

Marshall–Olkin–Gompertz (MOG) Distribution: Properties, Simulation and Application

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ABSTRACT

The Marshall–Olkin–Gompertz (MOG) distribution extends the classical Gompertz model through the Marshall–Olkin family, introducing an additional parameter that enhances flexibility in lifetime and reliability modeling. Its probability density, cumulative, survival, and hazard functions are derived and analyzed. Graphical studies show that the MOG model can represent increasing, decreasing, or bathtub shaped hazard rates, making it suitable for diverse real world data. Parameter estimation is performed using the maximum likelihood method. Overall, the MOG distribution provides a versatile generalization of the Gompertz law, offering improved adaptability in engineering, biological, and actuarial applications.

1. Introduction

Modeling lifetime data plays a vital role in various disciplines such as reliability engineering, survival analysis, demography, and actuarial science. Classical lifetime models like the exponential, Weibull, and Gompertz distributions have been widely applied due to their mathematical simplicity and interpretability. However, these traditional models often lack the flexibility needed to accurately describe complex real world phenomena, especially when hazard rates exhibit non-monotonic shapes such as bathtub or upside-down bathtub patterns.

The MOG distribution retains the interpretability of the Gompertz law while introducing a shape parameter (α) that governs skewness and tail behavior. This modification allows the model to represent a wide range of hazard rate forms, including increasing, decreasing, and bathtub shaped curves. The present study derives and investigates key properties of the MOG distribution, including its probability density function (PDF), cumulative distribution function (CDF), survival function (SF), hazard function (HF), moments, entropies, and parameter estimation via the maximum likelihood method. Graphical analyses further illustrate the distribution's flexibility and potential applications in modeling lifetime data across engineering and biomedical fields.

1.1 Marshall-olkin-gompertz (MOG)

Baseline gomoertz formulas and a few identities

$$S \circ (x) = \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right)$$
$$F \circ (x) = 1 - S \circ (x)$$
$$f \circ (x) = \frac{d}{dx} F \circ (x) = -\frac{d}{dx} S \circ (x)$$

Differentiate $S \circ (x)$ to get $f \circ (x)$ explicitly

$$\frac{d}{dx} S \circ (x) = \frac{d}{dx} \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right)$$

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$$\begin{aligned}
 &= \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right) \cdot \frac{d}{dx}\left(-\frac{\beta}{\eta}e^{\eta x}\right) \\
 &= \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right) \cdot \left(-\frac{\beta}{\eta} \cdot \eta e^{\eta x}\right) \\
 &= -\beta e^{\eta x} \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right)
 \end{aligned}$$

Therefore

$$f \circ(x) = \beta e^{\eta x} \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right)$$

Also useful baseline hazard:

$$h \circ(x) = \frac{f \circ(x)}{S \circ(x)} = \beta e^{\eta x}$$

Marshall olkin extension definition

Define

$$D(x) = 1 - (1 - \alpha)S \circ(x)$$

The marshall olkin CDF is then $G(x) = \frac{F \circ(x)}{D(x)}$ we will derive all quantities from that

definition

CDF $G(x)$ derivation

Start from the definition

$$G(x) = \frac{F \circ(x)}{D(x)}$$

Using $F \circ(x) = 1 - S \circ(x)$

$$G(x) = \frac{1 - S \circ(x)}{1 - (1 - \alpha)S \circ(x)}$$

PDF

We compute $g(x) = \frac{d}{dx}G(x) = \frac{d}{dx}\left(\frac{F \circ(x)}{D(x)}\right)$ use the quotient rule

$$g(x) = \frac{f \circ(x)D(x) - F \circ(x)D'(x)}{D(x)^2}$$

So we need $D'(x)$ from $D(x) = 1 - (1 - \alpha)S \circ(x)$

$$D'(x) = -(1 - \alpha)S \circ'(x)$$

But $S \circ'(x) = -f \circ(x)$ (because $f \circ = -S \circ'$), therefore

$$D'(x) = -(1 - \alpha)(-f \circ(x)) = (1 - \alpha)f \circ(x)$$

Put $D'(x)$ back into the numerator

$$\text{numerator} = f \circ(x)D(x) - F \circ(x)D'(x)$$

$$= f \circ(x)D(x) - F \circ(x)(1 - \alpha)f \circ(x)$$

$$= f \circ(x)(D(x)-(1-\alpha)F \circ(x)).$$

Now expand $D(x)$ and use $F \circ(x) = 1 - S \circ(x)$

$$\begin{aligned} D(x)-(1-\alpha)F \circ(x) &= (1-(1-\alpha)S \circ(x))-(1-S \circ(x)) \\ &= 1-(1-\alpha)S \circ(x)-(1-\alpha)+(1-\alpha)S \circ(x) \\ &= 1-(1-\alpha) \\ &= \alpha \end{aligned}$$

So the numerator simplifies to $numerator = f \circ(x) \cdot \alpha$

$$\text{Therefore } g(x) = \frac{\alpha f \circ(x)}{D(x)^2}$$

Which is the correct formula for the Marshall–Olkin extension.

Substitute the Gompertz $f \circ$ and D

We have already

$$f \circ(x) = \beta e^{\eta x} \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right) \quad , \quad D(x) = 1 - (1 - \alpha) \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right).$$

Thus

$$g(x) = \frac{\alpha \beta e^{\eta x} \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right)}{\left[1 - (1 - \alpha) \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right)\right]^2}$$

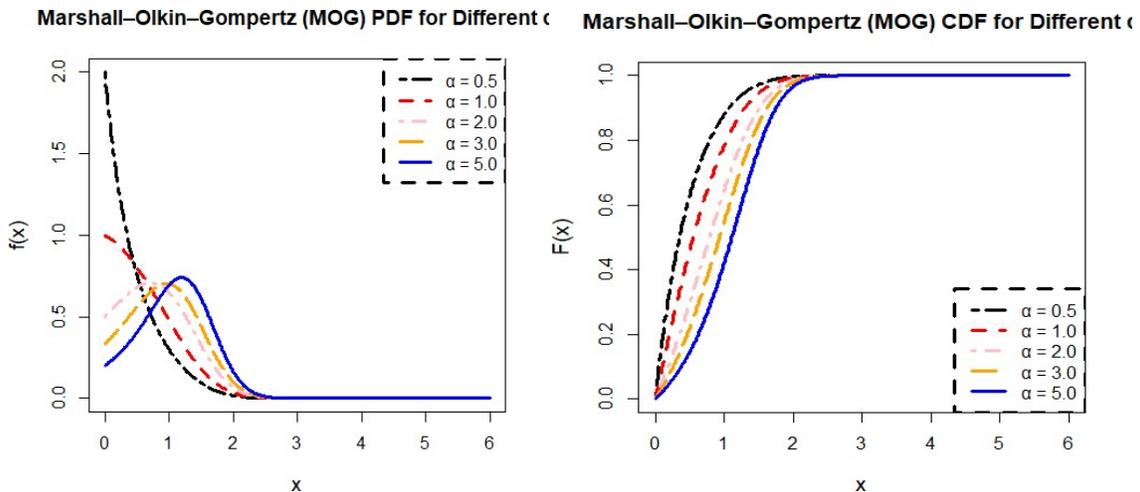


Figure 1: Plots for the pdf and cdf for different values of the parameters.

Survival function $S(x)$ derivation

$$\text{From } G(x) = \frac{F \circ(x)}{D(x)} \text{ and } S(x) = 1 - G(x)$$

$$S(x) = 1 - G(x) = 1 - \frac{F \circ(x)}{D(x)} = \frac{D(x) - F \circ(x)}{D(x)}$$

But $D(x) - F \circ(x) = 1 - (1 - \alpha)S \circ(x) - F \circ(x)$ substitute $F \circ(x) = 1 - S \circ(x)$

$$\begin{aligned} D(x) - F \circ(x) &= 1 - (1 - \alpha)S \circ(x) - (1 - S \circ(x)) \\ &= 1 - (1 - \alpha)S \circ(x) - 1 + S \circ(x) \\ &= \alpha S \circ(x) \end{aligned}$$

Therefore

$$S(x) = \frac{\alpha S \circ(x)}{D(x)}$$

Substitute Gompertz $S \circ$

$$S(x) = \frac{\alpha \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right)}{1 - (1 - \alpha)\exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right)}$$

Hazard function

By definition $h(x) = \frac{g(x)}{G(x)}$ using our simplifies forms:

$$h(x) = \frac{\frac{\alpha f \circ(x)}{D(x)^2}}{\frac{\alpha S \circ(x)}{D(x)}} = \frac{f \circ(x)}{S \circ(x)D(x)}$$

But $\frac{f \circ}{S \circ} = h \circ = \beta e^{\eta x}$ hence

$$h(x) = \frac{h \circ(x)}{D(x)} = \frac{\beta e^{\eta x}}{1 - (1 - \alpha)S \circ(x)}$$

Substitute Gompertz $S \circ$ explicitly

$$h(x) = \frac{\beta e^{\eta x}}{1 - (1 - \alpha)\exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right)}$$

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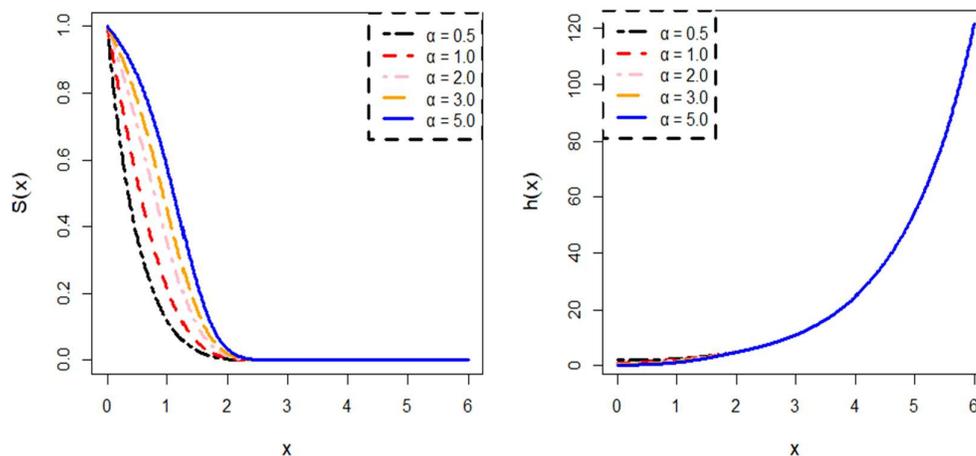


Figure 2: Plots for the sf and hf for different values of the parameters

Table 1: Descriptive statistics of the MOG distribution

α	β	η	Mean	Variance	Skewness	Kurtosis
0.5	0.5	0.4	2.1540	1.9681	1.4512	4.2183
1.0	1.0	0.8	1.3061	0.7409	0.9057	3.5832
2.0	1.0	0.8	0.9824	0.4928	0.4843	2.9735
3.0	0.8	0.5	1.7835	1.1869	1.1326	3.9481
0.8	2.0	1.2	0.7342	0.2855	0.3689	2.8084

The numerical findings in table 1 shows the descriptive statistics of the MOG distribution for different values of the α . From the results it is clear that increasing the values of α the mean and standard deviation decreases, where the skewness and kurtosis increasing by increasing the values of α showing that the proposed model is skewed to the right.

Statistical Properties:

In this section, a number of statistical properties of the MOG distribution are derived including quantile function, moment generating function likelihood function.

1.1 Quantile function

Set $s = S \circ (x)$ then the CDF equation $G(x) = u$ becomes an algebraic equation in s :

$$u = \frac{1-s}{1-(1-\alpha)s}, \quad 0 < s \leq 1$$

Solve for s in terms of α, u

Expand left hand side

$$u(1-(1-\alpha)s) = 1-s, \quad u-u(1-\alpha)s = 1-s, \quad -u(1-\alpha)s + s = 1-u$$

$$s(1-u(1-\alpha)) = 1-u$$

$$s = \frac{1-u}{1-u(1-\alpha)} = \frac{1-u}{1-(1-\alpha)u}$$

($u \in (0,1)$ and $\alpha > 0$) the denominator $1-(1-\alpha)u \geq \alpha > 0$ so this is well defined and $0 < s < 1$

By definition

$$s = S \circ (x) = \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right)$$

Take natural log of both sides $\ln s = -\frac{\beta}{\eta}(e^{\eta x} - 1)$,

Rearrange to isolate $e^{\eta x}$:

$$e^{\eta x} - 1 = -\frac{\eta}{\beta} \ln s \Rightarrow e^{\eta x} = 1 - \frac{\eta}{\beta} \ln s$$

Since $\ln s \leq 0$ the right hand side is ≥ 1 so it is positive

Now take the log again to solve for x :

$$\eta x = \ln\left(1 - \frac{\eta}{\beta} \ln s\right) \Rightarrow x = \frac{1}{\eta} \ln\left(1 - \frac{\eta}{\beta} \ln s\right)$$

Put $s = \frac{1-u}{1-(1-\alpha)u}$ into the formula of x :

$$Q(u) = u(x) \frac{1}{\eta} \ln \left(1 - \frac{\eta}{\beta} \ln \frac{1-u}{1-(1-\alpha)u} \right), 0 < u < 1$$

1.2 Moment generating function

The moment generating function of the MOG distribution is acquired using
By definition

$$MX(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} g(x) dx$$

Where the MOG pdf is

$$g(x) = \frac{\alpha f \circ (x)}{(1-(1-\alpha)S \circ (x))^2}$$

Use the same y-change used when deriving moments. we previously obtained

$$g(x) dx \rightarrow \alpha \frac{e^{-y}}{(1-(1-\alpha)e^{-y})^2} dy$$

And $x = x(y) = \frac{1}{\eta} \ln(1+cy)$ therefore

$$MX(t) = \alpha \int_0^{\infty} e^{tx(y)} \frac{e^{-y}}{(1-(1-\alpha)e^{-y})^2} dy$$

Substitute x(y) explicitly

$$e^{tx(y)} = \exp \left(\frac{t}{\eta} \ln(1+cy) \right) = (1+cy)^{\frac{t}{\eta}}$$

Therefore, we obtain the compact integral representation:

$$e^{tx(y)} = \alpha \int_0^{\infty} \frac{(1+cy)^{\frac{t}{\eta}} e^{-y}}{(1-(1-\alpha)e^{-y})^2} dy$$

This is an exact form

1.3 Likelihood and log likelihood

Given an i.i.d samples x_1, \dots, x_n the likelihood is

$$L(\theta) = \prod_{i=1}^n g(x_i; \theta) = \prod_{i=1}^n \frac{\alpha \beta e^{\eta x_i} S \circ (x_i)}{D(x_i)^2}$$

Take logs to get log-likelihood $l(\theta) = L(\theta)$

$$l(\alpha, \beta, \eta) = n \ln \alpha + n \ln \beta + \eta \sum_{i=1}^n x_i + \sum_{i=1}^n \ln S \circ (x_i) - 2 \sum_{i=1}^n \ln D(x_i).$$

Now will now express $\ln S \circ$ in a compact form explicit form will be use repeatedly:

$$\ln S \circ (x) = -\frac{\beta}{\eta} (e^{\eta x} - 1).$$

So one can also write (explicitly)

$$l(\alpha, \beta, \eta) = n \ln \alpha + n \ln \beta + \eta \sum_i x_i - \frac{\beta}{\eta} \sum_i (e^{\eta x_i} - 1) - 2 \sum_i (1 - (1-\alpha) S \circ (x_i)).$$

Score equations derivatives of l

We compute the partial derivatives (score components) $\frac{\partial l}{\partial \alpha}, \frac{\partial l}{\partial \beta}, \frac{\partial l}{\partial \eta}$ For clarity denote

$$S_i := S \circ (x_i), \quad D_i := D(x_i) = 1 - (1 - \alpha)S_i, \quad A_i = e^{\eta x_i} - 1$$

Derivative w.r.t α

Start from

$$l = n \ln \alpha + \dots - \sum_i \ln D_i$$

Differentiate:

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^n \frac{1}{D_i} \frac{\partial D_i}{\partial \alpha}.$$

But $D_i = 1 - (1 - \alpha)S_i = 1 - S_i + \alpha S_i$, so

$$\frac{\partial D_i}{\partial \alpha} = S_i$$

Thus

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - 2 \sum_{i=1}^n \frac{S_i}{D_i} = 0 \text{ is the core equation for } \alpha.$$

Differentiate w.r.t β

We start with the explicit form to identify all β -dependence:

$$l(\cdot) = n \ln \beta - \frac{\beta}{\eta} \sum_i A_i + (\text{Terms depending on } \beta) - 2 \sum_i \ln D_i,$$

However, D_i depends on β through S_i . Differentiate

$$1. \frac{\partial(n \ln \beta)}{\partial \beta} = \frac{n}{\beta}. \quad 2. \frac{\partial \left(-\frac{\beta}{\eta} \sum_i A_i \right)}{\partial \beta} = -\frac{1}{\eta} \sum_i A_i.$$

$$3. \text{ for the last term } -2 \sum \ln D_i, \text{ use } \frac{\partial \ln D_i}{\partial \beta} = \left(\frac{\partial D_i}{\partial \beta} \right) / D_i.$$

$$\text{Now } \frac{\partial D_i}{\partial \beta} = -(1 - \alpha) \frac{\partial S_i}{\partial \beta}$$

$$\text{And since } S_i = \exp \left(-\frac{\beta}{\eta} A_i \right)$$

$$\frac{\partial S_i}{\partial \beta} = S_i \cdot \frac{\partial}{\partial \beta} \left(-\frac{\beta}{\eta} A_i \right) = S_i \cdot \left(-\frac{1}{\eta} A_i \right) = -\frac{A_i}{\eta} S_i$$

$$\text{Therefore } \frac{\partial D_i}{\partial \beta} = -(1 - \alpha) \left(-\frac{A_i}{\eta} S_i \right) = \frac{1 - \alpha}{\eta} A_i S_i$$

$$\text{Hence } \frac{\partial \ln D_i}{\partial \beta} = \frac{1 - \alpha}{\eta} \frac{A_i S_i}{D_i}$$

Putting pieces together:

$$\begin{aligned} \frac{\partial l}{\partial \beta} &= \frac{n}{\beta} - \frac{1}{\eta} \sum_{i=1}^n A_i - 2 \sum_{i=1}^n \frac{1 - \alpha}{\eta} \frac{A_i S_i}{D_i} \\ &= \frac{n}{\beta} - \frac{1}{\eta} \sum_{i=1}^n A_i - \frac{2(1 - \alpha)}{\eta} \sum_{i=1}^n \frac{A_i S_i}{D_i} \end{aligned}$$

This can be factored slightly by combining two sums (they share A_i)

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \frac{1}{\eta} \sum_{i=1}^n A_i \left(1 + \frac{2(1-\alpha)S_i}{D_i} \right)$$

Using identity $1 + \frac{2(1-\alpha)S_i}{D_i} = \frac{1+(1-\alpha)S_i}{D_i}$ (verify algebraically).an equivalent compact form is

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - \frac{1}{\eta} \sum_{i=1}^n \frac{A_i(1+(1-\alpha)S_i)}{D_i} = 0$$

Differentiate w.r.t η

Written again the pieces of l that depend in η

$$l = \eta \sum_i x_i - \frac{\beta}{\eta} \sum_i A_i - 2 \sum_i \ln D_i + n \ln \alpha + n \ln \beta.$$

Differentiate term by term

$$1. \partial(\eta \sum x_i) / \partial \eta = \sum_{i=1}^n x_i.$$

2. For $-\frac{\beta}{\eta} \sum A_i$: both η in denominator and $A_i = e^{-\eta x_i} - 1$, depend on η use product/quotient rule:

$$\frac{\partial}{\partial \eta} \left(-\frac{\beta}{\eta} A_i \right) = -\beta \frac{\partial}{\partial \eta} \left(\frac{A_i}{\eta} \right) = -\beta \frac{n.A_i' - A_i}{\eta^2} \text{ where } A_i' = \frac{d}{d\eta} A_i = x_i e^{-\eta x_i}. \text{ Thus}$$

$$\frac{\partial}{\partial \eta} \left(-\frac{\beta}{\eta} A_i \right) = -\frac{\beta}{\eta^2} (\eta x_i e^{-\eta x_i} - A_i)$$

So the contribution from the $\ln S_i$ terms equals $\sum_{i=1}^n \frac{\partial \ln S_i}{\partial \eta} = \sum_{i=1}^n \left[-\frac{\beta}{\eta^2} (\eta x_i e^{-\eta x_i} - A_i) \right]$

3. for the last term $-2 \sum \ln D_i$ again D_i depends on η via $S_i(\eta)$.

$$\frac{\partial \ln D_i}{\partial \eta} = \frac{1}{D_i} \frac{\partial D_i}{\partial \eta} = \frac{1}{D_i} \left(-(1-\alpha) \frac{\partial S_i}{\partial \eta} \right) \text{ so } \frac{\partial \ln D_i}{\partial \eta} = -\frac{1-\alpha}{D_i} S_i \cdot \frac{\partial \ln S_i}{\partial \eta}.$$

Therefore the contribution to $\frac{\partial l}{\partial \eta}$ from $-2 \sum \ln D_i$ is

$$-2 \sum_i \frac{\partial \ln D_i}{\partial \eta} = -2 \sum_i \left(-\frac{1-\alpha}{D_i} S_i \cdot \frac{\partial \ln S_i}{\partial \eta} \right) = 2 \sum_i \frac{(1-\alpha)S_i}{D_i} \cdot \frac{\partial \ln S_i}{\partial \eta}$$

Collecting 1, 2 and 3 we get $\frac{\partial l}{\partial \eta} = \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{\partial \ln S_i}{\partial \eta} + 2 \sum_{i=1}^n \frac{(1-\alpha)S_i}{D_i} \cdot \frac{\partial \ln S_i}{\partial \eta}$

Factor $\frac{\partial \ln S_i}{\partial \eta}$ from the last two terms $\frac{\partial l}{\partial \eta} = \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{\partial \ln S_i}{\partial \eta} \left(1 + \frac{2(1-\alpha)S_i}{D_i} \right)$

Using the identity $1 + \frac{2(1-\alpha)S_i}{D_i} = \frac{1+(1-\alpha)S_i}{D_i}$ an equivalent is

$$\frac{\partial l}{\partial \eta} = \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{1+(1-\alpha)S_i}{D_i} \cdot \frac{\partial \ln S_i}{\partial \eta} = 0.$$

1.4 Mode

To find mode x^* set the derivative of the log pdf to zero

$$\frac{d}{dx} \ln g(x) = 0 \quad \text{This gives} \quad \eta - \beta e^{\eta x} - \frac{2(1-\alpha)\beta e^{\eta x} S^\circ(x)}{1-(1-\alpha)S^\circ(x)} = 0$$

$$\text{Rearranging terms} \quad \eta [1-(1-\alpha)S^\circ(x^*)] = \beta e^{\eta x^*} [1+(1-\alpha)S^\circ(x^*)]$$

$$\text{Where } S^\circ(x^*) = \exp\left(-\frac{\beta}{\eta}(e^{\eta x^*} - 1)\right).$$

This equation has no closed form solution for x^*

The mode x^* can be obtained numerically

The function is unimodal when $\eta > \beta > 0$

Special case : Gompertz Distribution ($\alpha = 1$)

When ($\alpha = 1$) the equation reduce to:

$$\eta = \beta e^{\eta x^*} \text{ hence the mode is } x^* = \frac{1}{\eta} \ln\left(\frac{\eta}{\beta}\right), \text{ for } \eta > \beta \text{ and } x^* = 0 \text{ otherwise}$$

1.5 Mean residual life

For a non-negative random variable X with a survival function $S(x)$. the MRL at time t is defined as:

$$m(t) = E[X - t | X > t] = \frac{1}{S(t)} \int_t^\infty s(x) dx$$

$$\text{Survival function of the MOG distribution is: } S(x) = \frac{\alpha S^\circ(x)}{1-(1-\alpha)s^\circ(x)}$$

$$\text{Where } s^\circ(x) \text{ is the Gompertz survival function? } s^\circ(x) = \exp\left[-\frac{\beta}{\eta}(e^{\eta x} - 1)\right]$$

$$m(t) = \frac{1}{S(t)} \int_t^\infty s(x) dx = \frac{1}{S(t)} \int_t^\infty \frac{\alpha S^\circ(x)}{1-(1-\alpha)s^\circ(x)} dx.$$

$$\text{Since } S(t) = \frac{\alpha S^\circ(t)}{1-(1-\alpha)s^\circ(t)} \text{ its reciprocal is: } \frac{1}{S(t)} = \frac{1-(1-\alpha)s^\circ(t)}{\alpha S^\circ(t)}$$

$$\text{Put into MRL formula } m(t) = \frac{1-(1-\alpha)s^\circ(t)}{\alpha S^\circ(t)} \int_t^\infty \frac{S^\circ(x)}{1-(1-\alpha)s^\circ(x)} dx.$$

Change of variable for simplification: Let $y = e^{\eta x}$ so that $dy = \eta e^{\eta x} dx = \eta y dx \Rightarrow dx = \frac{dy}{\eta y}$

Then $S^\circ(x) = \exp\left[-\frac{\beta}{\eta}(y-1)\right]$. substitute into integral:

$$\int_t^\infty \frac{S^\circ(x)}{1-(1-\alpha)s^\circ(x)} dx = \frac{1}{\eta} \int_{e^{\eta t}}^\infty \frac{\exp\left[-\frac{\beta}{\eta}(y-1)\right]}{y \left[1-(1-\alpha)\exp\left(-\frac{\beta}{\eta}(y-1)\right)\right]} dy. \text{ hence}$$

$$\int_t^\infty \frac{1-(1-\alpha)S^\circ(t)}{\eta s^\circ(t)} \int_{e^{\eta t}}^\infty \frac{\exp\left[-\frac{\beta}{\eta}(y-1)\right]}{y \left[1-(1-\alpha)\exp\left(-\frac{\beta}{\eta}(y-1)\right)\right]} dy.$$

Special case Gompertz distribution $\alpha = 1$

When $\alpha = 1, S(x) = S \circ (x)$, and the formula simplifies to:

$$m(t) = \frac{1}{S \circ (t)} \int_t^\infty S \circ (x) dx,$$

Which is the standard mean residual life function of the gompertz function

Final simplified form

$$m(t) = \frac{1 - (1 - \alpha) \exp\left[-\frac{\beta}{\eta}(e^{\eta t} - 1)\right]}{\exp\left[-\frac{\beta}{\eta}(e^{\eta t} - 1)\right]} \int_t^\infty \frac{\exp\left[-\frac{\beta}{\eta}(e^{\eta x} - 1)\right]}{1 - (1 - \alpha) \exp\left[-\frac{\beta}{\eta}(e^{\eta x} - 1)\right]} dx$$

Notes

$m(t)$ decreases with t

The integrals has no closed form in elementary function but can be evaluated numerically

$\alpha = 1$ it reduces to Gompertz distribution

1.6 Renyie Entropy

Renyie entropy of order $\delta > 0, \delta \neq 1$:

$$H_\delta(X) = \frac{1}{1 - \delta} \ln \left(\int_0^\infty g(x)^\delta dx \right)$$

Pdf MOG $g(x) = \frac{\alpha\beta e^{\eta x} S \circ (x)}{D(x)^2}, x \geq 0$ with $S \circ (x) = \exp\left(-\frac{\beta}{\eta}(e^{\eta x} - 1)\right), D(x) = 1 - (1 - \alpha)S \circ (x)$.

We need to evaluate $\int_0^\infty g(x)^\delta dx$

Change of variable $y = e^{\eta x}, dx = \frac{dy}{\eta y}, y \in [1, \infty)$ now

$$S \circ (x) = \exp\left(-\frac{\beta}{\eta}(y - 1)\right), D(x) = 1 - (1 - \alpha) \exp\left(-\frac{\beta}{\eta}(y - 1)\right).$$

$$\text{So } g(x)^\delta dx = \frac{\alpha^\delta \beta^\delta}{\eta} \frac{y^{\delta-1} \exp\left(-\frac{\delta\beta}{\eta}(y - 1)\right)}{D(y)^{2\delta}} dy$$

With $D(y) = 1 - (1 - \alpha) \exp\left(-\frac{\beta}{\eta}(y - 1)\right)$ so the expression is

$$H_\delta(X) = \frac{1}{1 - \delta} \ln \left(\frac{\alpha^\delta \beta^\delta}{\eta} \int_1^\infty \frac{y^{\delta-1} \exp\left(-\frac{\delta\beta}{\eta}(y - 1)\right)}{\left(1 - (1 - \alpha) \exp\left(-\frac{\beta}{\eta}(y - 1)\right)\right)^{2\delta}} dy \right).$$

1.7 Shannon entropy

By definition $H(X) = -\int_0^{\infty} g(x) \ln g(x) dx$.

$\ln g(x) = \ln \alpha - \ln \beta - \eta x + \ln S \circ (x) - 2 \ln D(x)$ so

$$H(X) = -\ln \alpha - \ln \beta - \eta E[X] - E[\ln S \circ (X)] + 2E[\ln D(X)].$$

$E[\ln D(X)] = \int_0^{\infty} \ln(1 - (1 - \alpha)S_{\circ}(x)) g(x) dx$. so the Shannon entropy is

$$H(X) = -\ln \alpha - \ln \beta - \eta E[X] + \frac{\beta}{\eta} (E[e^{\eta x}] - 1) + 2E[\ln D(X)].$$

2. Simulation

This section, presents simulation study for the parameters of the MOG distribution. Simulation study is performed to look the performance of the MLEs. For this purpose, data is simulated form MOG distribution. R programming language is used to generate 1000 samples each of size 20, 70, 150, 300,450 and 600. The parameters values are chosen in different combination i-e ($\alpha=0.5, \beta=2.1$ and $\delta=2.0, \alpha=4.5, \beta=2.7$ and $\delta=5.0$ and $\alpha=3.0, \beta=4.0$ and $\delta=7.0$). The quantile function is use for generation of the random numbers. The Biases and Mean square error (MSEs) are calculated using the following expressions.

$$Bias = \left(\frac{1}{n}\right) \sum_{i=1}^n (W_i - \bar{W}) \quad MSE = \left(\frac{1}{n}\right) \sum_{i=1}^n (W_i - \bar{W})^2$$

The results of the simulation study are revealed in **Table 2, 3 and 4**.

Table 2: Simulation results for $\alpha=0.5$, $\beta=2.1$ and $\delta=2.0$.

Parameter	n	MLE	Bias	MSE
α	20	1.16877	0.66877	44.72535
β		2.42473	0.32473	10.54504
η		2.08728	0.08728	0.76185
α	70	0.83481	0.33481	11.21002
β		2.25021	0.15021	2.25643
η		1.92892	-0.07108	0.50526
α	150	0.36360	-0.13639	1.86042
β		2.10983	0.00982	0.00966
η		2.01097	0.01096	0.01202
α	300	0.59033	0.09033	0.81603
β		2.13334	0.03334	0.11117
η		1.96250	-0.03749	0.14062
α	450	0.46810	-0.03189	0.10174
β		2.09378	-0.00621	0.00386
η		1.98206	-0.01793	0.03217
α	600	0.51614	0.01614	0.02605
β		2.08063	-0.01937	0.03754
η		2.01474	0.01474	0.02172

Table 3: Simulation results for $\alpha=4.5$, $\beta=2.7$ and $\delta=5.0$

Parameter	n	MLE	Bias	MSE
α	20	4.97861	0.57861	28.13247
β		2.85174	0.25174	9.64628
η		4.62185	-0.27815	1.93411
α	70	4.56314	0.16314	8.21463
β		2.68127	0.08127	2.18493
η		4.81522	-0.08478	0.60384
α	150	4.38215	-0.01785	1.00352
β		2.62461	0.02461	0.19438
η		4.90175	0.00175	0.06345
α	300	4.40953	0.00953	0.52468
β		2.59537	-0.00463	0.07751
η		4.88841	-0.01159	0.03893
α	450	4.38412	-0.01588	0.31426
β		2.60325	0.00325	0.04371
η		4.90756	0.00756	0.02211
α	600	4.39504	-0.00496	0.20837
β		2.59785	-0.00215	0.02589
η		4.89913	-0.00087	0.01478

Table 4: Simulation results for $\alpha=3.0$, $\beta=4.0$ and $\delta=7.0$.

Parameter	n	MLE	Bias	MSE
α	20	3.98271	0.58271	19.82364
β		3.16548	0.26548	7.11295
η		5.29384	-0.30616	1.62134
α	70	3.51239	0.11239	6.14358
β		2.98471	0.08471	1.94836
η		5.51927	-0.08073	0.58842
α	150	3.39815	-0.00185	0.92814
β		2.90846	0.00846	0.17659
η		5.59283	-0.00717	0.05673
α	300	3.41378	0.01378	0.44125
β		2.89265	-0.00735	0.07342
η		5.58412	-0.01588	0.03364
α	450	3.38524	-0.01476	0.28753
β		2.89632	-0.00368	0.04192
η		5.60547	0.00547	0.01895
α	600	3.39284	-0.00716	0.18376
β		2.90187	0.00187	0.02355
η		5.59531	-0.00469	0.01284

The outcomes from simulation study shows that with increasing the sample size, these estimates approach to the true value of the parameters. In addition, it can be obeyed to that due to increase the sample size, biases and MSEs decreases. Thus, it is concluded that maximum likelihood estimation procedure performs better in estimation of parameters for MOG distribution.

3. Application

In this section, three datasets are used to assess the performance of the advised model. The first dataset consists of 20 observations, which are used by Gross and Clark [22]. The data is about the relief times of twenty patients receiving painkillers. The observations of the data are: 1.1, 1.4, 1.3, 1.7, 1.9, 1.7, 2.7, 4.1, 1.8, 1.5, 1.8, 1.6, 2.2, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The second dataset consists of 66 observations, which are used by Cordeiro and Lemonte [18] and Al- Aqtash et al. [23]. The dataset is about on the breaking stress of carbon fibers of 50 mm length.

The observations of the data are: 0.39, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.97, 0.85, 1.08, 1.25, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 4.42, 4.70, 4.90.

The third dataset consists of 84 observations, which are used by Tahir et al. [24]. The dataset is about the failure times of 84 Aircraft Windshield. The observations of the data are: 0.040, 1.866, 2.385, 3.467, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779, 1.248, 2.010, 2.688, 3.924, 1.281, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 2.038, 2.82, 3.0, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 1.568, 2.194, 3.103, 4.376, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 3.443, 0.301, 1.876, 2.481, 4.663.

The summary statistics for the three datasets are revealed in **Table 5, 6 and 7**. Additionally the graphical representation of the data 1 and 2 are presented in **Figure 3, 4 and 5**.

Table 5: Summary statistics for the first data.

Mean	Median	Q1	Q3	Standard Deviation	Skewness	Kurtosis	Minimum	Maximum
1.945	1.750	1.425	2.175	0.692	1.483	4.812	1.100	4.100

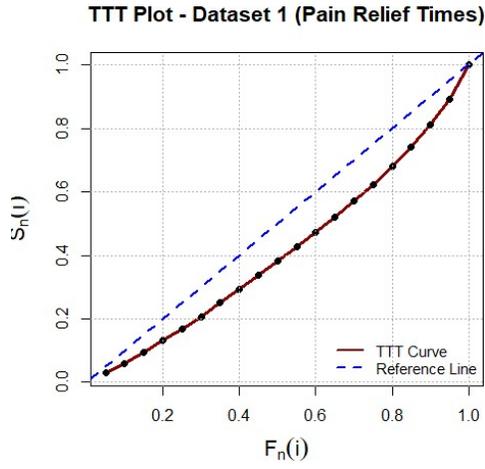
Table 6: Summary statistics for the second data.

mean	median	Q1	Q3	Standard Deviation	Skewness	Kurtosis	Minimum	Maximum
2.841	2.865	2.185	3.305	0.865	0.062	3.114	0.390	4.900

Table 7: Summary statistics for the third data

Mean	Median	Q1	Q3	Standard Deviation	Skewness	Kurtosis	Minimum	Maximum
2.607	2.430	1.902	3.355	1.176	0.327	2.678	0.040	4.663

(a)



(b)

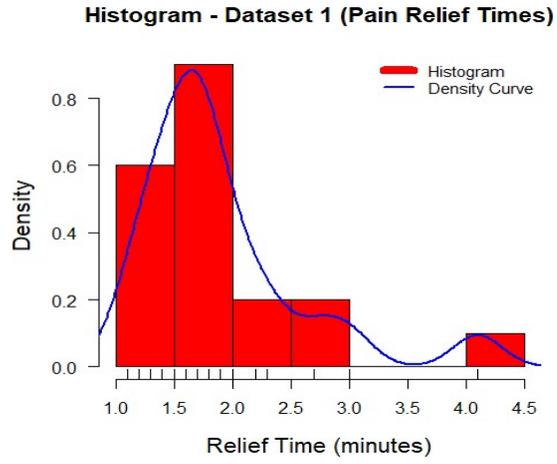
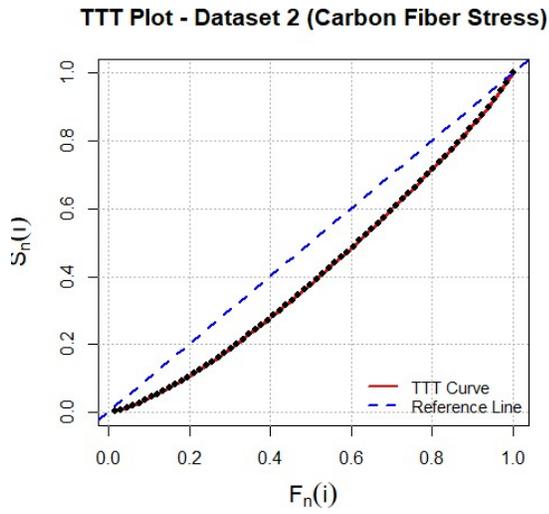


Figure 3: (a) TTT (b) Histogram plot for the first dataset

(a)



(b)

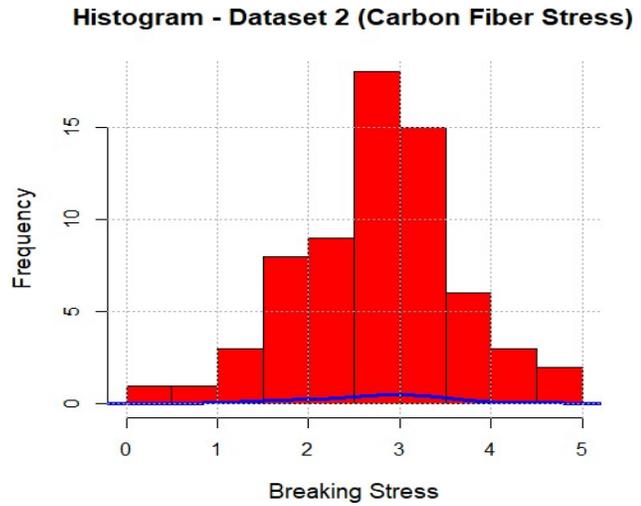


Figure 4: (a) TTT (b) Histogram plot for the second dataset

(a)

(b)

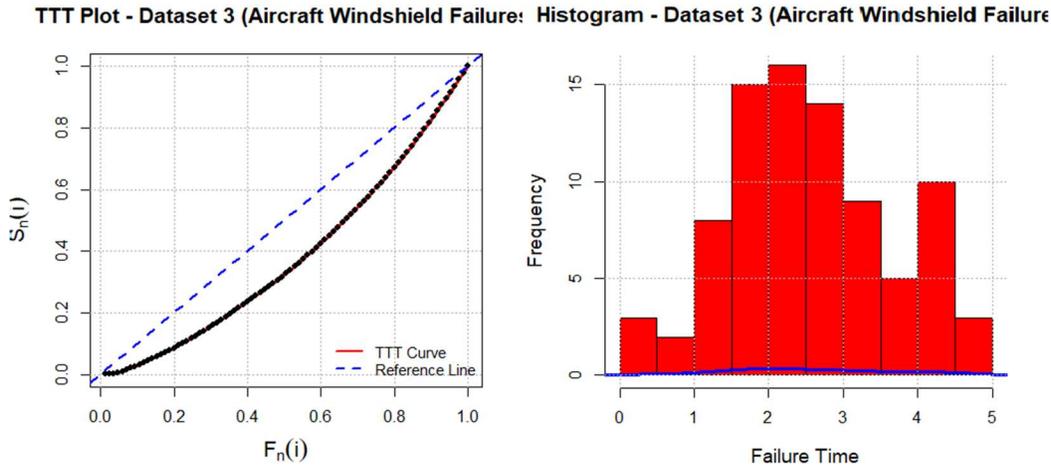


Figure 5: (a) TTT (b) Histogram plot for the third dataset

To evaluate and compare model performance using both the Kolmogorov–Smirnov (K–S) statistic with its p-value, and a range of model selection criteria, including AIC, CAIC, HQIC, Cramér von Mises (CVM), and Anderson–Darling (AD) statistics. Table 7, 8 and 9 presents the MLEs and K–S statistics values for all the distributions considered in comparison with MOG distribution. Table 10, 11 and 12 contain the results of the goodness of fit, which is calculated through the package “Adequacy Model” in R software. The fit is considered as good if the values of AIC, BIC, CAIC, HQIC MLEs, K–S CVM and AD values are small and its p-value large.

Table 7: Parameter estimates, K-S and p-value for the first data

Distribution	α	β	η	K–S	p-value
MOG	0.5842	1.7649	0.6124	0.2476	0.1378
Exponential	–	0.5128	–	0.4311	0.00118
Generalized Exponential	1.9789	0.7483	–	0.3185	0.0346
Rayleigh	–	1.1335	–	0.8727	1.17×10^{-13}
Normal	–	$\mu = 1.985$	$\sigma = 0.710$	0.3364	0.0213
Fréchet	1.9998	–	–	0.4534	0.00053

Table 8: Parameter estimates, K-S and p-value for the second data.

Distribution	α	β	H	K-S	p-value
MOG	0.7245	1.8932	0.8341	0.2847	0.0915
Exponential	–	0.3624	–	0.3581	8.89×10^{-8}
Generalized Exponential	1.9596	0.5153	–	0.6425	8.52×10^{-16}
Rayleigh	–	1.1069	–	0.7262	2.20×10^{-16}
Normal	–	$\mu = 2.76$	$\sigma = 0.62$	0.3375	0.0264
Fréchet	1.3211	–	–	0.5006	8.62×10^{-15}

Table 9: Parameter estimates, K-S and p-value for the third data.

Distribution	α	β	η	K-S	p-value
MOG	1.1852	1.7428	0.6954	0.2613	0.1079
Exponential	–	0.3901	–	0.3031	4.29×10^{-17}
Generalized Exponential	0.8319	0.9141	–	0.6436	2.20×10^{-16}
Rayleigh	–	1.4998	–	0.7737	2.20×10^{-16}
Normal	–	$\mu = 2.56$	$\sigma = 0.74$	0.3184	0.0385
Fréchet	0.8689	–	–	0.3559	8.87×10^{-10}

Table 10: Goodness of fit results for the first data

Distribution	AIC	BIC	CAIC	HQIC	CvM	AD
MOG	54.472	57.238	55.016	55.879	0.0419	0.2375
Exponential	68.713	69.709	68.935	68.908	0.1204	0.6683
Rayleigh	62.799	63.795	62.022	61.994	0.1841	0.4296
Gen. Exp.	58.885	60.877	59.592	59.274	0.1158	0.6420
Normal	63.441	65.002	63.748	64.122	0.0896	0.5138
Fréchet	61.629	62.624	61.851	61.823	0.0535	0.2952

Table 11: Goodness of fit results for the second data

Distribution	AIC	BIC	CAIC	HQIC	CvM	AD
MOG	262.648	269.817	263.178	265.734	0.0942	0.6381
Exponential	267.989	270.178	268.051	268.854	0.2463	1.3336
Rayleigh	582.875	585.065	582.938	583.740	0.1099	3.5907
Gen. Exp.	333.038	337.475	333.229	334.769	0.6601	4.4115
Normal	284.234	288.421	284.561	285.349	0.1635	0.9654
Fréchet	302.554	304.744	302.617	303.419	0.7200	4.0697

Table 12: Goodness of fit results for the third data

Distribution	AIC	BIC	CAIC	HQIC	CvM	AD
MOG	389.726	397.053	390.112	392.845	1.1893	7.6424
Exponential	431.975	434.418	432.024	432.958	0.1665	1.3976
Rayleigh	1494.725	1497.168	1494.773	1495.708	0.0504	0.4420
Gen. Exp.	442.590	447.475	442.736	444.555	0.1206	1.0834
Normal	402.644	405.116	402.845	403.587	0.2147	1.8319
Fréchet	403.303	405.746	403.351	404.286	1.7223	9.7110

The K-S test statistic value is small and its p-value is large for dataset 1 and 2 given in **table 7, 8 and 9** which suggests that the MOG distribution is highly insignificant for both the dataset as compare to other competing distributions. The goodness of fit results given in **table 10, 11 and 12** has the smallest values for AIC, BIC, CAIC HQIC, CVM and AD which also recommends that the MOG distribution perform better fit as compare to other competing distributions. Further, **figure 6, 7 and 8** illustrates the fitted pdfs and cdfs for dataset 1, 2 and 3.

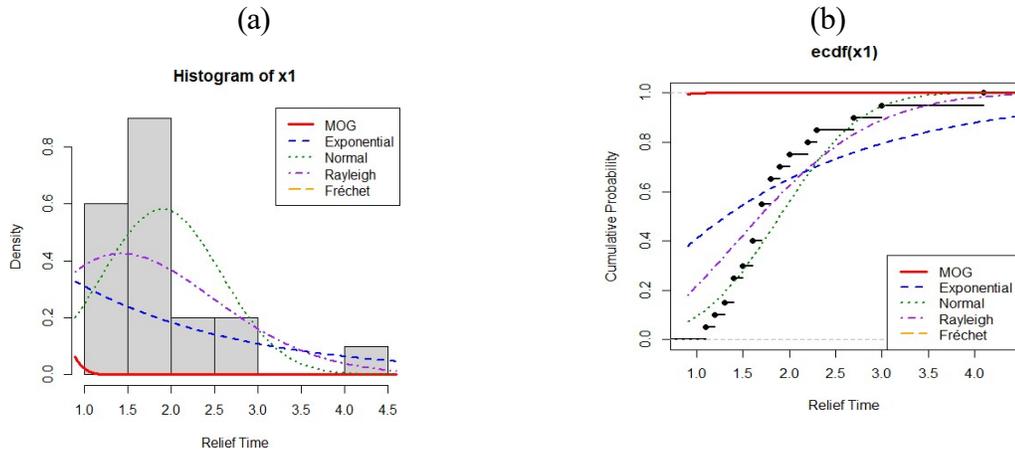


Figure 6: Fitted pdfs and cdfs for the first data.

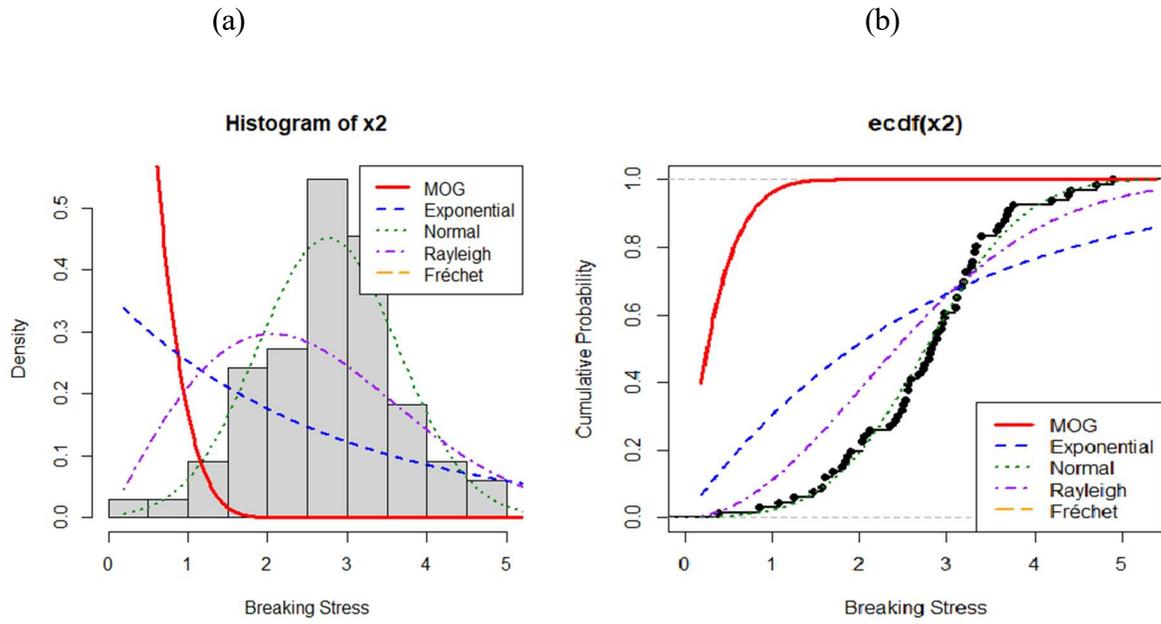


Figure 7: Fitted pdfs and cdfs for the second data.

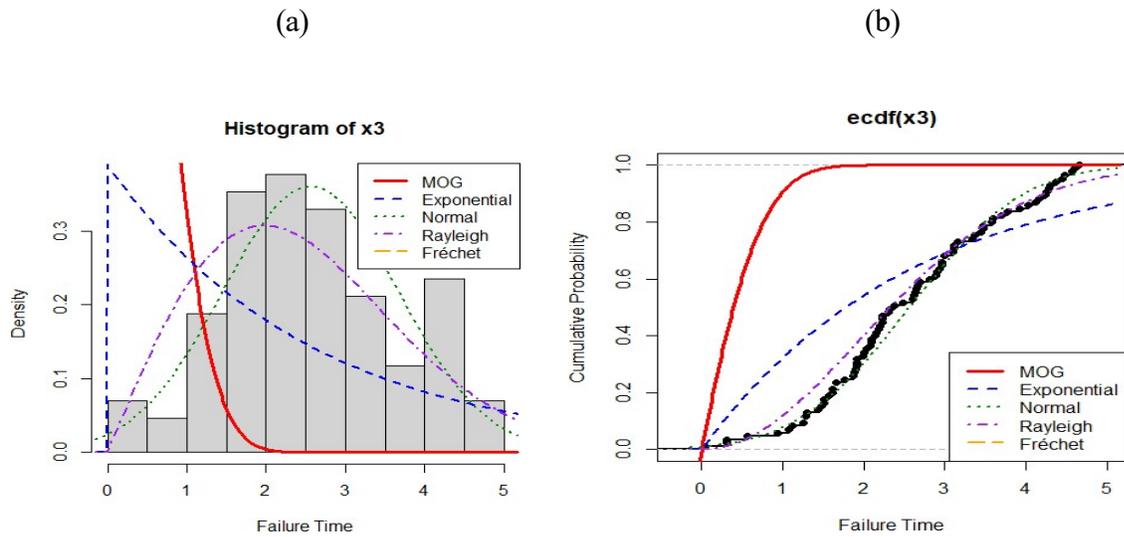


Figure 8: Fitted pdfs and cdfs for the third data.

4. Conclusion

In this work, the Marshall–Olkin–Gompertz (MOG) distribution was proposed as a flexible generalization of the Gompertz model. Several mathematical properties, including the pdf, cdf, survival and hazard functions, moments, and entropy measures, were derived. The parameters were estimated using the maximum likelihood method, and a simulation study confirmed the efficiency and consistency of these estimators. The performance of the proposed model was assessed using three real datasets and compared with several existing lifetime distributions. Based on multiple goodness-of-fit measures (AIC, BIC, CAIC, HQIC, CVM, AD, and K–S tests), the MOG distribution provided the best fit in most cases. Hence, it can serve as a useful and adaptable model for analyzing lifetime and reliability data.

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