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A Critique on Neutrosophic Estimators of Finite Population Mean Using Interval-Valued Data

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ABSTRACT
Numerous recent studies have proposed neutrosophic ratio-type estimators of the finite population mean for interval-valued data using auxiliary information. These studies report neutrosophic bounds of the mean-squared errors of the estimators. This study employs simulation-based empirical distributions to show that such bounds are often inaccurate and misleading, highlighting the need for more reliable inference when analyzing interval-valued data.
Keywords: Finite population; Mean-squared error; Population mean; Ratio-type estimator; Neutrosophic statistics.

1. Introduction

In many practical situations, study variables are observed as intervals rather than precise values—such as income ranges, temperature limits, or grouped survey responses. Such interval-valued data arise from rounding, measurement error, or imprecise reporting. Incorporating interval-valued observations into estimation procedures helps preserve the inherent uncertainty in the data rather than relying on point values.

Since the introduction of *Neutrosophic Statistics* by [30], several mathematical disciplines —such as topology, algebra, and geometry—have been extended under the neutrosophic framework. In recent years, this concept has also been applied to statistical problems involving interval-valued data, including analysis of variance, experimental design, process monitoring, capability analysis, and reliability assessment. Examples include neutrosophic regression analysis [14], neutrosophic experimental-design approaches [11], and neutrosophic estimators in survey sampling [32].

In survey sampling involving interval-valued data, several authors have proposed auxiliary information-based estimators for finite population parameters, particularly the population mean. The supplementary information is utilized in terms of one or more interval-valued auxiliary variables. These include works by [16], [32], [2], [3], [37], [15], [13], [24], [4], [33], [6], [39], [25], [40], [12], [22], [41], [23], [34], [27], [20], [21], [42], [1], [9], [18], [38], [28], [10], [7], including the relevant references cited therein. In all of these studies, the neutrosophic bounds for the mean squared errors (MSEs) of the estimators were expressed as intervals. However, the authors of these works did not clearly specify what underlying statistical quantity the neutrosophic intervals were intended to represent.

In recent years, several researchers have critically evaluated neutrosophic statistical methods. For instance, [35] and [31] reviewed neutrosophic process monitoring approaches and identified key conceptual and methodological issues that must be addressed before such methods

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can be considered practical. Building on this critique, [36] demonstrated through examples that neutrosophic analyses often yield results that are inaccurate or misleading and recommended simulation-based approaches as a clearer and more interpretable alternative for analyzing interval-valued data. Similarly, [5] examined recent applications of neutrosophic statistics to designed experiments and showed that the reported neutrosophic bounds on analysis of variance statistics were often incorrect or lacked sufficient precision to be practically useful.

Following the works of [36] and [5], this study employs a simulation-based approach to obtain empirical distributions of the MSEs of estimators for several finite interval-valued populations. The results indicate that the neutrosophic bounds of the MSE estimators are generally too narrow or inaccurate to provide reliable inference.

The remainder of the paper is organized as follows: Section 2 introduces the neutrosophic observations and notations. Section 3 presents the neutrosophic ratio-type estimators of the finite population mean and their mathematical expressions for the MSEs. Section 4 reports the simulation-based results, and Section 5 concludes the paper.

2. Neutrosophic Observation and Notations

Suppose that a variable Z takes a value within an interval $[a, b]$, say $Z \in [a, b]$. A corresponding neutrosophic variable Z_N is said to take a value within the neutrosophic interval $[Z_L, Z_U]$ when $Z_N = Z_L + Z_U I_N$ with $I_N \in [I_L, I_U]$, where (Z_L, Z_U) and (I_L, I_U) denote the (lower, upper) bounds of Z_N and I_N, respectively. Here, I_N denotes the indeterminacy in Z_N. It is conventional to take I_N as $I_N \in [0,1]$. $Z_N \in [a, b]$ when $I_N \in [0,1]$. Since Z_N is defined in terms of an interval, any subsequent computation involving Z_N may yield an interval rather than a single value.

Consider a finite neutrosophic population U_N that comprises M_N units, denoted by $U_{i,N}$ for $i = 1,2, \dots, M$. Let $Y_{i,N}$ and $X_{i,N}$ denote the neutrosophic study and auxiliary variables for the ith unit of U_N, respectively. Let $(\bar{Y}_N, \bar{X}_N)$, $(S^2_{Y,N}, S^2_{X,N})$ and denote neutrosophic population means and variances of the neutrosophic study and auxiliary variables, respectively. Similarly, let $(C_{X,N}, B_{2X,N})$ denotes the coefficients of variation and kurtosis for the neutrosophic auxiliary variable, respectively. Here, $\bar{Y}_N \in [\bar{Y}_L, \bar{Y}_U]$, $\bar{X}_N \in [\bar{X}_L, \bar{X}_U]$, $S^2_{Y,N} \in [S^2_{Y,L}, S^2_{Y,U}]$, $S^2_{X,N} \in [S^2_{X,L}, S^2_{X,U}]$, $C_{X,N} \in [C_{X,L}, C_{X,U}]$, and $B_{2X,N} \in [B_{2X,L}, B_{2X,U}]$. Similarly, let $R_{YX,N}$ denote the neutrosophic coefficient of correlation between Y_N and X_N, where $R_{YX,N} \in [R_{YX,L}, R_{YX,U}]$.

Suppose that a neutrosophic sample of size $n_N \in [n_L, n_U]$ is drawn from a finite neutrosophic population U_N using simple random sampling without replacement scheme. Then the neutrosophic sample means of Y_N and X_N are $\hat{\bar{Y}}_N \in [\hat{\bar{Y}}_L, \hat{\bar{Y}}_U]$ and $\hat{\bar{X}}_N \in [\hat{\bar{X}}_L, \hat{\bar{X}}_U]$, respectively. Here, $\hat{\bar{Y}}_N$ and $\hat{\bar{X}}_N$ are conventional estimators of $\bar{Y}_N$ and $\bar{Y}_N$, respectively. The variances and covariance of $\hat{\bar{Y}}_N$ and $\hat{\bar{X}}_N$ are respectively given by

$$\mathrm{Var}(\hat{\bar{Y}}_N) = Q_N \bar{Y}_N^2 C^2_{Y,N}, \tag{1}$$

$$\mathrm{Var}(\hat{\bar{X}}_N) = Q_N \bar{X}_N^2 C^2_{X,N} \text{ and} \tag{2}$$

$$\mathrm{Cov}(\hat{\bar{Y}}_N, \hat{\bar{X}}_N) = Q_N \bar{Y}_N \bar{X}_N R_{YX,N} C_{Y,N} C_{X,N}, \tag{3}$$

where $Q_N = (1/n_N - 1/M_N)$ with $Q_N \in [Q_L, Q_U]$. For more details, we refer to [32].

3. Neutrosophic Estimators

[32] proposed several neutrosophic ratio-type estimators of the finite population mean based on a single neutrosophic auxiliary variable.

The first five estimators of $\bar{Y}_N$, suggested by [32], use information on neutrosophic sample means $(\hat{\bar{X}}_N, \hat{\bar{X}}_N)$, along with some known parameters of the neutrosophic auxiliary variable X_N, say $C_{X,N}$ and $B_{2X,N}$, are given by

$$\hat{\bar{Y}}_{1,N} = \hat{\bar{Y}}_N \left(\frac{\bar{X}_N}{\hat{\bar{X}}_N}\right),$$
$$\hat{\bar{Y}}_{2,N} = \hat{\bar{Y}}_N \left(\frac{\bar{X}_N + C_{X,N}}{\hat{\bar{X}}_N + C_{X,N}}\right),$$
$$\hat{\bar{Y}}_{3,N} = \hat{\bar{Y}}_N \left(\frac{\bar{X}_N + B_{2X,N}}{\hat{\bar{X}}_N + B_{2X,N}}\right),$$
$$\hat{\bar{Y}}_{4,N} = \hat{\bar{Y}}_N \left(\frac{\bar{X}_N B_{2X,N} + C_{X,N}}{\hat{\bar{X}}_N B_{2X,N} + C_{X,N}}\right) \text{ and}$$
$$\hat{\bar{Y}}_{5,N} = \hat{\bar{Y}}_N \exp\left(\frac{\bar{X}_N - \hat{\bar{X}}_N}{\bar{X}_N + \hat{\bar{X}}_N}\right).$$

[32] derived mathematical expressions for the neutrosophic biases and MSEs of these estimators under a first-order of approximation. For brevity, only the the neutrosophic MSEs of the estimators are discussed here.

The neutrosophic MSEs of the estimators $\hat{\bar{Y}}_{j,N}$ for $j = 1,2,\ldots,5$ under first order of the approximation are given by

$$\text{MSE}(\hat{\bar{Y}}_{1,N}) \approx Q_N^2 \bar{Y}_N^2 \left(C_{Y,N}^2 + C_{X,N}^2 - 2R_{YX,N} C_{Y,N} C_{X,N}\right),$$
$$\text{MSE}(\hat{\bar{Y}}_{2,N}) \approx Q_N^2 \bar{Y}_N^2 \left(C_{Y,N}^2 + \left(\frac{\bar{X}_N}{\bar{X}_N + C_{X,N}}\right)^2 C_{X,N}^2 - 2\left(\frac{\bar{X}_N}{\bar{X}_N + C_{X,N}}\right) R_{YX,N} C_{Y,N} C_{X,N}\right),$$
$$\text{MSE}(\hat{\bar{Y}}_{3,N}) \approx Q_N^2 \bar{Y}_N^2 \left(C_{Y,N}^2 + \left(\frac{\bar{X}_N}{\bar{X}_N + B_{2X,N}}\right)^2 C_{X,N}^2 - 2\left(\frac{\bar{X}_N}{\bar{X}_N + B_{2X,N}}\right) R_{YX,N} C_{Y,N} C_{X,N}\right),$$
$$\text{MSE}(\hat{\bar{Y}}_{4,N}) \approx Q_N^2 \bar{Y}_N^2 \left(C_{Y,N}^2 + \left(\frac{\bar{X}_N B_{2X,N}}{\bar{X}_N B_{2X,N} + C_{X,N}}\right)^2 C_{X,N}^2 - 2\left(\frac{\bar{X}_N B_{2X,N}}{\bar{X}_N B_{2X,N} + C_{X,N}}\right) R_{YX,N} C_{Y,N} C_{X,N}\right),$$
$$\text{MSE}(\hat{\bar{Y}}_{5,N}) \approx Q_N^2 \bar{Y}_N^2 \left(C_{Y,N}^2 + \frac{1}{4} \times C_{X,N}^2 - R_{YX,N} C_{Y,N} C_{X,N}\right).$$

Further derivations and proofs can be found in [32].

4. Numerical study

This section considers four finite populations containing interval-valued data on both study and auxiliary variables. These populations have been used in several previous studies on neutrosophic estimators of the population mean based on auxiliary information. Following [36], a simple and informative simulation-based approach is employed to obtain the empirical distributions of the MSEs for the considered ratio-type estimators.

1. **First population:** It is taken from the R package 'neutroSurvey' of [17], containing yearly interval-valued data for Japan from 1975 to 2015. The dataset comprises 31 observations on the variables: 'Country', 'Sex', 'Year', 'Auxili_min', 'Auxili_max', 'Study_min', and 'Study_max'. Here, $M_N = 82$ and $n_N = 25$.
2. **Second population:** A real-life dataset on natural growth rate, taken from [8] and [26], contains 36 interval-valued observations on 'birthrate' (auxiliary variable) and 'natural growth rate' (study variable). Here, $M_N = 36$ and $n_N = 9$.
3. **Third population:** This dataset, taken from [29], contains records of patients who visited the Basic Health Unit (BHU) reporting gastritis between June 2021 and August 2021. It comprises 70 interval-valued observations on four variables: 'Systolic blood pressure', 'Diastolic blood pressure', 'Heartbeat rate' and 'Body temperature'. The 'heartbeat rate' and 'Body temperature' are considered as study and auxiliary variables, respectively. Here, $M_N = 70$ and $n_N = 30$.
4. **Forth population:** This dataset, taken from [19], consists of interval-valued climate data for state of Alabama, USA, recorded during the month of May. It comprises 18 interval-valued observations on three variables: 'Hourly Temperature', 'Dew Point Temperature' and 'Relative Humidity'. The 'Hourly Temperature' and 'Dew Point temperature' are considered as study and auxiliary variables, respectively. Here, $M_N = 18$ and $n_N = 4$.

In the neutrosophic survey sampling literature, the bounds of the MSEs for auxiliary information-based estimators are typically obtained using two methods. The first method employs the R Package 'neutroSurvey' to calculate neutrosophic parameters, which are then substituted into the MSE formulas. The second, most commonly used method, relies soly on the lower and upper endpoints of the interval-valued data. Here, we use both of these approaches to calculate the neutrosophic bounds of the MSEs of estimators.

Following [36], a probability distribution may be assumed within each interval of the data to generate simulated observations. Although any beta-type distribution can be adopted, assuming a uniform distribution over each interval is reasonable when the underlying distribution is unknown. Using 100,000 simulated datasets for each population, empirical MSE values of the estimators were computed based on the expressions obtained under first-order of approximation. Figures 1–4 compare the empirical MSE distributions with the corresponding neutrosophic bounds.

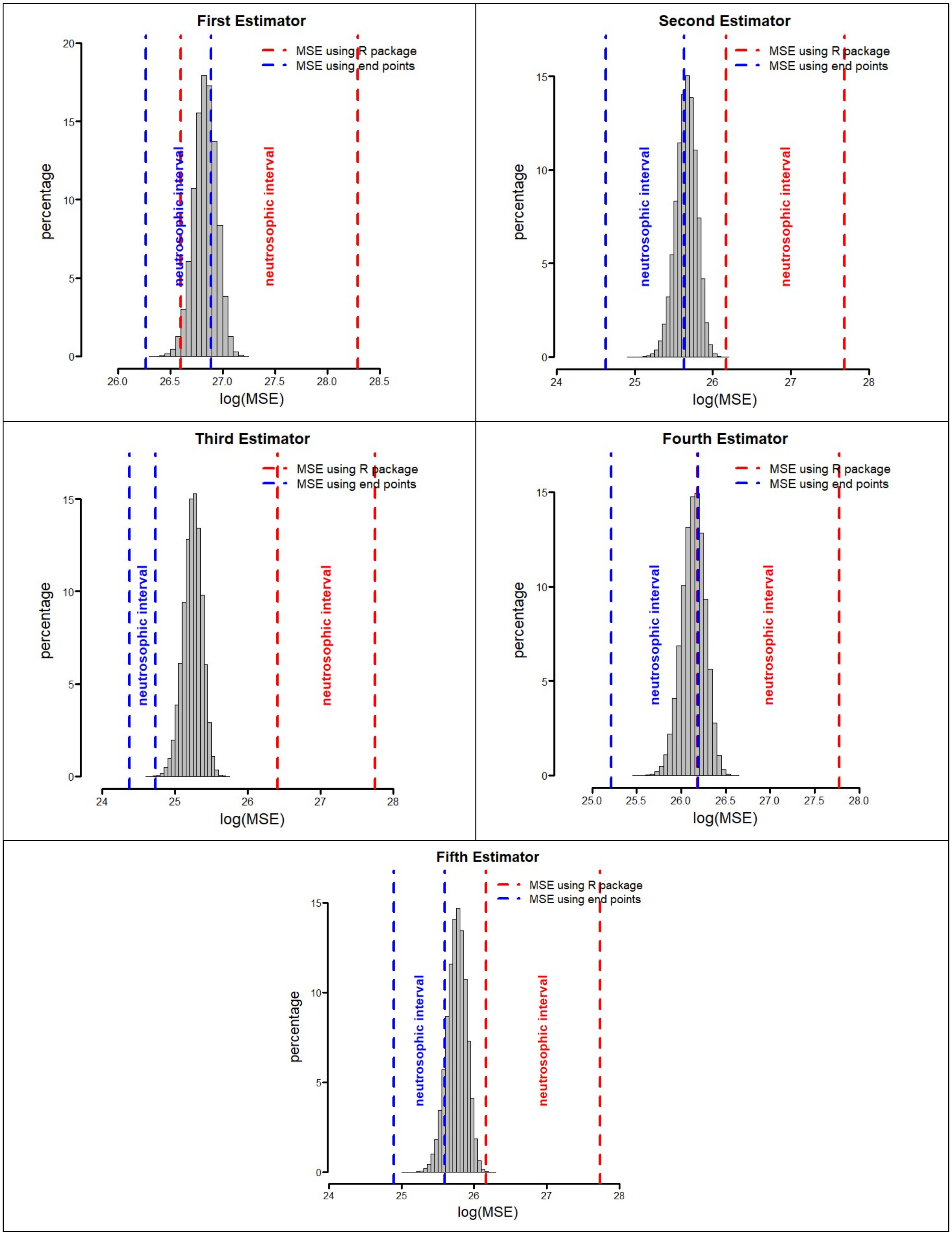

Figure 1: Empirical histograms and neutrosophic bounds of the MSEs of estimators for Population-I

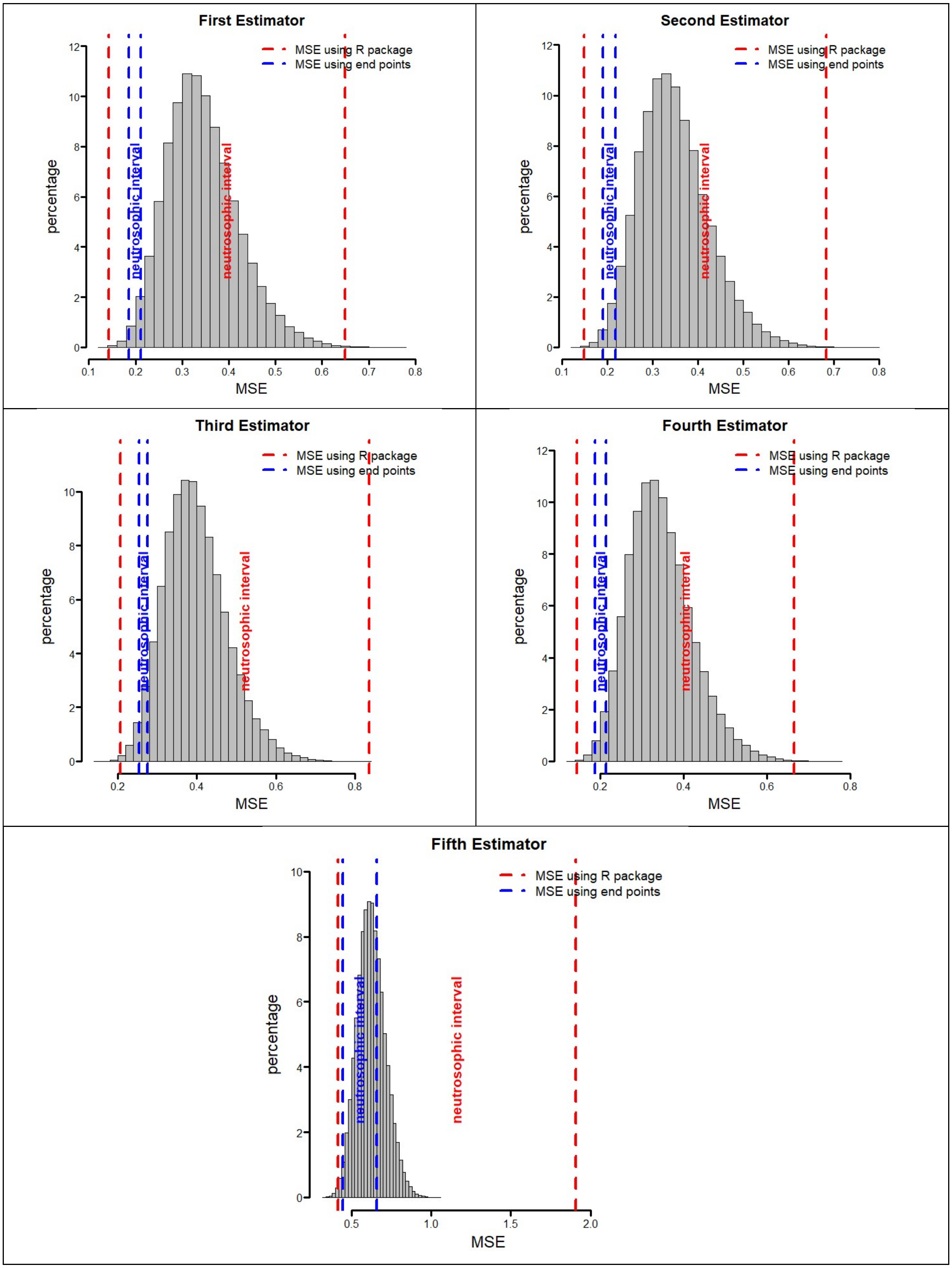

Figure 2: Empirical histograms and neutrosophic bounds of the MSEs of estimators for Population-II

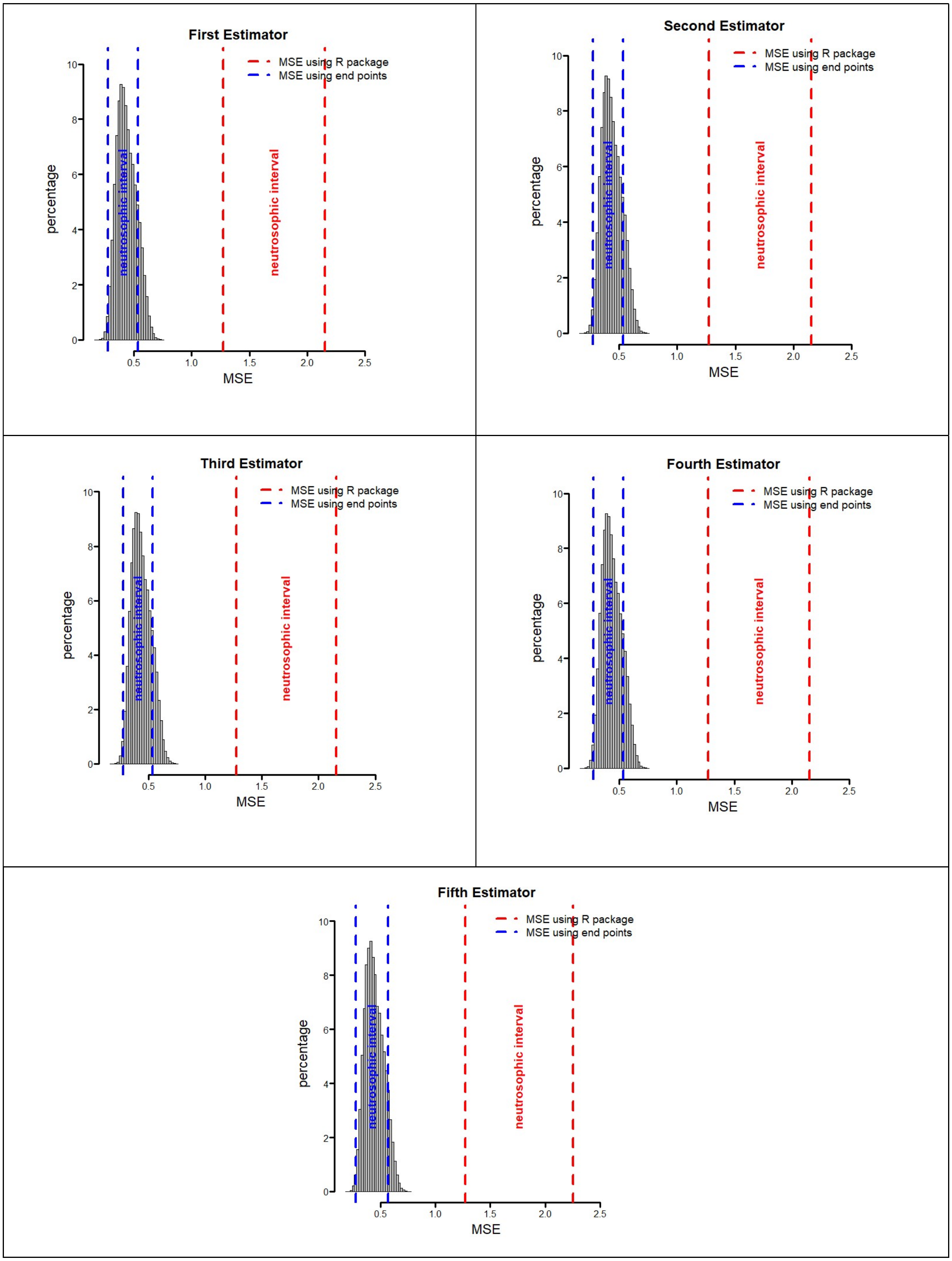

Figure 3: Empirical histograms and neutrosophic bounds of the MSEs of estimators for Population-III

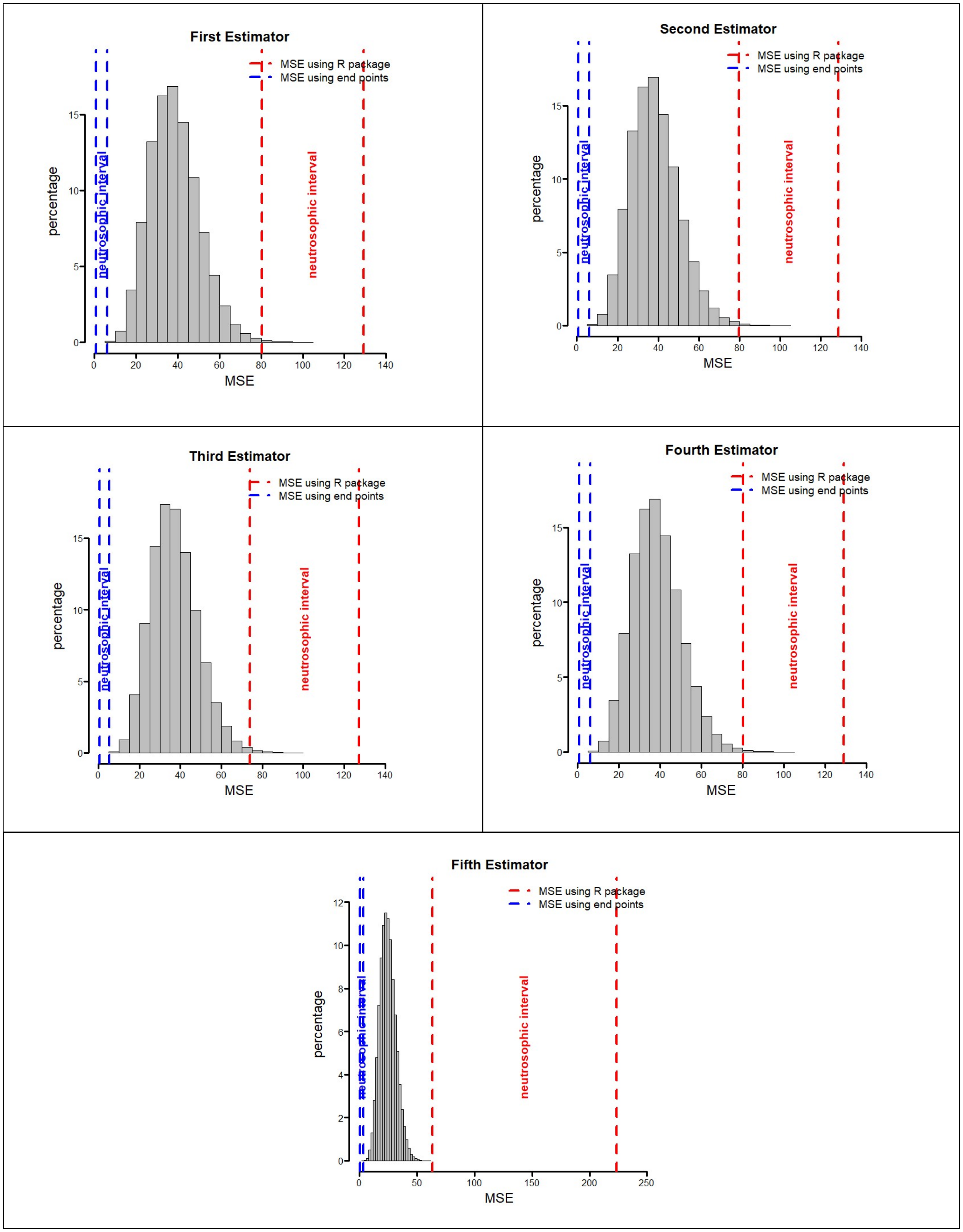

Figure 4: Empirical histograms and neutrosophic bounds of the MSEs of estimators for Population-IV

Figures 1–4 reveal that the neutrosophic MSE bounds obtained using the 'neutroSurvey' R package are often overly narrow or misleading, underscoring the need for simulation-based methods to achieve accurate and interpretable inference.

A fundamental shortcoming of the neutrosophic approach is that it derives one bound using only the lower endpoints and another using only the upper endpoints of the data. Although the original rationale for this procedure is not clearly documented, it appears to reproduce the published results despite its mathematical inadequacy.

5. Conclusion

This study demonstrated, through simulation, that the neutrosophic bounds for the MSEs of estimators are frequently inaccurate and fail to represent the true variability present in interval-valued populations. The simulation-based approach yields empirical MSE distributions that offer a more reliable and informative basis for inference, thereby highlighting the limitations of neutrosophic bounds and providing a practical alternative for analyzing interval-valued data.

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