

A New Family of Distribution Based on the Bounded Marshall-Olkin Exponential Distribution: Properties and Applications

Abdul-Latif Alhassan
Tamale College of Education
Tamale, Ghana

Nasiru Suleman
University of Technology and Applied Sciences,
Navrongo-Ghana

Albert Luguterah
University of Technology and Applied Sciences
Navrongo-Ghana

ABSTRACT

A new family of distributions based on the bounded Marshall-Olkin exponential distribution. The new distribution known as the bounded Marshall-Olkin extended exponentially generated family of distributions. Various statistical properties, including the quantile function, characteristic function, moments and moment-generating function were derived for this new family. The distribution parameters were estimated using the maximum likelihood, ordinary least squares, and Cramér-von Mises procedures. In addition, the study explores three special distributions derived from the newly bounded Marshall-Olkin extended exponential-generated family. The hazard rates of these special distributions show their suitability for datasets exhibiting diverse non-monotonic failure rates. Monte Carlo simulations demonstrated the consistency of the estimators, with the maximum likelihood estimator showing superior performance compared to the other estimators. A location-scale regression model was conducted with data on long-term interest against direct foreign investment. It was observed that long-term interest rates positively affect foreign direct investment with a 5% confidence level of significance for bounded Marshall-Olkin extended exponential Weibull and all the competing location-scale regression models.

Keywords: Location-scale regression, bounded Marshall-Olkin, estimation, non-monotonic failure rate, quantile, Distribution theory

1. Introduction

Real-world phenomena are effectively modeled and predicted using statistical distributions. Different statistical distributions have been used to represent data across various domains over time, including the social sciences and applied sciences. A few fields in which they find are used are environmental science (for example, modeling extreme weather patterns), engineering (forecasting annual flood discharge), finance (forecasting monthly tax revenues), medical science (analyzing survival times after surgery), and computer science for evaluating software failure rates (Nasiru, 2018).

One challenge with existing distributions is their inability to effectively handle data with non-monotonic failure rates. Additionally, they often struggle to accurately represent datasets that are highly skewed or have heavy tails (Alahmadi et al., 2025). Ideally, distributions should be chosen to best suit the original dataset rather than resorting to data transformation. Hence, the limitations of classical statistical distributions in data modeling have spurred researchers to develop new, more flexible distributions. Over the years, several techniques have been introduced to modify existing distributions to improve their goodness-of-fit. These modifications focus on enhancing features such as tractable cumulative distribution functions, adaptability to skewed and

- Received March 8, 2026, in final form August 2026.
- Abdul-Latif Alhassan (corresponding author) is affiliated with Tamale College of Education, Tamale, Ghana
Email: alatif@tace.edu.gh

heavy-tailed data, and the ability to handle both monotonic and non-monotonic failure rates. For instance, the log-logistic tan generalized family (Altun et al., 2019) and the generalized odd Gamma-G (Hosseini et al., 2018) were developed with such objectives in mind.

As a result, researchers have continued to propose novel families of distributions to extend existing models and improve their applicability to real-life data. This movement has seen further contributions such as the work by Abdelall et al., 2025, who proposed proportional data on continuous distributions defined on the unit interval. Their study proposes the unit inverse power Lomax distribution, deriving its key statistical properties and assessing ten parameter estimation methods. Comparative analyses with existing models on two real datasets show that their study was superior and has the potential for broad application. A study on the Kumaraswamy Weibull regression model conducted by Ahmed et al., (2025) which compared five parameter estimation methods and found maximum likelihood estimation to be the most effective based on multiple information criteria. The model outperformed the standard Weibull model in fitting economic value-added data, showing stronger explanatory power and relationships consistent with economic theory. Mustapha et al., (2025) developed a quantile regression model based on the tan exponentiated odd log-logistic Weibull distribution to handle both complete and censored data in survival and reliability analysis. Simulation and real-data applications showed that the model provides accurate parameter estimates and a superior fit compared to the existing approaches. Gomes-Silva et al. (2016), emphasized enhancing flexibility in practical data modeling. Subsequent proposals include the exponentiated generalized power series family (Nasiru et al., 2019), the sine inverse Lomax family (Fayomi et al., 2021), and the sine Kumaraswamy-G family (Chesneau and Jamal, 2021). Additional developments include the cosine-G family (Muhammad et al., 2021), the Khalil new generalized family (KNGF) (Zaidi et al., 2021), and the U family of distributions (Jamal et al., 2022). Roozegar et al. (2022) introduced the generalized-X family of distributions, while Zamanah et al. (2022) proposed the Harmonic mixture Weibull-G family. More recently, Keganne et al. (2023) introduced the Type II half logistic Topp-Leone exponentiated generalized-G distributions, and Ahmad et al. (2024) proposed the Unified generalized family of distributions.

Despite the myriads of distribution models now available, the demand for more flexible and accurate tools for lifetime data modeling persists. This ongoing pursuit drives continuous innovation in the statistical distribution theory. However, the applicability of these modified distributions may be limited across different fields due to the varying dataset characteristics. To address this, researchers continuously adapt existing distributions to suit the specific challenges posed by different datasets. The bounded Marshall-Olkin exponentials (BMOE) distribution is a beneficial and adaptable distribution that can be used to describe several datasets with parameters that may be changed. It makes statistical inference easier by providing closed-form equations for both its cumulative distribution function (CDF) and probability density function (PDF). The bathtub-shaped hazard function of the distribution is especially helpful for survival and reliability studies. Comprehensive statistical analysis by the explicit derivation of moments and the moment generating function (MGF). Maximum likelihood estimation (MLE) and other similar techniques are useful for carrying out parameter estimation. The BMOE distribution is particularly well-suited for applications in dependability and financial modeling. It can also handle heavy-tailed data for risk assessment and model interdependence between variables (Ghosh et al., 2019). Because of the above motivation, the study aims to propose a novel family of distributions called the new bounded Marshall-Olkin extended exponential generated (BMOEEG) family of distributions.

The rest of the sections in this article include; Sections 2 and 3 introduce the bounded Marshall-Olkin Extended Exponential Generated (BMOEEG) distribution and its statistical properties, respectively. Section 4 discusses the parameter estimation method (MLE). Section 5 looks at the special distributions developed. Section 6 examines the performance of the estimators through Monte Carlo simulations. Section 7 shows the applications of the BMOEEG distribution. Finally, Section 8 presents the study's conclusion.

2. Data and methods

2.1 Bounded Marshall-Olkin Exponential Distributions

Suppose $G(x, \xi)$ and $g(x, \xi)$ are baseline CDF and PDF, respectively. The CDF of the BMOEEG family of distributions is given by

$$F(x, \alpha, \lambda, \xi) = \frac{\alpha G(x, \xi)^\lambda}{[1 - (1 - \alpha)G(x, \xi)^\lambda]}, x \in R, \quad (1)$$

where $\alpha > 0$ and $\lambda > 0$ are the scale and shape parameters respectively and ξ is a $k \times 1$ vector of the parameters for the baseline $G(x, \xi)$. The PDF of the new family is obtained by differentiating equation (1) and is given by

$$f(x, \alpha, \lambda, \xi) = \frac{\alpha \lambda g(x, \xi) G(x, \xi)^{\lambda-1}}{[1 - (1 - \alpha)G(x, \xi)^\lambda]^2}, x \in R. \quad (2)$$

The survival function is particularly useful in reliability analysis, survival analysis and risk analysis. Hence, the survival function of the new family by equation (3)

$$S(x, \alpha, \lambda, \xi) = \frac{1 - G(x, \xi)^\lambda}{[1 - (1 - \alpha)G(x, \xi)^\lambda]}, x \in R. \quad (3)$$

The associated hazard rate function of the BMOEEG family is given by

$$h(x) = \frac{\alpha \lambda g(x, \xi) G(x, \xi)^{\lambda-1}}{[1 - G(x, \xi)^\lambda][1 - (1 - \alpha)G(x, \xi)^\lambda]}, x \in R. \quad (4)$$

The quantile function of the BMOEEG family of distribution is given by

$$x_u = G^{-1} \left\{ \left[\frac{u}{\alpha + u(1 - \alpha)} \right]^{\frac{1}{\lambda}}; \xi \right\}, u \in [0, 1]. \quad (5)$$

Lemma 1. The linear representation of the CDF of the BMOEEG family of distribution is given as

$$F(x, \alpha, \lambda, \xi) = \begin{cases} \sum_{i=1}^{\infty} (1 - \alpha)^i G(x; \xi)^{\lambda(i+1)}, & x \in R, \lambda > 0, \alpha \in (0, 1) \\ \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^i G(x; \xi)^{\lambda(i+1)}, & x \in R, \lambda > 0, \alpha > 1 \end{cases}. \quad (6)$$

Proof. Using the binomial expansion, $(1 - \tau)^{-s} = \sum_{i=1}^{\infty} \binom{s+i-1}{i} \tau^i, |\tau| \leq 1$, the denominator of the CDF can be written as

$$\left[1 - (1 - \alpha)G(x, \xi)^\lambda\right]^{-1}, \quad \alpha \in (0, 1).$$

Hence,

$$\left[1 - (1 - \alpha)G(x, \xi)^\lambda\right]^{-1} = \sum_{i=1}^{\infty} \binom{1+i-1}{i} (1 - \alpha)^i G(x, \xi)^{\lambda-1}, \quad \alpha \in (0, 1).$$

$$F(x, \alpha, \lambda, \xi) = \sum_{i=1}^{\infty} \alpha (1 - \alpha)^i G(x, \xi)^{\lambda(i+1)}, \quad \alpha \in (0, 1).$$

Further,

When $\alpha > 1$

$$\begin{aligned} \left[1 - (1 - \alpha)G(x, \xi)^\lambda\right]^{-1} &= [1 - G(x, \xi)^\lambda + \alpha G(x, \xi)^\lambda]^{-1} \\ &= \alpha^{-1} \left[1 - \left(1 - \frac{1}{\alpha}\right)(1 - G(x, \xi)^\lambda)\right]^{-1}. \end{aligned}$$

Using the generalized binomial expansion,

$$\begin{aligned} &= \alpha^{-1} \left[1 - \left(1 - \frac{1}{\alpha}\right)(1 - G(x, \xi)^\lambda)\right]^{-1} \\ &= \alpha^{-1} \sum_{i=1}^{\infty} \left(1 - \frac{1}{\alpha}\right)^i (1 - G(x, \xi)^\lambda)^i. \end{aligned}$$

Applying the binomial expansion, gives

$$F(x, \alpha, \lambda, \xi) = \alpha^{-1} \sum_{i=1}^{\infty} \sum_{j=0}^i (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^i G(x, \xi)^{\lambda(j+i)}, \quad \alpha > 1.$$

Hence,

$$F(x, \alpha, \lambda, \xi) = \begin{cases} \sum_{i=1}^{\infty} (1 - \alpha)^i G(x, \xi)^{\lambda(i+1)}, & x \in R, \lambda > 0, \alpha \in (0, 1) \\ \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^i G(x, \xi)^{\lambda(i+1)}, & x \in R, \lambda > 0, \alpha > 1 \end{cases}.$$

Lemma 2. The mixture representation of the PDF of the BMOEEG family of distribution is given as follows

$$f(x, \alpha, \lambda, \xi) = \begin{cases} \sum_{i=1}^{\infty} (1 - \alpha)^i \lambda(i+1) g(x, \xi) G(x, \xi)^{\lambda(i+1)-1}, & x \in R, \lambda > 0, \alpha \in (0, 1) \\ \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^j \lambda(i+1) g(x, \xi) G(x, \xi)^{\lambda(i+1)-1}, & x \in R, \lambda > 0, \alpha > 1 \end{cases} \quad (7)$$

Proof. The PDF is obtained by finding the first derivative of the expanded form of the CDF with respect to x . Hence,

$$f(x, \alpha, \lambda, \xi) = \frac{\partial F(x, \alpha, \lambda, \xi)}{\partial x}. \quad (8)$$

2.2 Statistical Properties

In this session, the statistical properties of the BMOEEG family of distributions are discussed.

2.2.1 Quantile Function

Quantile functions play a crucial role in generating random samples from a specific distribution, making them highly valuable in simulations. In addition, important distribution characteristics such as kurtosis, skewness, and median can be extracted using quantile functions.

Proposition 2. The quantile function of the BMOEEG distributions for $u \in [0, 1]$ is given by

$$x_u = G^{-1} \left\{ \left[\frac{u}{\alpha + u(1-\alpha)} \right]^{\frac{1}{\lambda}}; \xi \right\}, u \in [0, 1]. \quad (9)$$

Proof. Using the CDF of the BMOEEG family of distributions defined in equation (1), the quantile function is derived as follows,

$$\frac{\alpha G(x, \xi)^\lambda}{[1 - (1-\alpha)G(x, \xi)^\lambda]} = u.$$

This implies

$$\begin{aligned} \alpha G(x, \xi)^\lambda + u(1-\alpha)G(x, \xi)^\lambda &= u, \\ G(x, \xi)^\lambda [\alpha + u(1-\alpha)] &= u \end{aligned}$$

Further simplifying

$$G(x, \xi) = \left[\frac{u}{[\alpha + u(1-\alpha)]} \right]^{\frac{1}{\lambda}}.$$

Hence,

$$x_u = G^{-1} \left(\left[\frac{u}{[\alpha + u(1-\alpha)]} \right]^{\frac{1}{\lambda}}, \xi \right) \quad u \in [0, 1].$$

This completes the proof.

2.2.2 Moments

Moments are useful in statistical research, particularly when studying the properties of distributions such as measures of central tendency, skewness, and kurtosis.

Proposition 3. The n^{th} non-central moment of the BMOEEG family of distributions is

$$\mu'_n = \begin{cases} \sum_{i=1}^{\infty} (1-\alpha)^i \lambda(i+1) \int_{-\infty}^{\infty} x^n g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha \in (0, 1), n = 1, 2, 3, \dots \\ \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^j \lambda(i+1) \int_{-\infty}^{\infty} x^n g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha > 1, n = 1, 2, 3, \dots \end{cases}. \quad (10)$$

Proof. The n^{th} non-central moment is given by

$$\mu'_n = \sum x^n = \int_{-\infty}^{\infty} x^n f(x, \alpha, \lambda, \xi) dx. \quad (11)$$

Substituting the expanded form of the PDF given in equation (7) into equation (11) gives the non-central moment below.

$$\mu'_n = \begin{cases} \sum_{i=1}^{\infty} (1-\alpha)^i \lambda(i+1) \int_{-\infty}^{\infty} x^n g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha \in (0,1), n=1,2,3,\dots \\ \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^j \lambda(i+1) \int_{-\infty}^{\infty} x^n g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha > 1, n=1,2,3,\dots \end{cases}.$$

2.2.3 Moment-generating Function

The moment-generating function (MGF) proves valuable in estimating the moment of a random variable. If a random variable follows the BMOEEG family of distributions and the MGF exists, it can be expressed using the following proposition

Proposition 4. The moment generating function of the BMOEEG distributions is expressed as

$$M_X(t) = \begin{cases} \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} (1-\alpha)^i \lambda(i+1) \int_{-\infty}^{\infty} \frac{(xt)^n}{n!} g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx & \lambda > 0, \alpha \in (0,1) \quad n=1,2,\dots \\ \sum_{i=0}^{\infty} \sum_{j=0}^i \sum_{n=0}^{\infty} (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^j \lambda(i+1) \int_{-\infty}^{\infty} \frac{(xt)^n}{n!} g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx & \lambda > 0, \alpha > 1, n=1,2,\dots \end{cases}. \quad (12)$$

Proof. Using the identity $e^{tX} = \sum_{n=0}^{\infty} \frac{t^n X^n}{n!}$, the moment generating function of a random variable is

$$M_X(t) = E[e^{tX}] = \sum_{n=0}^{\infty} E\left[\frac{t^n X^n}{n!}\right].$$

But $E[X^n]$ is the n^{th} moment about the origin. This implies $M_X(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} \mu'_n$.

Substituting the n^{th} moment about the origin gives;

$$M_X(t) = \begin{cases} \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} (1-\alpha)^i \lambda(i+1) \int_{-\infty}^{\infty} \frac{x^n t^n}{n!} g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha \in (0,1), n=1,2,\dots \\ \sum_{i=0}^{\infty} \sum_{j=0}^i \sum_{n=0}^{\infty} (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^j \lambda(i+1) \int_{-\infty}^{\infty} \frac{x^n t^n}{n!} g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha > 1, n=1,2,\dots \end{cases}.$$

The proof is complete.

2.2.4 Incomplete moment

The utility of the incomplete moment becomes evident when estimating the median deviation, mean deviation, and measures of inequalities such as Lorenz and Bonferroni curves.

Proposition 5. The n^{th} incomplete moment of the BMOEEG family of distributions is

$$M_n(y) = \begin{cases} \sum_{i=1}^{\infty} (1-\alpha)^i \lambda(i+1) \int_{-\infty}^y x^n g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha \in (0,1), n=1,2,\dots \\ \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^j \lambda(i+1) \int_{-\infty}^y x^n g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha > 1, n=1,2,\dots \end{cases} \quad (13)$$

Proof. The n^{th} incomplete moment is defined as

$$E[X | X < y] = \int_{-\infty}^y x^n f(x) dx.$$

Substituting the mixture representation of the density function into the definition and simplifying yields

$$M_n(y) = \begin{cases} \sum_{i=0}^{\infty} (1-\alpha)^i \lambda(i+1) \int_{-\infty}^y x^n g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha \in (0,1), n=1,2,\dots \\ \sum_{i=0}^{\infty} \sum_{j=0}^i (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^j \lambda(i+1) \int_{-\infty}^y x^n g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha > 1, n=1,2,\dots \end{cases}.$$

The proof is complete.

2.2.5 Characteristics function

In statistics and probability theory, the characteristic function is crucial. It becomes more useful when the moment-generating function does not exist. The characteristic function of the BMOEEG family of distributions is

Proposition 6. The characteristic function of the BMOEEG family of distributions is given by

$$\phi_X(t) = \begin{cases} \sum_{i=0}^{\infty} \sum_{n=0}^{\infty} (1-\alpha)^i \lambda(i+1) \int_{-\infty}^{\infty} \frac{(ixt)^n}{n!} g(x, \xi) G(x, \xi)^{\lambda(i+1)-1} dx & \lambda > 0, \alpha \in (0,1), n=1,2,\dots \\ \sum_{i=0}^{\infty} \sum_{j=0}^i \sum_{n=0}^{\infty} (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^j \lambda(i+1) \int_{-\infty}^{\infty} \frac{(ixt)^n}{n!} g(x, \xi) G(x, \xi)^{\lambda(i+1)-1} dx & \lambda > 0, \alpha > 1, n=1,2,\dots \end{cases} \quad (14)$$

Proof. Using the identity

$$e^{itX} = \sum_{n=0}^{\infty} \frac{i^n t^n X^n}{n!},$$

$$E[e^{itX}] = \sum_{n=0}^{\infty} E\left[\frac{i^n t^n X^n}{n!}\right].$$

This implies

$$\phi_X(t) = \sum_{n=0}^{\infty} \frac{i^n t^n}{n!} \mu'_n.$$

Hence,

$$\phi_X(t) = \begin{cases} \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} (1-\alpha)^i \lambda(i+1) \int_{-\infty}^{\infty} \frac{(ixt)^n}{n!} g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha \in (0,1), n=1,2,\dots \\ \sum_{i=0}^{\infty} \sum_{j=0}^i \sum_{n=0}^{\infty} (-1)^j \binom{i}{j} \left(1 - \frac{1}{\alpha}\right)^j \lambda(i+1) \int_{-\infty}^{\infty} \frac{(ixt)^n}{n!} g(x; \xi) G(x; \xi)^{\lambda(i+1)-1} dx, & \lambda > 0, \alpha > 1, n=1,2,\dots \end{cases}$$

2.3 Parameter Estimation

In this section, we utilized three estimation approaches to estimate the unknown parameters of the BMOEEG family. These estimation methods include maximum likelihood estimation (MLE), ordinary least squares estimation (OLS), and Cramér-von Mises estimation (CVM).

2.3.1 Maximum likelihood Estimation

Assuming $X_1, X_2, \dots, X_n$ constitutes a random sample of n independent and identically distributed random variables from the BMOEE family of distributions, the total log-likelihood function is expressed as

$$\ell(\lambda) = n \log(\alpha \lambda) \sum_{i=1}^n \log g(x_i) + n(\lambda - 1) \sum_{i=1}^n \log G(x_i) - 2n \log \sum_{i=1}^n \left[1 - (1 - \alpha) G(x_i)^{\lambda} \right] \quad (15)$$

The maximum likelihood estimates are obtained by directly maximizing the log-likelihood function with respect to the parameters α, λ, ξ respectively.

2.3.2 Ordinary Least Squares Estimation

The ordinary least squares (OLS) estimation method estimates parameters by minimizing an objective function. If $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ are the order statistics of a random sample of size obtained from the BMOEEG family, then the OLS estimates for the parameters can be obtained by minimizing the objective function

$$\psi_{(\alpha, \lambda, \xi)} = \sum_{i=1}^n \left[F(x_{(i)} | \alpha, \lambda, \xi) - \frac{i}{n+1} \right]^2 \quad (16)$$

With respect to the parameters α, λ, ξ respectively.

2.3.3 Cramér-Von Mises Minimum Distance Estimation Method

The Cramér-von Mises (CVM) distance estimation method, also known as the maximum goodness-of-fit estimation method. This is characterized by a minimal bias compared to other minimum distance estimation methods. This approach relies on assessing the disparity between the estimates of the cumulative distribution function (CDF) and the empirical distribution. Considering $X_{(1)}, X_{(2)}, \dots, X_{(n)}$ as the order statistics of a random sample of size obtained from the BMOEEG distribution, the estimators of the CVM for the parameters of the BMOEEG family can be obtained by minimizing the corresponding function

$$V_{(\alpha, \lambda, \xi)} = \frac{1}{12n} + \sum_{i=1}^n \left[F(x_i | \alpha, \lambda, \xi) - \frac{2i-1}{2n} \right]^2, \quad (17)$$

With respect to the parameters α, λ, ξ respectively.

3. Special distributions

In this section, three new distributions within the BMOEEG family are developed and their applications are studied.

3.1 Bounded Marshall-Olkin Extended Exponential Weibull distribution (BMOEEW)

Suppose the baseline PDF and CDF given by Sabor et al. as (Saboor et al., 2014)] respectively as $g(x, \theta, \gamma) = \theta x^{\theta-1} e^{-\gamma x^\theta}$ and $G(x, \theta, \gamma) = 1 - e^{-\gamma x^\theta}$. Then, the PDF and the hazard functions of the BMOEEW distribution are respectively given in equations (18) and (19)

$$f(x, \alpha, \lambda, \theta, \gamma) = \frac{\alpha \lambda \theta x^{\theta-1} e^{-\gamma x^\theta} \left[1 - e^{-\gamma x^\theta}\right]^{\lambda-1}}{\left[1 - (1-\alpha) \left(1 - e^{-\gamma x^\theta}\right)^\lambda\right]^2}, x > 0 \quad (18)$$

$$h(x, \alpha, \lambda, \theta, \gamma) = \frac{\alpha \lambda \theta x^{\theta-1} e^{-\gamma x^\theta} \left(1 - e^{-\gamma x^\theta}\right)^{\lambda-1}}{\left[1 - (1-\alpha) \left(1 - e^{-\gamma x^\theta}\right)^\lambda\right] \left[\left[1 - (1-\alpha) \left(1 - e^{-\gamma x^\theta}\right)^\lambda\right] - \alpha \left(1 - e^{-\gamma x^\theta}\right)^\lambda\right]}, x > 0. \quad (19)$$

The density and the hazard plots of the BMOEEW distribution for some selected parameter values are displayed in Figure 1. The density is characterized by different shapes such as symmetric, right-skewed, reverse J-shaped, decreasing and increasing shapes. Also, the hazard plot shows evidence of constant, reversed-J, increasing, and decreasing shapes.

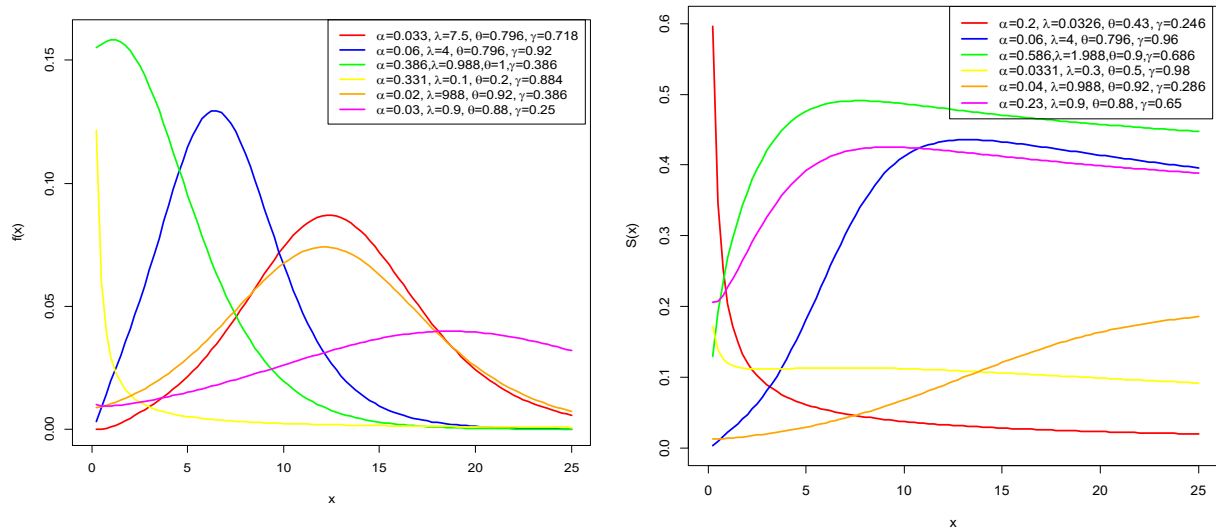


Figure 1: Plots of the density (left) and hazard (right) rate function of BMOEEW distribution

3.2 Bounded Marshall-Olkin Extended Exponential Inverse Weibull distribution (BMOEEIW)

If the baseline CDF and PDF of the inverse Weibull distribution are given by (Marshall and Olkin, 1997) and (Cordeiro et al., 2014) respectively as $G(x) = e^{-\theta x^{-\beta}}$ and $g(x) = \beta \theta x^{-\beta-1} e^{-\theta x^{-\beta}}$. The PDF and the hazard functions of the BMOEEIW are given in equations (20) and (21).

$$f(x, \alpha, \lambda, \beta, \theta) = \frac{\alpha \lambda \theta \beta x^{-(\beta+1)} (e^{-\theta x^{-\beta}})^{\lambda-1}}{\left[1 - (1-\alpha)(e^{-\theta x^{-\beta}})^{\lambda}\right]^2}, \quad x > 0, \quad (20)$$

$$h(x) = \frac{\alpha \lambda \theta \beta x^{-(\beta+1)} (e^{-\theta x^{-\beta}})^{\lambda-1}}{\left[1 - (1-\alpha)(e^{-\theta x^{-\beta}})^{\lambda}\right] \left[1 - e^{-\theta x^{-\beta}}\right]}, \quad x > 0. \quad (21)$$

Figure 2 shows the density and the hazard plots of the BMOEEIW distribution for some chosen parameter values that exhibit reverse-J, upside bathtub almost symmetric and right-skewed shapes. The hazard plot also shows an increasing, decreasing, and modified upside bathtub.

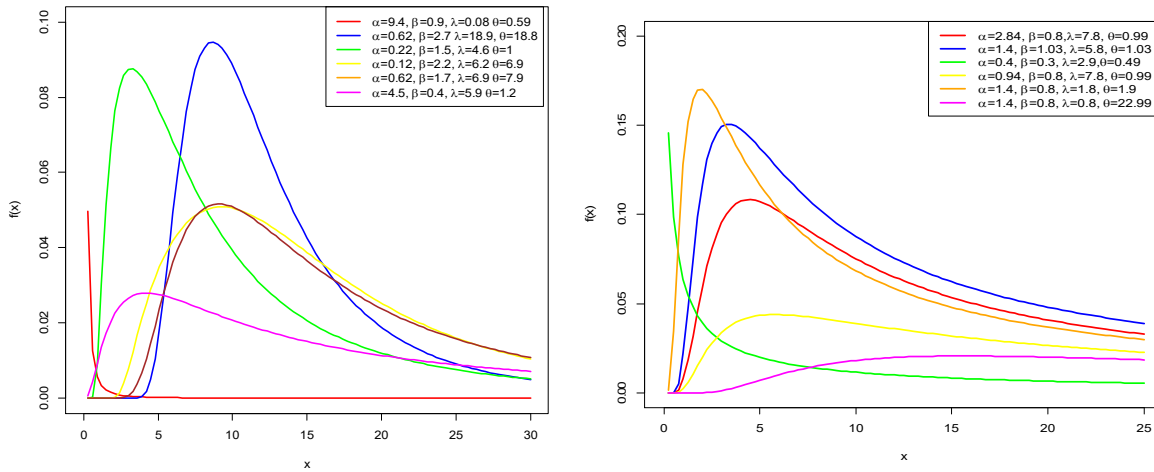


Figure 2: Plots of the density(left) and hazard(right) functions of the BMOEEIW distribution

3.3 Bounded Marshall-Olkin Extended Exponential Nadarajah-Haghighi distribution (BMOEENH)

If the baseline PDF and the CDF of the Nadarajah-Haghighi distribution as given by Nadarajah and Kotz (2007) and (Nadarajah et al., 2011) respectively as $g(x) = \theta \beta (1 + \beta x)^{\theta-1} e^{1-(1+\beta x)^{\theta}}$ and $G(x) = 1 - e^{1-(1+\beta x)^{\theta}}$. The PDF and the hazard functions of the BMOEENH distributions are, respectively given in equations (22) and (23)

$$f(x, \alpha, \lambda, \theta, \beta) = \frac{\alpha \lambda \theta \beta (1 + \beta x)^{\theta-1} e^{1-(1+\beta x)^\theta} (1 - e^{1-(1+\beta x)^\theta})^{\lambda-1}}{\left[1 - (1 - \alpha)(1 - e^{1-(1+\beta x)^\theta})^\lambda\right]^2}, x > 0 \quad (22)$$

$$h(x, \alpha, \lambda, \theta, \beta) = \frac{\alpha \lambda \theta \beta (1 + \beta x)^{\theta-1} e^{1-(1+\beta x)^\theta} (1 - e^{1-(1+\beta x)^\theta})^{\lambda-1}}{\left[1 - (1 - \alpha)(1 - e^{1-(1+\beta x)^\theta})^\lambda\right]^2 \left[(1 - (1 - \alpha)(1 - e^{1-(1+\beta x)^\theta})^\lambda) - \alpha(1 - e^{1-(1+\beta x)^\theta})^\lambda\right]}, x > 0. \quad (23)$$

Figure 3 shows the density and the hazard plots of the BMOEENH distributions. The density plot exhibits various shapes, including reversed-J, upside-down bathtub, decreasing, right-skewed, and almost symmetrical. Also, the hazard plot shows its ability to produce a decreasing, reversed-J, modified bathtub, increasing, and constant hazard rate shapes.

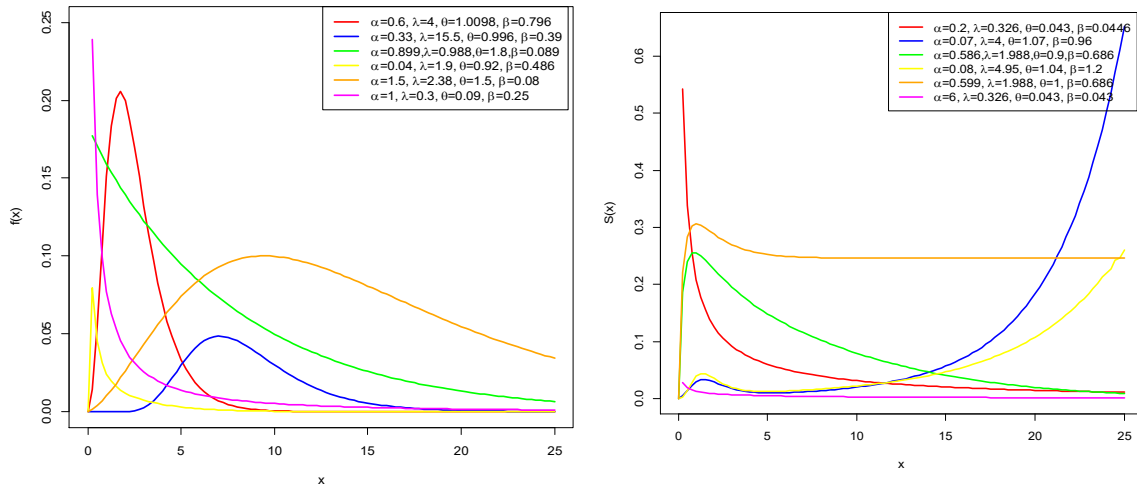


Figure 3: Plots of density(left) and hazard(right) rate functions of BMOEENH distributions

3.2 Monte Carlo Simulation

This section uses the Monte Carlo simulation to study the characteristics of the estimators for the BMOEEW distribution. Random samples were generated from the BMOEEW distribution using its quantile function. Tables 1 and 2 show the average bias (AB) and the root mean square error (RMSE) of the maximum likelihood estimators (MLE), ordinary least squares (OLS) and Cramér-von Mises (CVM) estimators for the parameters for some selected parameter values. The simulation study was replicated $N = 5,000$ times with sample sizes $N = 30, 50, 80, 120, 200, 150$, and 250 with parameter values $(\alpha, \lambda, \theta, \gamma) = (0.8, 1.4, 0.3, 0.9)$ and $(0.6, 1.0, 0.1, 0.7)$ was conducted. The simulation results for $(\alpha, \lambda, \theta, \gamma) = (0.8, 1.4, 0.3, 0.9)$, and $(0.6, 1.0, 0.1, 0.7)$ for the three estimation methods (MLE, OLS, and CVM) are displayed in the tables. The ABs for all the parameters in were positive and tended to drop to approximately zero as the sample size increased. Additionally, the RMSEs of the parameter decrease with increasing sample size, indicating that, the estimators of the parameters are consistent. Nonetheless, the maximum likelihood estimator was seen to be the best because it recorded the lowest AB and RMSE values.

Table 1: Simulation results for $(\alpha, \lambda, \theta, \gamma) = (0.8, 1.4, 0.3, 0.9)$

Parameter	N	AB			RMSE		
		MLE	CVM	OLS	MLE	CVM	OLS
α	30	0.68730	1.38297	4.37492	0.10376	1.75624	36.7050
	50	0.26811	0.94948	3.24537	0.07523	1.36255	25.4291
	80	0.24628	0.92901	2.91913	0.03965	1.33501	24.8563
	120	0.19812	0.83808	2.59267	0.03636	1.27982	21.2540
	150	0.18150	0.81924	2.00909	0.03628	1.23335	16.9675
	200	0.18043	0.73012	1.14309	0.03383	1.04460	8.55782
	250	0.16540	0.71993	1.08137	0.02984	0.96657	8.49420
λ	30	0.59955	53.0712	3.84910	0.42711	163.057	30.0152
	50	0.58680	2.33562	3.76390	0.39992	6.38471	29.7621
	80	0.55014	1.99116	2.80732	0.30662	3.71942	20.1715
	120	0.50420	0.86131	2.53306	0.26497	1.15376	15.8107
	150	0.45074	0.82322	1.95676	0.21935	0.97128	12.4197
	200	0.42595	0.72343	1.30267	0.20276	0.91532	6.31652
	250	0.40775	0.67897	1.20796	0.20187	0.76890	5.49130
θ	30	0.24446	0.36014	0.29982	0.08989	0.09413	0.33512
	50	0.24139	0.33201	0.29915	0.08949	0.09101	0.28496
	80	0.18590	0.31005	0.29899	0.08940	0.09031	0.22098
	120	0.17805	0.30721	0.29886	0.08932	0.09004	0.20580
	150	0.15446	0.30040	0.29795	0.08879	0.09010	0.20386
	200	0.14048	0.29887	0.29771	0.08657	0.08933	0.20164
	250	0.13689	0.29725	0.29410	0.08805	0.08842	0.17498
γ	30	0.31581	0.47229	1.84200	0.10225	0.26068	2.75368
	50	0.30512	0.46775	1.69814	0.09800	0.25844	2.53428
	80	0.29436	0.41650	1.24984	0.09415	0.20279	1.86239
	120	0.27277	0.39840	1.11062	0.08161	0.17120	1.54382
	150	0.23687	0.34642	1.02036	0.06567	0.13861	1.34930
	200	0.23508	0.34140	0.88034	0.06558	0.12484	1.17889
	250	0.22030	0.28349	0.75656	0.05962	0.09662	0.86316

Table 2: Simulation results for $(\alpha, \lambda, \theta, \gamma) = (0.6, 1.0, 0.1, 0.7)$

Parameter	N	AB			RMSE		
		MLE	CVM	OLS	MLE	CVM	OLS
α	30	0.21682	1.00287	6.31978	0.06910	1.29371	56.2905
	50	0.20489	0.84447	4.77706	0.05507	1.11951	43.7337
	80	0.16837	0.73526	4.39132	0.03472	0.90735	36.7851
	120	0.15516	0.60432	3.53472	0.02673	0.73385	30.0447
	150	0.12400	0.57990	1.99496	0.01719	0.69911	16.9211
	200	0.11281	0.43027	1.86465	0.01561	0.55194	15.4811
	250	0.11117	0.40356	1.03104	0.01292	0.45919	7.49185
λ	30	0.39445	5.96848	17.7781	0.20941	42.1483	51.0919
	50	0.29833	2.45516	4.63362	0.20169	3.92827	40.5596
	80	0.21019	0.80502	4.46705	0.16520	1.13060	36.7617
	120	0.19432	0.73941	3.68169	0.07695	1.05886	32.3099
	150	0.11588	0.44819	1.98412	0.07060	0.51969	16.2751
	200	0.07844	0.40927	1.97355	0.02107	0.47750	15.4007
	250	0.02673	0.32150	1.09736	0.01443	0.35665	8.18242
θ	30	0.09199	0.10504	0.09995	0.01300	0.01842	0.09991
	50	0.08045	0.10100	0.09962	0.01050	0.01212	0.09924
	80	0.06741	0.09998	0.09961	0.01021	0.08581	0.09922
	120	0.05934	0.09995	0.09948	0.01000	0.07712	0.09896
	150	0.05269	0.09990	0.09941	0.00999	0.07406	0.09883
	200	0.04751	0.09942	0.09875	0.00986	0.06177	0.09753
	250	0.04492	0.06936	0.09863	0.00981	0.05071	0.09719
γ	30	0.26992	0.61027	2.64503	0.12565	0.47362	3.33557
	50	0.17309	0.54627	1.67129	0.04400	0.41828	2.17393
	80	0.15869	0.52672	0.96453	0.03815	0.37743	1.50703
	120	0.15477	0.39978	0.78419	0.03752	0.30876	1.22219
	150	0.12325	0.32192	0.74318	0.03367	0.19549	0.86199
	200	0.11828	0.29990	0.64834	0.01946	0.19044	0.72551
	250	0.08691	0.24965	0.58098	0.01110	0.12607	0.69141

3.4 Application of the BMOEEG family

This section illustrates the practical utility of the proposed distributions; bounded Marshall-Olkin extended exponential Weibull (BMOEEW), bounded Marshall-Olkin extended exponential Nadarajah-Haghighi (BMOEENH), and bounded Marshall-Olkin extended exponential inverse Weibull (BMOEEIW). To achieve this, these special distributions were fitted to datasets and compared against other distributions such as Odd generalized exponential Weibull (OGEW) by Muhammad and Abed (2019), extended generalized inverse Weibull (EGIW) by Hanagal, and Bhalerao (2021), Kumaraswamy (KUMA) by Kumaraswamy (1980), exponentiated Top-Leon (ETL) by Pourdarvish (2015), extended odd Fréchet Weibull (EOFW) by Nasiru (2018), and bounded odd inverse Pareto exponential (BOIPE) by Dumonceaux and Antle (1976). Additionally, the goodness-of-fit was assessed using the following tests; Anderson-Darling (AD), Cramér-von Mises (CVM), and Kolmogorov-Smirnov (K-S), along with information criteria like Akaike information criterion (AIC), corrected AIC (AICc), Bayesian information criterion BIC, and the log-likelihood. The P-values from the K-S test were provided, and the model with the lowest goodness-of-fit measures and highest log-likelihood was considered the best fit for the datasets.

3.5 First Application

The first dataset represents the maximum flood data. It was first used by Dumonceaux and Antle (1976) and later by Nasiru et al., (2021)). It comprises the maximum flood level (in millions of cubic feet per second) for the Susquehanna River at Harrisburg, Pennsylvania. Each data point is the maximum flood level for four years, comprising 20 data points as outlined in Table 3. In Table 4, the maximum likelihood parameter estimations for the fitted distributions are presented with at least every model having at least one of its parameters to be significant, showing the suitability of the data to the models. Table 5 presents the goodness-of-fit metrics for these distributions with respect to the flood dataset. Overall, the outcomes suggest that the BMOEEW distribution offers superior fits to the maximum flood data compared to other competing models, as shown by its lower values of AIC, AICc, BIC, and corresponding higher P-value.

Table 3: Data on the maximum flood level

0.654	0.613	0.315	0.449	0.297
0.402	0.379	0.423	0.379	0.324
0.269	0.740	0.418	0.412	0.494
0.416	0.338	0.392	0.484	0.26

Table 4: Maximum Likelihood Estimates for the maximum flood dataset

<i>Model</i>	<i>Estimates</i>	<i>SE</i>	<i>z-value</i>	<i>p-value</i>
BMOEEW	$\hat{\alpha}=0.2069$	0.21405	0.9665	0.3338
	$\hat{\lambda}=0.3357$	0.14535	2.3096	0.0209*
	$\hat{\theta}=0.2111$	1.43173	2.2428	0.0249*
	$\hat{\gamma}=8.4809$	1.56712	5.4118	<0.0001***
BMOEEK	$\hat{\alpha}=69.960$	0.30661	228.17	<0.0001***
	$\hat{\lambda}=34.661$	8.07445	4.2927	<0.0001***
	$\hat{\theta}=0.0263$	0.00983	2.6752	0.0075**
	$\hat{\gamma}=0.5510$	0.04709	11.698	<0.0001***
BMOEEN	$\hat{\alpha}=55.466$	3.02355	18.345	<0.0001***
	$\hat{\lambda}=0.1254$	0.15229	0.8233	0.4103
	$\hat{\theta}=1.4532$	0.28138	5.1645	<0.0001***
	$\hat{\beta}=24.718$	23.1661	1.0670	0.2860
EOFW	$\hat{\alpha}=5.5290$	30.0617	0.1839	0.8541
	$\hat{\lambda}=1.4492$	1.16595	1.2429	0.2139
	$\hat{\theta}=9.3102$	22.7253	0.4097	0.6820
	$\hat{\gamma}=0.8779$	1.89727	0.4627	0.6436
EGIW	$\hat{\alpha}=0.0255$	0.04359	5.8418	<0.0001***
	$\hat{\lambda}=0.0174$	0.01992	8.7544	<0.0001***
	$\hat{\theta}=16.446$	0.000675	24378	<0.0001***
	$\hat{\gamma}=19.589$	0.000023	86679	<0.0001***
OGEW	$\hat{\alpha}=80.851$	0.09209	878.02	<0.0001***
	$\hat{\lambda}=40.166$	43.0606	0.9328	0.3509
	$\hat{\theta}=0.2482$	0.23994	1.0344	0.3009
	$\hat{\beta}=0.8733$	0.15061	5.7984	<0.0001***
KUMA	$\hat{\alpha}=11.793$	0.60357	5.5745	0.0279*
	$\hat{\lambda}=3.3639$	5.36644	2.1990	0.6034
ETL	$\hat{\alpha}=4.6079$	6.38721	0.8176	0.0250*
	$\hat{\beta}=4.0457$	4.56302	1.3876	1.4757
	$\hat{\alpha}=0.5764$	0.74351	1.7864	0.0740
BOIPE	$\hat{\beta}=5.4591$	1.18367	4.6120	<0.0001***
	$\hat{\lambda}=91.894$	149.779	0.6135	0.5813
BOME	$\hat{\alpha}=73.152$	46.2599	1.5813	0.1138
	$\hat{\lambda}=0.0914$	0.01350	6.7734	<0.0001***

* Means significant at the 5% significance level

Table 5: Goodness-of-fit statistics and information criteria for the maximum flood dataset

MODEL	$\hat{\eta}$	AIC	AICc	BIC	A*	W*	K-S	p-value
BMOEEN	13.34	-18.69	-16.019	-14.703	2.2921	0.39237	1.0000	< 0.0001
BMOEENW	16.23	-24.46	-23.988	-22.472	0.2580	0.04229	0.1220	0.9473
BMOEEK	16.19	-24.41	-21.605	-20.428	0.2974	0.05437	0.1342	0.9358
EOFW	16.05	-22.90	-22.234	-21.917	0.2897	0.05052	1.0000	< 0.0001
EGIW	16.07	-24.15	-23.279	-21.103	0.2992	0.04966	0.1298	0.8893
OGEW	16.18	-24.24	-21.571	-20.255	0.2973	0.04844	0.1291	0.8926
BOIPE	16.02	-22.93	-22.531	-20.457	0.2737	0.16370	0.8297	0.5583
KUMA	12.89	-21.74	-21.031	-20.178	0.9326	0.12710	0.1913	0.3525
BMOEE	12.90	-21.80	-21.089	-21.198	1.4356	0.14340	0.1898	0.4710
ETL	13.60	-23.19	-22.485	-20.598	0.8147	0.13672	0.1525	0.4132

* Bold means best according to the chosen criteria

Plots of Figure 4 are the empirical density against the fitted distributions and the empirical distribution's CDF against the fitted distribution's CDFs. Figure 4 fitted distributions attempt to mimic the CDF's shape and the empirical distribution's density. This mimic nature gives an indication that the dataset was appropriate for the model. Additionally, the total time on test (TTT) plot in Figure 5 indicates an increasing hazard rate. This is evident from the concave shape of the TTT curve, which bends downward from the origin toward the upper right corner of the plot. Such a pattern indicate that the probability of observing larger flood values increases over time.

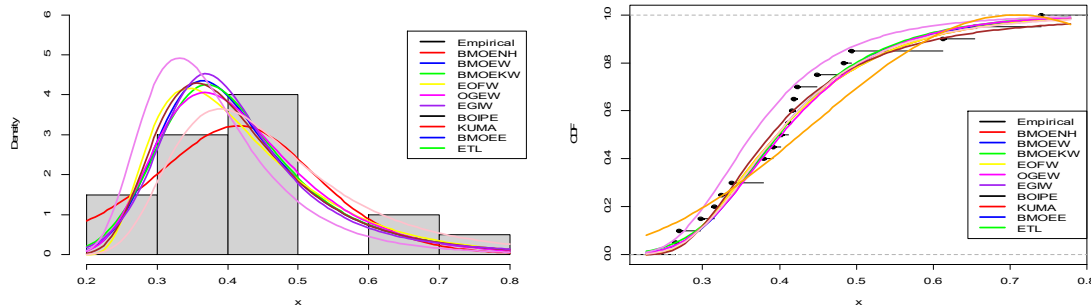


Figure 4: Plots of the fitted densities and CDFs of the maximum flood data

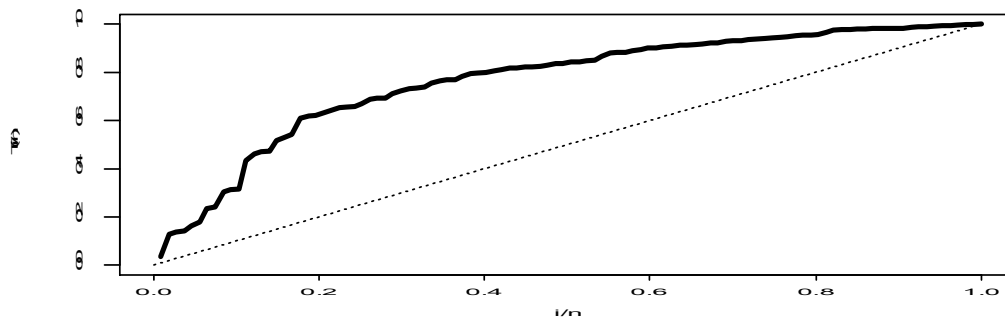


Figure 5: TTT transform plot for data on the maximum flood

3.6 Second Application

The second dataset was used by (Cordeiro et al., 2011; Cordeiro et al., 2012 & Festus and Blessing 2021). Their studies made used the 48 rock samples sourced from a petroleum reservoir as provided in Table 6. The maximum likelihood parameter estimates generated for the fitted distributions are provided in Table 7, with at least each parameter in the competing models being significant. Table 8 offers an in-depth examination of the goodness-of-fit metrics for these models in relation to the petroleum data. The goodness-of-fit results established that the BMOEEIW model presents better fits to the petroleum dataset compared to the competing models as seen by its lower AIC, AICc and BIC. The high P-value also indicate superiority.

Table 6: Data comprising 48 rock samples from a petroleum reservoir

0.0903	0.2037	0.2043	0.2809	0.1977	0.3286	0.1486	0.1624	0.2627
0.1795	0.3266	0.2301	0.1833	0.1509	0.2001	0.1918	0.1542	0.4641
0.1171	0.1481	0.1448	0.1331	0.2760	0.4205	0.1224	0.2286	0.1139
0.2252	0.1770	0.2007	0.1671	0.2316	0.2910	0.3413	0.4387	0.2627
0.1897	0.1726	0.2401	0.3117	0.1636	0.1825	0.1641	0.1535	0.1619-
0.2760	0.2538	0.2005						

Table 7: Maximum likelihood Estimates for the petroleum dataset

<i>Model</i>	<i>Estimates</i>	<i>SE</i>	<i>z-value</i>	<i>p-value</i>
BMOEEW	$\hat{\alpha} = 3.5005$	6.56740	0.5330	0.5940

	$\hat{\lambda}=11.025$	24.2154	0.4553	0.6489
	$\hat{\theta}=1.1199$	0.86005	1.3022	0.0199*
	$\hat{\gamma}=12.521$	5.36378	2.3343	0.0196*
EGIW	$\hat{\alpha}=9.3058$	58.5003	0.1591	0.8736
	$\hat{\lambda}=3.2066$	23.3779	0.1372	0.8909
	$\hat{\theta}=0.3876$	1.56360	0.2479	0.8042
	$\hat{\gamma}=0.9118$	3.71870	0.2452	0.8063
BMOEEN	$\hat{\alpha}=88.655$	68.8436	1.2878	0.0178*
	$\hat{\lambda}=2.9335$	6.97235	0.4207	0.6739
	$\hat{\theta}=0.8150$	0.38202	2.1333	0.0329*
	$\hat{\gamma}=10.424$	21.4373	0.4863	0.6268
BMOEEK	$\hat{\alpha}=4.5678$	7.31529	0.6244	0.5324
	$\hat{\lambda}=17.302$	42.3899	0.4082	0.6832
	$\hat{\theta}=0.7733$	0.84734	0.9126	0.3615
	$\hat{\gamma}=6.9560$	4.73800	3.4681	0.0268*
OGEW	$\hat{\alpha}=51.464$	82.9396	0.6205	0.5349
	$\hat{\lambda}=27.347$	29.2677	0.9344	0.3501
	$\hat{\theta}=0.6987$	0.28691	2.4353	0.0149*
	$\hat{\gamma}=0.4477$	0.40556	1.1038	0.0267*
BMOEEIW	$\hat{\alpha}=114.830$	0.00500	23392	<0.0001**
	$\hat{\beta}=1.00982$	0.12590	8.0216	<0.0001**
	$\hat{\theta}=0.72764$	1.48300	0.4971	0.6237
	$\hat{\gamma}=1.26683$	2.58180	0.5907	0.3357
MOETL	$\hat{\alpha}=0.0034$	0.26420	0.0813	0.1768
	$\hat{\lambda}=5.5834$	0.12290	0.0553	0.0482*
KUMA	$\hat{\alpha}=2.7189$	0.31700	0.1143	0.1283
	$\hat{\beta}=44.674$	0.14910	0.0886	0.0552
BETA	$\hat{\alpha}=0.7248$	0.10250	2.2183	0.0124*
	$\hat{\beta}=0.3714$	2.74910	8.2665	0.8613
EOFW	$\hat{\alpha}=0.0829$	0.13544	0.6122	0.5404
	$\hat{\lambda}=6.2538$	7.78741	0.8031	0.4219
	$\hat{\theta}=11.515$	1.32427	8.6951	<0.0001**
	$\hat{\gamma}=2.8348$	1.21979	2.3240	0.0201*

* Means significant at the 5% significance level

Table 8: Goodness-of-fit statistics and information criteria for thd petroleum dataset

MODEL	$\hat{\ell}$	AIC	AICc	BIC	A*	W*	K-S	p-value
BMOEEW	58.46	-108.92	-107.99	-105.44	0.1721	0.027	0.076	0.9435
EGIW	58.27	-108.50	-107.57	-106.81	0.1889	0.029	0.083	0.8956
OGEW	58.30	-108.65	-107.72	-107.01	0.2053	0.033	0.088	0.8521
BMOEEK	56.84	-105.67	-104.74	-104.22	32.907	6.727	0.942	<0.9120
BMOEEN	53.70	-99.769	-98.999	-98.284	0.1902	0.024	0.688	<0.0001
BMOEIW	58.55	-109.10	-108.17	-108.00	0.1581	0.020	0.050	0.9511
EOFW	58.45	-108.90	-107.97	-106.11	0.2071	0.031	0.084	0.8893
KUMA	52.49	-100.98	-100.00	-99.124	0.5030	0.206	0.153	0.2094
MOETL	57.66	-101.33	-100.34	-100.05	0.1887	0.039	0.083	0.8938
BETA	55.60	-107.20	-106.58	-106.15	1.5421	0.130	0.143	0.2828

* Bold means best according to the chosen criteria

Plots of Figure 6 are the empirical density against the fitted distributions and the empirical distribution's CDF against the fitted distribution's CDFs. The fitted distributions attempt to mimic the CDF's shape and the empirical distribution's density. Figure 7 shows the total time on test (TTT) plot of the petroleum data, which provides insight into the underlying failure behavior of the dataset. The plot displays a concave shape that rises above the 45° reference line (the diagonal from (0,0) to (1,1)). This characteristic pattern indicates an increasing hazard rate, which means, that, the likelihood of failure or occurrence of events increases with time.

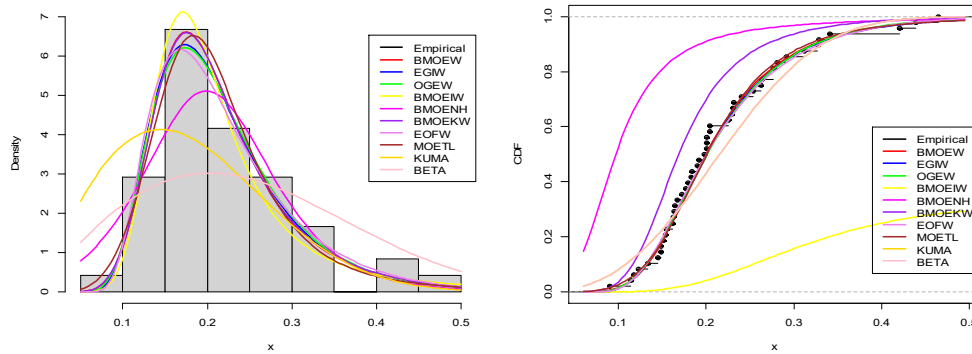


Figure 6: Plots of fitted densities and CDFs for data on petroleum

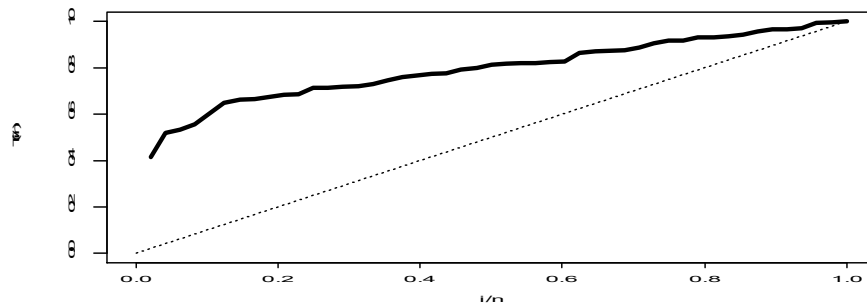


Figure 7: TTT transform plot of the petroleum dataset

3.7 Third Application

A study conducted by Caramanis et al., (1983) and Mazumdar and Gaver (1984) made use of the third dataset which, evaluated two distinct algorithms, SC16 and P3, for estimating unit capacity

factors. The data was later used by Marshall and Olkin, (1997) and are presented in Table 9. The values in Table 10 contain the maximum likelihood parameter estimates for the fitted distributions showing that, at least one of each model parameter is significant. Table 11 provide a detailed analysis of the goodness-of-fit for these distributions. The results confirmed that the BMOEENH model offers superior fits to the algorithm dataset compared to other competing models.

Table 9: Dataset on the algorithm

0.853	0.759	0.866	0.809	0.717
0.544	0.492	0.403	0.344	0.213
0.116	0.048	0.092	0.070	0.059
0.021	0.014	0.011	0.008	0.006
0.116	0.036	0.029		

Table 10: Maximum likelihood estimates for the Algorithm dataset

<i>Model</i>	<i>estimates</i>	<i>SE</i>	<i>z-value</i>	<i>p-value</i>
--------------	------------------	-----------	----------------	----------------

BMOEEN	$\hat{\alpha}$	=0.13990	0.18323	0.7635	0.4452
	$\hat{\lambda}$	=0.73234	1.49243	0.4907	0.6236
	$\hat{\theta}$	=12.5032	20.7021	0.6040	0.5459
	$\hat{\gamma}$	=0.09719	0.16883	0.5757	0.5648
BMOEEK	$\hat{\alpha}$	=3.44180	4.05808	0.8481	0.3964
	$\hat{\lambda}$	=0.69425	1.94023	0.3578	0.7205
	$\hat{\theta}$	=1.00902	2.85805	0.3530	0.7241
	c	=0.76892	0.56714	1.3558	0.0352**
BMOEEW	$\hat{\alpha}$	=0.13013	1.70965	1.2396	0.2151
	$\hat{\lambda}$	=0.10140	0.04519	2.2440	0.0248*
	$\hat{\theta}$	=6.44295	1.73384	3.7160	<0.0001***
	$\hat{\beta}$	=2.09699	0.68310	3.0698	<0.0001***
BMOEEIW	$\hat{\alpha}$	=0.07719	0.20628	0.6309	0.5281
	$\hat{\lambda}$	=0.92357	0.20604	4.4825	<0.0001**
	$\hat{\theta}$	=0.05875	1.42084	0.0413	0.9670
	$\hat{\lambda}$	=0.27001	6.52988	0.0413	0.9670
OGEW	$\hat{\alpha}$	=0.06858	0.02890	2.3726	0.0177 *
	$\hat{\lambda}$	=0.13885	0.06493	2.1384	0.0325*
	$\hat{\theta}$	=6.58540	2.86904	2.2953	0.0217*
	$\hat{\beta}$	=5.62524	2.54514	2.2102	0.0271*
BOIPE	$\hat{\alpha}$	=0.56790	0.29076	1.9532	0.0508
	$\hat{\beta}$	=14.1115	27.7342	0.5088	0.6109
	$\hat{\lambda}$	=0.93022	0.37310	2.4932	0.0127*
KUMA	$\hat{\alpha}$	=2.19056	0.22211	9.8623	<0.0001***
	$\hat{\lambda}$	=3.41688	0.57790	5.9126	<0.0001***
EOFW	$\hat{\alpha}$	=0.05992	0.01893	3.1658	0.0015**
	$\hat{\lambda}$	=6.45850	1.35512	4.7660	<0.0001***
	$\hat{\theta}$	=7.19178	2.09514	3.4326	0.0006***
	$\hat{\beta}$	=0.61348	0.13519	4.5378	<0.0001***
TL	$\hat{\alpha}$	=0.59430	0.12390	8.7283	0.0273*
EGIW	$\hat{\alpha}$	=7.59615	2.03688	0.8785	0.3797
	$\hat{\beta}$	=0.12933	0.27221	2.7758	0.0055**
	$\hat{\lambda}$	=4.75023	0.17631	0.6018	0.5473
	$\hat{\theta}$	=0.51131	3.76011	0.6363	0.5246

* Means significant at the 5% level of significance

Table 11: Goodness-of-fit and information criteria for the algorithm dataset

<i>MODEL</i>	$\hat{\ell}$	<i>AIC</i>	<i>AICc</i>	<i>BIC</i>	<i>A</i> *	<i>W</i> *	<i>K-S</i>	<i>p-value</i>
--------------	--------------	------------	-------------	------------	------------	------------	------------	----------------

BMOEEK	7.560	-7.119	-4.8967	-3.5770	0.7212	0.1137	0.5503	<0.0001
BMOEEN	10.27	-12.55	-11.325	-10.705	0.4171	0.0815	0.1237	0.8312
BMOEEW	9.510	-11.02	-10.799	-9.9796	0.6107	0.0950	0.1443	0.7242
EGIW	7.610	-7.273	-5.0504	-5.0106	0.5132	0.0995	0.1633	0.5720
BMOEIW	6.410	-4.825	-3.6031	-3.0034	33.986	7.3029	0.8347	<0.0001
OGEW	10.02	-11.03	-10.110	-9.0013	0.6709	0.1993	0.1936	0.3549
BOIPE	10.13	-11.46	-10.932	-10.018	0.6471	0.1057	0.1423	0.5583
KUMA	8.010	-10.78	-9.6714	-9.0263	0.8372	0.1096	0.1572	0.4876
EOFW	9.070	-11.14	-10.013	-9.5931	0.5635	0.0893	0.2073	0.2763
TL	8.120	-10.23	-9.1720	-9.0951	0.7456	0.1197	0.1690	0.5273

*Bold means best based on the selection criteria

Plots of Figure 8 are the empirical density against the fitted distributions and the empirical distribution's CDF against the fitted distribution's CDFs. Figure 8 fitted distributions attempt to imitate the CDF's shape and the empirical distribution's density. Figure 9 shows the total time on test (TTT) transform plot for the algorithm data, which offers valuable insights into the behavior of the underlying failure rate. Initially, the TTT curve exhibits a concave shape that lies below the 45° reference line, indicating a period of increasing hazard rate. Where the likelihood of failure rises over time. However, this was followed by a change in the curvature of the plot, eventually forming an inverted bathtub shape. This characteristic pattern suggests that the failure rate of the algorithm data increases at first, reaches a peak, and then begins to decline, implying that the risk of failure is highest at an intermediate point before decreasing.

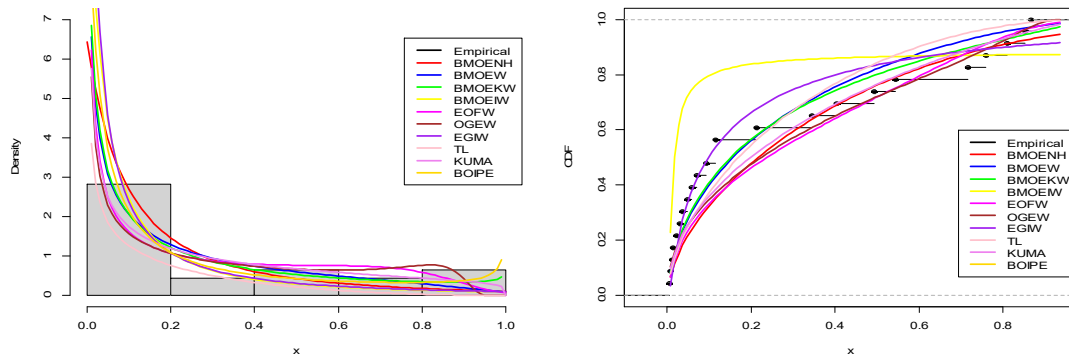


Figure 6: Plots of fitted densities and CDFs for the algorithm dataset.

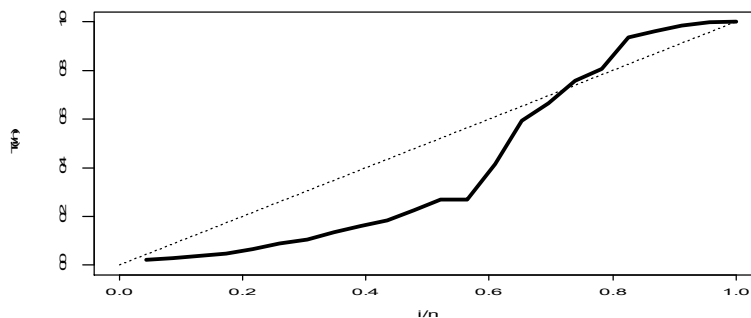


Figure 9: TTT transform plot of the algorithm dataset

3.8 Log BMOEEW Location-Scale Regression Model

This section discusses the location-scale regression model for the log-transformed BMOEEW model. If a random variable X is distributed according to the BMOEEW, the logarithm of the variable Y , denoted as $Y = \log(X)$, follows the logarithm of the BMOEEW distribution.

Following the assumptions, $\gamma = e^{-\frac{\mu}{\sigma}}$ and $\theta = \frac{1}{\sigma}$, then we can represent the density function for the location-scale regression BMOEEW distribution as

$$f(y, \mu, \sigma, \lambda, \alpha) = \frac{\alpha \lambda \exp\left(\frac{y - \mu}{\sigma}\right) \exp\left[-\exp\left(\frac{y - \mu}{\sigma}\right)\right] \left[1 - \exp\left[\exp\left(\frac{y - \mu}{\sigma}\right)\right]\right]^{\lambda-1}}{\sigma \left[1 - (1 - \alpha) \left(1 - \exp\left[-\exp\left(\frac{y - \mu}{\sigma}\right)\right]\right)\right]^{\lambda}}^2, y \in \mathbb{R}, \quad (24)$$

where $\alpha > 0$, $\lambda > 0$, and $\mu \in \mathbb{R}$ are shape parameters and $\sigma > 0$ is a scale parameter.

The associated survival function of the location-scale regression is given as

$$S(y, \alpha, \mu, \lambda, \sigma) = 1 - \frac{\alpha \left[1 - \exp\left[-\exp\left(\frac{y - \mu}{\sigma}\right)\right]\right]^{\lambda}}{\left[1 - (1 - \alpha) \left(1 - \exp\left[-\exp\left(\frac{y - \mu}{\sigma}\right)\right]\right)\right]^{\lambda}}, y \in \mathbb{R}. \quad (25)$$

Using the PDF in equation (21), the location-scale regression model of the BMOEEW is defined with the following regression feature.

$$y_i = q_i^r \phi + \sigma z_i, i=1,2,3,...,n,$$

where $\mu = q_i^r \phi$ is the location parameter depending on a set of covariates $\phi = (\phi_0, \phi_1, \dots, \phi_k)$ are the regression parameters, k represents the number of covariates, $q_i = (q_{i1}, q_{i2}, \dots, q_{ik})$ represents the covariates and z_i is the random error term that follows the log BMOEEW distribution. The maximum likelihood estimation method was used to estimate the regression model's parameters. The location-scale regression model log-likelihood function is given as

$$\ell(\lambda) = n \log\left(\frac{\alpha \lambda}{\sigma}\right) + \sum_{i=1}^n \left(\frac{y_i - \mu_i}{\sigma}\right) - \sum_{i=1}^n \exp\left(\frac{y_i - \mu_i}{\sigma}\right) + (\lambda - 1) \sum_{i=1}^n \log\left(1 - \exp\left[\exp\left(\frac{y_i - \mu_i}{\sigma}\right)\right]\right) - 2 \sum_{i=1}^n \log\left(1 - (1 - \alpha) \left(1 - \exp\left[-\exp\left(\frac{y_i - \mu_i}{\sigma}\right)\right]\right)\right). \quad (26)$$

The suitability of the BMOEEW location-scale regression model was evaluated using Cox-Snell residuals. The Cox-Snell residual of the BMOEEW location-scale regression model is $\hat{r}_i = -\log(S(y_i / \hat{\alpha}, \hat{\lambda}, \hat{\sigma}, \hat{\mu}))$, where $S(y_i / \hat{\alpha}, \hat{\lambda}, \hat{\sigma}, \hat{\mu})$ is defined in equation (25). The Cox-Snell residuals of the BMOEEW location-scale regression model should exhibit a conventional exponential distribution if the model fits the provided data satisfactorily. Data on long-term interest (X_i) versus foreign direct investment (y_i) was modeled to illustrate the use of the BMOEEW

location-scale regression model. These data can be found in Marshall and Olkin, (1997) and modeled using the regression equation;

$$y_i = \phi_0 + \gamma_1 \phi_{1i}, \quad i = 1, 2, 3, \dots, 34,$$

where $y_i = \log(x_i)$ follows the BMOEEW distribution. The performance of the BMOEEW location-scale regression model was compared with Beta, Simplex, Unit-Lindley and Unit-ISDL location-scale regression models. The parameter estimates of the BMOEEW and the competing location-scale regression models as well as the goodness-of-fit statistics are provided in Tables 12 and 13. Table 12, which displays the likelihood estimates showed all the models with their parameter to be significant, an indication that the, dataset was very appropriate. From Table 13, it can be established that the BMOEEW location-scale regression model has performed better in fitting the dataset than the location-scale regression models of the Simplex, Beta, Unit-Lindley, and Unit-ISDL since it has the highest log-likelihood value and has the smallest values of AIC, AICc, and BIC statistics. It was also observed that long-term interest (LTI) rates positively affect foreign direct investment (FDI) with a 5% level of significance for BMOEEW and all the competing location-scale regression models.

Table 12: Likelihood Estimates for the Location Scale Regression with Other Models

<i>Model</i>	<i>estimates</i>	<i>SE</i>	<i>z-value</i>	<i>p-value</i>
BMOEEW	$\hat{\alpha} = 1788.9$	<0.0001	2477900	<0.0001***
	$\hat{\lambda} = 0.6700$	<0.0001	7959800	<0.0001***
	$\hat{\sigma} = -1.8885$	0.0618	-30.5550	<0.0001***
	$\hat{\mu} = -0.1846$	0.0332	-5.55700	<0.0001***
	$\hat{\gamma}_0 = 3.7591$	<0.0001	5401400	<0.0001***
	$\hat{\gamma}_1 = 0.5093$	<0.0001	9671000	<0.0001***
Beta	$\hat{\alpha} = -3.7591$	0.1686	-28.2023	0.001**
	$\hat{\lambda} = -0.3716$	0.1722	-35.3417	0.0309**
	$\hat{\sigma} = 71.200$	18.810	417.291	0.001**
Simplex	$\hat{\alpha} = -3.7986$	0.3286	-108.337	0.001**
	$\hat{\lambda} = -0.4421$	0.1213	-73.2810	0.001**
	$\hat{\sigma} = 2.5358$	0.1988	65.3790	0.0226*
Unit-Lindley	$\hat{\alpha} = -3.7171$	0.2022	-32.4810	0.001**
	$\hat{\lambda} = -0.5141$	0.1750	-24.3170	0.003**
Unit-ISDL	$\hat{\alpha} = -3.7242$	0.1998	-30.1170	0.001**
	$\hat{\lambda} = -.05093$	0.1737	-23.4180	0.003**

Table 13: Goodness of fit statistics of the Regression Models

<i>Model</i>	$\hat{\ell}$	<i>AIC</i>	<i>AICc</i>	<i>BIC</i>	<i>p-value</i>
BMOEEW	107.271	-253.344	-251.051	-248.713	0.02744
Beta	103.800	-201.600	-197.021	-195.186	0.00100
Simplex	94.3895	-182.799	-178.200	-176.540	0.02620
Unit-Lindley	103.981	-203.962	-200.909	-199.038	0.00330
Unit-ISDL	104.393	-204.785	-201.733	-162.103	0.00330

*Bold means best based on the selection criteria

4. Conclusions

In this study, the BMOEEG family of distributions was introduced and rigorously explored. A comprehensive analysis of its statistical properties was conducted, including derivations for the moments, moment generating function, characteristic function, incomplete moments, and quantile function. Within this newly proposed family, three notable sub-models, as BMOEEW, BMOEENH and BMOEEIW, were developed and analyzed in detail. The density plots of these distributions revealed a wide variety of shapes such as bimodal, left-skewed, right-skewed, and symmetric forms, indicating the models' adaptability to diverse data characteristics across different application areas. Moreover, the hazard function plots exhibited multiple patterns, including both monotonic and non-monotonic behaviors, underscoring the family's suitability for modeling a broad spectrum of lifetime and reliability data. To estimate the model parameters effectively, several estimation techniques were applied: Maximum Likelihood Estimation (MLE), Ordinary Least Squares (OLS), and the Cramér-von Mises (CVM) method. The performances of these estimators were assessed using a Monte Carlo simulation, confirming their consistency and reliability under different scenarios.

The practical relevance and applicability of the BMOEEG family were demonstrated through empirical analyses on real-world datasets. In these applications, the proposed distributions outperformed the existing competing models, offering superior goodness-of-fit to the datasets used. Furthermore, a new location-scale regression model based on the BMOEEG distribution framework showed improved parametric fitting compared to the existing models. This study provides a significant contribution to statistical modeling, especially in the domain of real data analysis and flexible distribution design. The BMOEEG family addresses the key limitations of classical models by offering enhanced flexibility, richer shape diversity, and improved estimation accuracy, making it a valuable tool for statisticians and applied researchers. Its ability to model real-world phenomena more precisely across disciplines such as engineering, medicine, economics, and environmental science marks a substantial advancement in the development of modern statistical methodologies.

5. Limitations of the proposed regression model

The BMOEEG distribution has some limitations that can affect its applicability and performance. Its complex parameter structure, involving location, scale, and flexibility parameters, can lead to computational challenges and identifiability issues, especially with overparameterization. The bounded nature of the distribution may not suit datasets with unbounded characteristics, leading to model misspecifications, and it can be sensitive to outliers near or beyond its bounds, which may complicate the modeling process. Additionally, the model's effectiveness relies on strict assumptions such as independence, homoscedasticity, and the appropriateness of the distribution for residuals, with violations potentially causing biased estimates. The parameters can also be difficult to interpret, posing challenges in communicating the results to non-statistical audiences. Moreover, the model may struggle with datasets featuring heavy tails or multimodality and may require large sample sizes for reliable parameter estimation, with incomplete data further reducing accuracy.

6. Future work

The BMOEEG family of distribution provided a robust foundation, but its potential is far from fully explored. By extending it to multivariate settings, dynamic systems, robust estimation, and machine learning, researchers can significantly expand its applicability across disciplines.

These directions also offer opportunities for theoretical advancements and practical contributions to modern data analysis.

7. Availability of the Data

The data used are online and appropriate citations are made to that effect.

8. Conflict of Interest

There are no conflicts of interest regarding this work among the authors.

References

- [1]. Abdelall, Y., Ismail, G., & Nagy, H. (2025). A New Bounded Distribution: Covid-19 Application. *The Egyptian Statistical Journal*, 69(1), 1-29. [https://doi: 10.21608/esju.2025.330520.1047](https://doi.org/10.21608/esju.2025.330520.1047)
- [2]. Ahmed, A., Mohamed, S., & Abdelfattah, A. (2025). Comparative Study of Estimation Methods for Kumaraswamy Weibull Regression Model: An Application to Economic Value-Added Data. *The Egyptian Statistical Journal*, 69(1), 87-102. doi: 10.21608/esju.2025.349713.1062.
- [3]. Ahmad, M., Jabeen, R., Zaka, A., Hamdi, W., and Alnssyan, B. (2024). A unified generalized family of distributions: Properties, inference, and real-life application. *AIP Advances*. <https://doi.org/10.1063/5.0187188>
- [4]. Alahmadi, A. A., Albalawi, O., Khashab, R. H., Johannssen, A., Nasiru, S., Almarzouki, S. M., & Elgarhy, M. (2025). Modeling of lifetime scenarios with non-monotonic failure rates. *PLOS ONE*, 20(1), e0314237. <https://doi.org/10.1371/journal.pone.0314237>
- [5]. Altun, E., Alizadeh, M., Afify, A. Z., & Özel, G. (2019). The generalized odd half-Logistic family of distributions with regression models. *International Journal of Statistics and Economics*, 20(1), 88–110.
- [6]. Caramanis, M., Stremmel, J.V., Fleck, W., and Daniel, S. (1983). Probabilistic production costing. *International Journal of Electrical Power and Energy Systems*, 5(3), 75-86.
- [7]. Chesneau, C., and Jamal, F. (2021). The sine Kumaraswamy-G family of distributions. *Journal of Mathematical Extension*, 15(2), 1-33.
- [8]. Cordeiro, G.M., and Brito, R.S. (2012). The Beta Power Distribution. *Brazilian Journal of Probability and Statistics*, 26(1), 88-112.
- [9]. Cordeiro, G.M., and de-Castro, M. (2011). A new family of generalized distributions. *Journal of Statistical Computation and Simulation*, 8(1), 883-893.
- [10]. Cordeiro, G.M., Ortega, E.M.M., and Lemonte, A.J. (2014). The exponential–Weibull lifetime distribution. *Journal of Statistical Computation and Simulation*, 8(4), 2592-2606.
- [11]. Dumonceaux, R., and Antle, C.E. (1973). Discrimination between the log-normal and Weibull distributions. *Technometrics*, 15(4), 923-926.
- [12]. Fayomi, A., Algarni, A., Almarashi, A.M., and Jan, N. (2021). Sine inverse Lomax generated family of distributions with applications. *Mathematical Problems in Engineering*. <https://doi.org/10.1155/2021>
- [13]. Festus, O., and Blessing, I. (2021). A New Marshall-Olkin Extended Family of Distributions with Bounded Support. *Journal of Science*, 34(3), 29-35.
- [14]. Ghosh, I., Dey, S., and Kumar, D. (2019). Bounded M-O Extended Exponential Distribution with Applications. *Stochastics and Quality Control*. <https://doi.org/10.1515/2018-0028>

- [15]. Gomes-Silva, F., Ramos, M.W., Percontini, A., and Venancio, R. (2016). The Odd Lindley-G Family of Distributions. *Austrian Journal of Statistics*. <https://doi.org/10.17713/ajs.v46i1.222>
- [16]. Hosseini, B., Afsharif, M., and Alizadeh, M. (2018). The Generalized Odd Gamma-G Family of Distributions: Properties and Applications. *Austrian Journal of Statistics*. <https://doi.org/10.17713/ajs.v47i2.580>
- [17]. Jamal, F., Chesneau, C., Saboor, A., Aslam, M., Tahir, M.H., and Mashwani, W.K. (2022). The U family of distributions: Properties and applications. *Mathematica Slovaca*, 72(1), 217-240.
- [18]. Keganne, K., Gabaitiri, L., and Makubate, B. (2023). A new extension of the exponentiated generalized-G family of distributions. *Scientific African*. <https://doi.org/10.1016/j.sciaf.2023.e01719>
- [19]. Kumaraswamy, P. (1980). A generalized probability density function for double-bounded random processes. *Journal of Hydrology*, 46(1), 79-88.
- [20]. Marshall, A.M., and Olkin, I. (1997). A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. *Biometrika*, 4(8), 641-652.
- [21]. Mazumdar, M., and Gaver, D.P. (1984). On the Computation of Power-Generating System Reliability Indexes. *Technometrics*, 26(2), 17-33.
- [22]. Muhammad, M., Bantan, R.A., Liu, L., Chesneau, C., Tahir, M.H., Jamal, F., and Elgarhy, M., (2021). A new extended cosine G distributions for lifetime studies. *Mathematics*, 9(21), 1-29.
- [23]. Muhammad, S., and Abed, K. (2019). Odd generalized exponential Weibull exponential distribution. *Journal of Engineering and Applied Sciences*, 14, 10360–10368.
- [24]. Mustpha, M. H., Jakperik, D., & Nasiru, S. (2025). Tangent Exponentiated Odd Log-Logistic Weibull Quantile Regression with Applications to Complete and Censored Data. *The Egyptian Statistical Journal*, 69(1), 173-205. doi: 10.21608/esju.2025.359062.1070
- [25]. Nadarajah, S., and Kotz, S. (2007). The gamma-Weibull distribution revisited. *NZIAM Journal*, 4(8), 405-407.
- [26]. Nadarajah, S., Cordeiro, G.M., and Ortega, M.M. (2011). General results for the beta-modified Weibull distribution. *Journal of Statistical Computation and Simulation*, 8(1), 121-132.
- [27]. Nasiru, S. (2018). Extended odd Fréchet-G family of distributions. *Journal of Probability and Statistics*. <https://doi.org/10.1155/2018/2931326>
- [28]. Nasiru, S., Abubakari, A.G., and Irene, D.A. (2021). Bounded Odd Inverse Pareto distributions. *International Journal of Mathematics and Mathematical Sciences*. <https://doi.org/10.1155/2021/9955657>
- [29]. Nasiru, S., Mwinta, P.N., and Ngesa, O. (2019). Exponentiated Generalized Power Series Family of Distributions. *Annals of Data Science*. <https://doi.org/10.1007/s40745-018-0170-3>
- [30]. Pourdarvish, A., MirMostafaei, S.M., and Naderi, K. (2015). The exponentiated Topp-Leone distribution: Properties and application. *Journal of Applied Environmental and Biological Sciences*, 5, 251-256.
- [31]. Roozegar, R., Tekle, G., and Hamedani, G. (2022). A New Generalized-X Family of Distributions: Applications, Characterization and a Mixture of Random Effect Models. *Pakistan Journal of Statistics and Operation Research*, 18(2), 483-504.

- [32]. Saboor, A., and Pogány, T.K. (2014). Marshall-Olkin gamma-Weibull distribution with applications. *Communications in Statistics-Theory and Methods*. <https://doi.org/10.13140/2.1.3722.3046>
- [33]. Zaidi, S.M., Al Sobhi, M.M., El-Morshedy, M., and Afify, A.Z. (2021). A new generalized family of distributions: Properties and applications. *AIMS Mathematics*, 6(1), 456-476.
- [34]. Zamanah, E., Nasiru, S., and Luguterah, A. (2022). Harmonic Mixture Weibull-G Family of Distributions: Properties, Regression and Applications to Medical Data. *Computational and Mathematical Methods*. <https://doi.org/10.1155/2022/2836545> Authors Biography: