaly and with distribution function $F_1(x)$and $F_2(y)$ respectively. Then the bivariate vector (X, Y) has a bivariate GT-L distribution with shape parameters $\beta_1, \beta_2, \theta_1,$ and θ_2 we will denote the bivariate generalized Topp-Leone distribution by BGT-L distribution. The bivariate FGM system of distributions is written as:

$$F(x,y) = F_1(x)F_2(y)\{1+\gamma\{[1-F_1(x)][1-F_2(y)]\};\ -1<\gamma<1$$

Its pdf can be obtained by a direct differentiation as

$$f(x,y) = f_1(x)f_2(y)\{1+\gamma\{[1-2F_1(x)][1-2F_2(y)]\};\ -1<\gamma<1$$

The cdf of the BGT-L distribution can be expressed as

$$F(x,y) = [1-\xi(x;\beta_1,\theta_1)][1-\xi(y;\beta_2,\theta_2)]\{1+\gamma[\xi(x;\beta_1,\theta_1)][\xi(y;\beta_2,\theta_2)]\}.$$

The pdf of the BGT-L distribution can be expressed as

$$f(x,y) = 4\beta_1\beta_2\theta_1\theta_2\,[\xi(x;\beta_1,\theta_1)]^{1-\frac{1}{\theta_1}}\,[\xi(y;\beta_2,\theta_2)]^{1-\frac{1}{\theta_2}}(1-x)(1-y)[x(2-x)]^{\beta_1-1}$$
$$.\,[y(2-y)]^{\beta_2-1}\{1+\gamma\{\big[1-2[1-\xi(x;\beta_1,\theta_1)]\big]\big[1-2[1-\xi(y;\beta_2,\theta_2)]\big]\}\}$$
$$;0<x<\ 1,0<y<1,\beta_1>0,\beta_2>0,\theta_1>0,\theta_2>0,-1<\gamma<1$$

Where $\xi(x;\beta_i,\theta_i) = \left[1-(2x-x^2)^{\beta_i}\right]^{\theta_i}; i = 1,2.$

7.1 Statistical Properties of BGT-L Distribution

The main purpose of this sub-section is to provide some basic properties of the BGT-L distribution. Such as marginal, conditional densities, different product moments and joint moment generating function.

7.1.1 The Marginal Distributions

The BGT-L distribution has GT-L distribution marginal. The marginal density function for X is as follows

$f_{GT-L}(x;\beta_1,\theta_1) = 2\beta_1\theta_1\,[\,1-(2x-x^2)^{\beta_1}]^{\theta_1-1}(1-x)[x(2-x)]^{\beta_1-1}.$

Proof.

Start with $f_{GT-L}(x;\beta_1,\theta_1) = \int_0^1 f(x,y)dy$

we get the marginal density

$$f(x,y) = f(x)\int_0^1 f(y)dy + 4\gamma f(x)F(x)\int_0^1 f(y)F(y)dy - 2\gamma f(x)F(x)\int_0^1 f(y)dy$$
$$-2\gamma f(x)\int_0^1 f(y)F(y)\,dy + \gamma f(x)\int_0^1 f(y)dy.$$

Since $\int_0^1 f(y)F(y)dy = \frac{1}{2}, \int_0^1 f(y)dy = 1.$

$f_{GT-L}(x;\beta_1,\theta_1) = f(x) + 2\gamma f(x)F(x) - 2\gamma f(x)F(x) - \gamma f(x) + \gamma f(x).$

Then $f_{GT-L}(x;\beta_1,\theta_1) = 2\beta_1\theta_1\,[\,1-(2x-x^2)^{\beta_1}]^{\theta_1-1}\,(1-x)[x(2-x)]^{\beta_1-1}.$

And Similarly,

$$f_{GT-L}(y;\beta_2,\theta_2) = 2\beta_2\theta_2\,[\,1-(2y-y^2)^{\beta_2}]^{\theta_2-1}\,(1-y)[y(2-y)]^{\beta_2-1}.$$

7.1.2 Conditional Distributions

The conditional probability distributions for X and Y are given by

$f(x|\,y) = 2\beta_1\theta_1\,[\,1-(2x-x^2)^{\beta_1}]^{\theta_1-1}\,(1-x)[x(2-x)]^{\beta_1-1}$.

And Similarly, $f(y|\,x) = \frac{f(x,y)}{f\,(x)} = 2\beta_2\theta_2\,[\,1-(2y-y^2)^{\beta_2}]^{\theta_2-1}\,(1-y)[y(2-y)]^{\beta_2-1}$.

7.1.3 Product Moments and Moment Generating Function

Let (X,Y) denote a random variable distributed as BGT-L distribution .Then, its r^{th} and s^{th} joint moments around zero denoted by μ'_{rs}can be expressed as follows:

$$\begin{aligned}\mu'_{rs} &= \mu'_r\mu'_s + \gamma\mu'_r\mu'_s\\ &+4\gamma[(\mu'_r - 2^{\beta_1+\beta_1 i-j}\beta_1\theta_1\textstyle\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{i+j}\Gamma(2\theta_1)\Gamma(\beta_1+\beta_1 i)}{i!j!\Gamma(2\theta_1-i)\Gamma(\beta_1+\beta_1 i-j)}B(r+j+1,2))\\ &.(\mu'_s - 2^{\beta_2+\beta_2 i-j}\beta_2\theta_2\textstyle\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{i+j}\Gamma(2\theta_2)\Gamma(\beta_2+\beta_2 i)}{i!j!\Gamma(2\theta_2-i)\Gamma(\beta_2+\beta_2 i-j)}B(s+j+1,2))]\\ &-2\gamma\mu'_s[(\mu'_r - 2^{\beta_1+\beta_1 i-j}\beta_1\theta_1\textstyle\sum_{i,j=1}^{\infty}\frac{(-1)^{i+j}\Gamma(2\theta_1)\Gamma(\beta_1+\beta_1 i)}{i!j!\Gamma(2\theta_1-i)\Gamma(\beta_1+\beta_1 i-j)}B(r+j+1,2))]\\ &-2\gamma\mu'_r[(\mu'_s - 2^{\beta_2+\beta_2 i-j}\beta_2\theta_2\textstyle\sum_{i,j=1}^{\infty}\frac{(-1)^{i+j}\Gamma(2\theta_2)\Gamma(\beta_2+\beta_2 i)}{i!j!\Gamma(2\theta_2-i)\Gamma(\beta_2+\beta_2 i-j)}B(s+j+1,2))].\end{aligned}$$

Where $\int_0^1 x^r f(x)F(x)dx = \mu'_r - 2^{\beta+\beta i-j}\beta\theta\sum_{i,j=1}^{\infty}\frac{(-1)^{i+j}\Gamma(2\theta)\Gamma(\beta+\beta i)}{i!j!\Gamma(2\theta-i)\Gamma(\beta+\beta i-\ )}B(r+j+1,2)$.

Such that $i<2\theta_2\ ,\ j<\beta_2+\beta_2 i,\ i<2\theta_1\ ,\ j<\beta_1+\beta_1 i$

Proof.

$$\mu'_{rs} = E(X^rY^s) = \int_0^1\int_0^1 x^r y^s\ f(x,y)dx\,dy.$$

$$\begin{aligned}&= \textstyle\int_0^1\int_0^1 x^r y^s f(x)f(y)dxdy + \gamma\int_0^1\int_0^1 x^r y^s f(x)f(y)dxdy\\ &+4\gamma\textstyle\int_0^1\int_0^1 x^r y^s f(x)f(y)F(x)F(y)dxdy - 2\gamma\int_0^1\int_0^1 x^r y^s f(x)f(y)F(x)dxdy\\ &-2\gamma\textstyle\int_0^1\int_0^1 x^r y^s f(x)f(y)F(y)]dx\,dy.\end{aligned}$$

we substitute μ'_r in product moment for BGT-L distribution

$$\begin{aligned}\mu'_{rs} &= \mu'_r\mu'_s + \gamma\mu'_r\mu'_s\\ &+4\gamma[(\mu'_r - 2^{\beta_1+\beta_1 i-j}\beta_1\theta_1\textstyle\sum_{i,j=1}^{\infty}\frac{(-1)^{i+j}\Gamma(2\theta_1)\Gamma(\beta_1+\beta_1 i)}{i!j!\Gamma(2\theta_1-i)\Gamma(\beta_1+\beta_1 i-j)}B(r+j+1,2))\\ &.(\mu'_s - 2^{\beta_2+\beta_2 i-j}\beta_2\theta_2\textstyle\sum_{i,j=1}^{\infty}\frac{(-1)^{i+j}\Gamma(2\theta_2)\Gamma(\beta_2+\beta_2 i)}{i!j!\Gamma(2\theta_2-i)\Gamma(\beta_2+\beta_2 i-j)}B(s+j+1,2))]\\ &-2\gamma\mu'_s[(\mu'_r - 2^{\beta_1+\beta_1 i-j}\beta_1\theta_1\textstyle\sum_{i,j=1}^{\infty}\frac{(-1)^{i+j}\Gamma(2\theta_1)\Gamma(\beta_1+\beta_1 i)}{i!j!\Gamma(2\theta_1-i)\Gamma(\beta_1+\beta_1 i-j)}B(r+j+1,2))]\\ &-2\gamma\mu'_r[(\mu'_s - 2^{\beta_2+\beta_2 i-j}\beta_2\theta_2\textstyle\sum_{i,j=1}^{\infty}\frac{(-1)^{i+j}\Gamma(2\theta_2)\Gamma(\beta_2+\beta_2 i)}{i!j!\Gamma(2\theta_2-i)\Gamma(\beta_2+\beta_2 i-j)}B(s+j+1,2))].\end{aligned}$$

Consequently, the joint moment generating function of the BGT-L distribution is given as:

$$\begin{aligned}M_{X,Y}(t_1,t_2) &= \sum_{n,m=0}^{\infty}\frac{t^n t^m}{n!\,m!}E(X^nY^m) = \sum_{n,m=0}^{\infty}\frac{t^n t^m}{n!\,m!}\mu'_r\mu'_s + \gamma\mu'_r\mu'_s\\ &+4\gamma[(\mu'_r - 2^{\beta_1+\beta_1 i-j}\beta_1\theta_1\textstyle\sum_{i,j=1}^{\infty}\frac{(-1)^{i+j}\Gamma(2\theta_1)\Gamma(\beta_1+\beta_1 i)}{i!j!\Gamma(2\theta_1-i)\Gamma(\beta_1+\beta_1 i-j)}B(r+j+1,2))\end{aligned}$$

$$.(\mu'_s - 2^{\beta_2+\beta_2 i-j}\beta_2\theta_2 \sum_{i,j=1}^{\infty} \frac{(-1)^{i+j}\Gamma(2\theta_2)\Gamma(\beta_2+\beta_2 i)}{i!j!\Gamma(2\theta_2-i)\Gamma(\beta_2+\beta_2 i-j)} B(s+j+1,2))]$$
$$-2\gamma\mu'_s[(\mu'_r - 2^{\beta_1+\beta_1 i-j}\beta_1\theta_1 \sum_{i,j=1}^{\infty} \frac{(-1)^{i+j}\Gamma(2\theta_1)\Gamma(\beta_1+\beta_1 i)}{i!j!\Gamma(2\theta_1-i)\Gamma(\beta_1+\beta_1 i-j)} B(r+j+1,2))]$$
$$-2\gamma\mu'_r[(\mu'_s - 2^{\beta_2+\beta_2 i-j}\beta_2\theta_2 \sum_{i,j=1}^{\infty} \frac{(-1)^{i+j}\Gamma(2\theta_2)\Gamma(\beta_2+\beta_2 i)}{i!j!\Gamma(2\theta_2-i)\Gamma(\beta_2+\beta_2 i-j)} B(s+j+1,2))].$$

7.2 Estimation the parameters of BGT-L Distribution

Two different estimation methods are used to estimate the parameters of BGT-L distribution which are the maximum likelihood estimation (MLE) and inference functions for margins (IFM) methods.

7.2.1 Maximum Likelihood Estimation (MLE)

Let $\{(x_i, y_i)|i = 1, \dots, n)\}$ are a random sample drawn from BGT-L distribution the log-likelihood function, in this case, can be written as :

$$logL(\Theta) = n[\log(4) + Log(\beta_1) + Log(\beta_2) + \log(\theta_1) + \log(\theta_2)]$$
$$+(\theta_1 - 1)\sum_{i=1}^{n} log[A(x_i;\beta_i)] + (\theta_2 - 1)\sum_{i=1}^{n} log[A(y_i;\beta_i)] + \sum_{i=1}^{n}(1 - x_i)$$
$$+\sum_{i=1}^{n}(1 - y_i) + (\beta_1 - 1)\sum_{i=1}^{n}\log(2x_i - {x_i}^2) + (\beta_2 - 1)\sum_{i=1}^{n} log(2y_i - {y_i}^2)$$
$$+\sum_{i=1}^{n} 1 + \gamma\{[1 - 2[1 - [A(x_i;\beta_i)]^{\theta_1}]][1 - 2[1 - [A(y_i;\beta_i)]^{\theta_2}]]\}.$$

Where $\Theta = (\beta_1, \theta_1, \beta_2, \theta_2, \gamma)$, and $A(x;\beta_i) = \left[1 - (2x - x^2)^{\beta_i}\right]; i = 1,2.$

All parameter estimators are derived by differentiating the log-likelihood function for each parameter individually and equating it to zero, as shown below.

$$\frac{\partial logL(\Theta)}{\partial \beta_1} = \frac{n}{\beta_1} + (\theta_1 - 1)\sum_{i=1}^{n} \frac{(2x_i - {x_i}^2)^{\beta_1}\log(2x_i - {x_i}^2)}{1-(2x_i - {x_i}^2)^{\beta_1}} + \sum_{i=1}^{n} log(2x_i - {x_i}^2)$$
$$+\sum_{i=1}^{n} 2\gamma\theta_1(1 - 2[1 - [A(y_i;\beta_i)]^{\theta_2}](2x_i - {x_i}^2)^{\beta_1} lo\, g(2x_i - {x_i}^2)\,[A(x_i;\beta_i)]^{\theta_1 - 1},$$

$$\frac{\partial logL(\Theta)}{\partial \beta_2} = \frac{n}{\beta_2} + (\theta_2 - 1)\sum_{i=1}^{n} \frac{(2y_i - {y_i}^2)^{\beta_2}\log(2y_i - {y_i}^2)}{1-(2y_i - {y_i}^2)^{\beta_2}} + \sum_{i=1}^{n} log(2y_i - {y_i}^2)$$
$$+\sum_{i=1}^{n} 2\gamma\theta_2(1 - 2\left(1 - \left(A(x_i;\beta_i)\right)^{\theta_1}\right)(2y - y^2)^{\beta_2} lo\, g(2y_i - {y_i}^2)\left(A(y_i;\beta_i)\right)^{\theta_2 - 1},$$

$$\frac{\partial logL(\Theta)}{\partial \theta_1} = \frac{n}{\theta_1} + \sum_{i=1}^{n} log[A(x_i;\beta_i)] + \sum_{i=1}^{n} 2\gamma\{1 - 2\{1 - [A(y_i;\beta_i)]^{\theta_2}\}\}$$
$$.\{1 - 2\{1 - [A(x_i;\beta_i)]^{\theta_1}\}\} log[A(x_i;\beta_i)],$$

$$\frac{\partial logL(\Theta)}{\partial \theta_2} = \frac{n}{\theta_2} + \sum_{i=1}^{n} log[A(y_i;\beta_i)] + \sum_{i=1}^{n} 2\gamma\{1 - 2\{1 - [A(x_i;\beta_i)]^{\theta_1}\}\}$$
$$.\{1 - 2\{1 - [A(y_i;\beta_i)]^{\theta_2}\}\} log[A(y_i;\beta_i)],$$

And

$$\frac{\partial logL(\Theta)}{\partial \gamma} = \sum_{i=1}^{n}\{1 - 2\{1 - [A(x_i;\beta_i)]^{\theta_1}\}1 - 2\{1 - [A(y_i;\beta_i)]^{\theta_2}\}\}.$$

The MLEs $\hat{\beta}_1, \hat{\beta}_2, \hat{\theta}_1, \hat{\theta}_1,$ and $\hat{\gamma}$ can be determined by solving the likelihood equations numerically by the R package.

7.2.2 Estimation by Inference Functions for Margins (IFM)

In 2005 Joe introduced another estimating method for more information on inference functions for margins (IFM). It is a parametric method with two-step of estimations. Firstly, estimating the marginal F_1 and F_2 respectively by maximizing the log-likelihood function of each marginal

$$ln\, L_{X_1} = \sum_{j=1}^{n} ln\, f_1(x_j) \text{ and } ln\, L_{X_2} = \sum_{j=1}^{n} ln\, f_2(y_j)$$

The second step is estimating the copula parameter by maximizing the log-likelihood function of the copula density using the ML estimates of the marginal $\hat{F}_1\,(x_i\,)$ and $\hat{F}_2\,(y_i\,)$ from the first step as given below

$$\hat{\gamma} = arg\, max \sum_{i=1}^{n} ln\, c\,(\hat{F}_1\,(x_i\,), \hat{F}_2\,(y_i\,)\,).$$

By considering the first step for the BGT-L distribution based on the FGM copula, the parameters of each marginal distribution will be estimated by MLE separately. Then log likelihood function of the marginal BGT-L distribution is defined as

$$log\, L_{x_i}(\Theta_i) = n[2 + Log(\beta_i) + Log(\theta_i)] + (\theta_i - 1)\sum_{i=1}^{n} log\left[1 - (2x_i - {x_i}^2)^{\beta_i}\right] + \sum_{i=1}^{n}(1 - x_i) + (\beta_i - 1)\sum_{i=1}^{n} log(2x_i - {x_i}^2).$$

$$\frac{\partial logl}{\partial \beta_i} = \frac{n}{\beta_i} - (\theta_i - 1)\sum_{i=1}^{n} \frac{(2x_i - {x_i}^2)^{\beta_i}\, log(2x_i - {x_i}^2)}{1 - (2x_i - {x_i}^2)^{\beta_i}} + \sum_{i=1}^{n} log(2x_i - {x_i}^2)\,; i = 1,2,$$

And $\frac{\partial logl}{\partial \theta_i} = \frac{n}{\theta_i} + \sum_{i=1}^{n} log\left[1 - (2x_i - {x_i}^2)^{\beta_i}\right]$.

Then, The log likelihood function of the copula function of the BGT-L distribution is defined as

$$logL_{\gamma} = \sum_{i=1}^{n} \log\{1 + \gamma\{1 - 2\{1 - [A(x_i; \hat{\beta}_i)]^{\hat{\theta}_1}\}1 - 2\{1 - [A(y_i; \hat{\beta}_i)]^{\hat{\theta}_2}\}\}$$

The estimates of all parameters are obtained by differentiating the log-likelihood function with respect to each parameter separately. Based on this, differentiating the log-likelihood function with respect to γ as

$$\frac{\partial logl}{\partial \gamma} = \sum_{i=1}^{n} \frac{\{1 - 2\{1 - [A(x_i;\hat{\beta}_i)]^{\hat{\theta}_i}\}1 - 2\{1 - [A(y_i;\hat{\beta}_i)]^{\hat{\theta}_i}\}\}}{\{1 + \gamma\{1 - 2\{1 - [A(x_i;\hat{\beta}_i)]^{\hat{\theta}_i}\}1 - 2\{1 - [A(y_i;\hat{\beta}_i)]^{\hat{\theta}_i}\}\}\}}.$$

Where $[A(x_i; \hat{\beta}_i)] = \left[1 - (2x_i - {x_i}^2)^{\hat{\beta}_i}\right]$.

The IFM estimators $\hat{\beta}_1, \hat{\beta}_2, \hat{\theta}_1, \hat{\theta}_1$ and $\hat{\gamma}$ can be obtained by solving simultaneously the likelihood equations.

$$\left.\frac{\partial logL}{\partial \beta_i}\right|_{\beta_i = \hat{\beta}_i} = 0, \left.\frac{\partial logL}{\partial \theta_i}\right|_{\theta_i = \hat{\theta}_i} = 0,\ and\ \left.\frac{\partial logL}{\partial \gamma}\right|_{\gamma = \hat{\gamma}} = 0\,; i = 1,2$$

The estimates of the parameters are handled numerically through statistical software.

7.3 Data Analysis

In this section, we use the real data set about the recurrence time of infection for 30 kidney patients which was discussed by Elaal and Jarwan (2017). Let X refer to the first recurrence time and Y to the second recurrence time but we will divide this data by 1000 to be suitable for the nature of our distributions the data for 30 patients are reported in Table (5).

The correlation between the two variables (bivariate data) is low as shown in Table (6). Through this, we arrive at an appropriate model that specializes in studying weak relationships and their existence effect, and efficacy. We compare BGT-L distribution with the bivariate generalized exponential (BGE) distribution which was discussed by Elaal and Jarwan (2017) and introduced below and use bivariate Weibull (BW) distribution in the comparison which was introduced by Almetwaly et al. (2020) and introduced below based on the maximum likelihood estimates ,Akaike information criterion (AIC) , Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC), and Hannan-Quinn information criterion (HQIC)

$$f_{BGE}(x,y) = \alpha_1\beta_1\alpha_2\beta_2 e^{-\beta_1 x} e^{-\beta_2 y}\left(1 - e^{-\beta_1 x}\right)^{\alpha_1 - 1}\left(1 - e^{-\beta_2 y}\right)^{\alpha_2 - 1} .1 + \theta\{[1 - 2\left(1 - e^{-\beta_1 x}\right)^{\alpha_1}][1 - 2\left(1 - e^{-\beta_2 y}\right)^{\alpha_2}]\}.$$

Where $x > 0,\ y > 0,\ \beta_1, \beta_2, \alpha_1, \alpha_2, > 0, -1 < \theta < 1.$

$$f_{BW}(x,y) = \frac{\alpha_1}{\beta_1}\frac{\alpha_2}{\beta_2}\left(\frac{x}{\beta_1}\right)^{\alpha_1 - 1}\left(\frac{y}{\beta_1}\right)^{\alpha_2 - 1} e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}} e^{-\left(\frac{y}{\beta_1}\right)^{\alpha_2}} .1 + \theta\{[1 - 2[1 - e^{-\left(\frac{x}{\beta_1}\right)^{\alpha_1}}]][1 - 2[1 - e^{-\left(\frac{y}{\beta_1}\right)^{\alpha_2}}]]\}.$$

Where $x > 0,\ y > 0,\ \beta_1, \beta_2, \alpha_1, \alpha_2, > 0, -1 < \theta < 1.$

From Table (7), the values of AIC, BIC, CAIC, and HQIC are less for the GT-L distribution than GE distribution and Weibull distribution, which reflect the better fit of the GT-L distribution to this data.

For the bivariate generalized Topp-Leone (BGT-L) distribution the log-likelihood value and AIC values are founded as (81.68 and -158.4) respectively. For the bivariate generalized exponential (BGE) distribution the log-likelihood value and AIC values are founded as (95.7 and -187) respectively. For the bivariate Weibull distribution the log-likelihood value and AIC values are founded as (92.7 and -180) respectively. In Table (8) it is observed that the BGT-L distribution performs better in this way of discussion, because it has the smallest value of the AIC compared with AIC of BGE distribution and AIC of bivariate Weibull distribution.

We obtained for the parameters of BGT-L distribution depending on methods of estimation such as MLEs and IFM methods by using the R package. In Table (9) it is observed that, the MLEs method provides a better fit than the IFM method because it has the smallest value of AIC of BGT-L distribution depending on the MLEs method.

Table 5: Kidney Infection Data 30 Patients

X	0.008	0.023	0.022	0.447	.030	0.024	0.007	0.511	0.053	0.015	0.007	0.013	0.039	0.132	0.034
Y	0.016	0.013	.028	.318	0.012	245	0.009	0.030	0.196	.154	0.333	.008	0.038	.070	025
X	.141	0.096	.149	.536	0.017	0.185	0.292	.022	.015	0.152	.402	.012	.113	.002	.130
Y	.004	.117	.114	.159	.108	.362	.024	.066	.046	.040	.201	.156	.030	.025	.026

Table 6: The Correlation Coefficient and Test of Correlation for Real Data.

	Correlation	p-value
Pearson	0.05105553	0.7887

Table 7: The Maximum Likelihood Estimation Parameters of GT-L Distributions, p-value, k-s, LL, AIC, BIC, CAIC, HQIC

Distribution X	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\theta}_1$	P-value	k-s	LL	-AIC	-BIC	-CAIC	-HQIC
GT-L	–	0.6297	2.063	0.3698	0.167	35.33	68.67	63.8	71.26	65.768
Generalized Exponential	0.66	–	6.19	0.34	0.17	35.2	68	63.6	71.0	71.04
Weibull	0.75	0.100	–	0.549	0.14	35.74	69.4	64.6	72.0	66.58
Distribution Y	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}_2$	P-value	k-s	LL	AIC	BIC	CAIC	HQIC
GT-L	–	0.860	3.626	0.5083	0.150	39.50	77	72.2	79.601	74.105
Generalized Exponential	0.92	–	9.56	0.46	0.15	39	76	72.0	79.4	73.91
Weibull	0.93	0.095	–	0.215	0.19	39	76	72.1	80	74.1

Table 8: The Estimates of Parameters of Bivariate Distributions

Distributions	$\hat{\alpha}_1$	$\hat{\beta}_1$	$\hat{\theta}_1$	$\hat{\alpha}_2$	$\hat{\beta}_2$	$\hat{\theta}_2$	γ	P-value	L-L	-AIC
BGT-L	–	1.0653	3.884	–	0.4548	1.545	0.70008	0.585	81.68	158.4
Generalized Exponential	0.409	0.8054		0.4064	0.689		0. 6040	0.585	95.7	187

Weibull	1.03 5	0.1386	–	0.8060	0.059	–	1.0196	0.585	92.7	18*0*

Table 9: The Estimates of Parameters of Bivariate Distribution under MLE, IFM

Distribution	$\hat{\beta}_1$	$\hat{\theta}_1$	$\hat{\beta}_2$	$\hat{\theta}_2$	γ	P-value	LL	-AIC
GT-L MLE	1.0653	3.884	0.4548	1.545	0.7001	0.585	81.68	158.4
GT-L IFM	0.6297	2.063	0.860	3.626	1.84×10^{-5}	0.58	30.002	59

8. Conclusion

In this paper, a generalization of the Topp-Leone distribution is introduced and studied which is called the generalized Topp-Leone (GT-L) distribution. The new distribution's statistical properties, including the behavior of hazard and reversed hazard functions, moments, quantile, mode, order statistics, and joint order statistics, are also obtained. The unknown parameters of the new distribution are estimated by the maximum likelihood estimation method and an approximate variance-covariance matrix has been obtained. A simulation study has been performed to see the performances of the MLEs. Depending on three data sets we compared the fitting of the generalized Topp-Leone distribution with some lifetime distributions which are used to describe bounded data. Moreover, a bivariate extension of the new distribution named the bivariate generalized Topp-Leone (BGT-L) distribution is introduced. The proposed bivariate distribution is of type Farlie–Gumbel–Morgenstern (FGM) copula. The BGT-L distribution has generalized Topp-Leone marginal distributions. The joint cumulative distribution function, the joint survival function, and the joint probability density function are also derived. Maximum likelihood and inference functions for margins approaches are used to estimate the bivariate model's unknown parameters. Finally, we illustrate the importance of the proposed bivariate modal using real data set in the medical field. Finally, For the future work Bayesian and non-Bayesian estimation for the GT-L and BGT-L distributions parameters can be studied for complete, censored and ranked set samples.

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